Seminar of I. Vekua Institute
of Applied Mathematics
REPORTS, Vol. 34, 2008

THE METHOD OF THE SMALL PARAMETER FOR NON-LINEAR SHALLOW CYLINDRICAL SHELLS

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Abstract. In the present paper we consider the geometrically non-linear shallow cylindrical shells. By means of I. Vekua method the system of equilibrium equations in two variables is obtained. Using complex variable functions and the method of the small parameter approximate solutions are constructed for $N = 1$ in the hierarchy by I. Vekua. Concrete problem is solved, when the components of the external force are constants.

Keywords and phrases: Shallow cylindrical shells, small parameter.

AMS subject classification (2000): 74K25.

I. Vekua has constructed refined theory of shallow shells [1],[2]. This method for non-shallow shells in case of geometrical and physical theory was generalized by T. Meunargia [3],[4].

In the present paper we consider the system of equilibrium equations of the two-dimensional geometrically non-linear shallow cylindrical shells which are obtained from the three-dimensional problems of the theory of elasticity for isotropic and homogeneous shell by the method of I. Vekua.

The displacement vector $\mathbf{U}(x^1, x^2, x^3)$ is expressed by the following formula [1]

$$\mathbf{U}(x^1, x^2, x^3) = \mathbf{u}(x^1, x^2) + \frac{x^3}{h}\mathbf{v}(x^1, x^2).$$

Here $\mathbf{u}(x^1, x^2)$ and $\mathbf{v}(x^1, x^2)$ are the vector fields on the middle surface $x^3 = 0$, $2h$ is the thickness of the shell, x^3 is a thickness coordinate $(-h \leq x^3 \leq h)$, x^1 and x^2 are isometric coordinates on the cylindrical surface.

Let us construct the solutions of the form [4]

$$u_i = \sum_{k=1}^{\infty} \overset{k}{u}_i \varepsilon^k, \qquad v_i = \sum_{k=1}^{\infty} \overset{k}{v}_i \varepsilon^k \quad (i = 1, 2, 3),$$

where u_i and v_i are the components of the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively, $\varepsilon = \frac{h}{R_0}$ is a small parameter, R_0 is the radius of the middle surface of the cylinder.

The system of equilibrium equations of the two-dimensional non-shallow cylindrical shells may be written in the following form [1], [5]

$$\begin{aligned}
\mu\Delta \overset{k}{u}_1 &+ (\lambda+\mu)\partial_1 \overset{k}{\theta} + \lambda\partial_1 \overset{k}{v}_3 = \overset{k}{X}_1,\\
\mu\Delta \overset{k}{u}_2 &+ (\lambda+\mu)\partial_2 \overset{k}{\theta} + \lambda\partial_2 \overset{k}{v}_3 = \overset{k}{X}_2,\\
\mu\Delta \overset{k}{v}_3 &- 3\left[\lambda \overset{k}{\theta} + (\lambda+2\mu) \overset{k}{v}_3\right] = \overset{k}{X}_3,
\end{aligned} \tag{1}$$

$$\begin{aligned}
\mu\Delta \overset{k}{v}_1 &+ (\lambda+\mu)\partial_1 \overset{k}{\Theta} - 3\mu(\partial_1 \overset{k}{u}_3 + \overset{k}{v}_1) = \overset{k}{X}_4,\\
\mu\Delta \overset{k}{v}_2 &+ (\lambda+\mu)\partial_2 \overset{k}{\Theta} - 3\mu(\partial_2 \overset{k}{u}_3 + \overset{k}{v}_2) = \overset{k}{X}_5,\\
\mu\Delta \overset{k}{u}_3 &+ \mu \overset{k}{\Theta} = \overset{k}{X}_6,\\
&(k=1,2,...),
\end{aligned} \tag{2}$$

where λ and μ are Lame's constants, $\overset{k}{X}_p$ $(p=1,...,6)$ are the components of external force and well-known quantities, defined by functions $\overset{0}{u}_i, ..., \overset{k-1}{u}_i, \overset{0}{v}_j, ..., \overset{k-1}{v}_j$.

The general solutions of systems (1) end (2) are written in the following form

$$\begin{aligned}
2\mu \overset{k}{u}_+ &= \frac{5\lambda+6\mu}{3\lambda+2\mu}\overset{k}{\varphi}(z) - z\overline{\overset{k}{\varphi}{}'(z)} - \overline{\overset{k}{\psi}(z)} - \frac{\lambda}{6(\lambda+\mu)}\frac{\partial \overset{k}{\chi}(z,\bar z)}{\partial \bar z} + \widehat{\overset{k}{u}}_+,\\
2\mu \overset{k}{v}_3 &= -\frac{2\lambda}{3\lambda+2\mu}\left(\overset{k}{\varphi}{}'(z) + \overline{\overset{k}{\varphi}{}'(z)}\right) + \overset{k}{\chi}(z,\bar z) + \widehat{\overset{k}{v}}_3,\\
2\mu \overset{k}{v}_+ &= \frac{4(\lambda+2\mu)}{3\mu}\overline{\overset{k}{f}{}''(z)} + z\overline{\overset{k}{f}{}'(z)} + \overset{k}{f}(z) - 2\overline{\overset{k}{g}{}'(z)} + i\frac{\partial \overset{k}{w}(z,\bar z)}{\partial \bar z} + \widehat{\overset{k}{v}}_+,\\
2\mu \overset{k}{u}_3 &= -\frac{1}{2}\left(\bar z \overset{k}{f}(z) + z\overline{\overset{k}{f}(z)}\right) + \overset{k}{g}(z) + \overline{\overset{k}{g}(z)} + \widehat{\overset{k}{u}}_3,
\end{aligned}$$

$$\Bigg(\overset{k}{u}_+ = \overset{k}{u}_1 + i\overset{k}{u}_2, \quad \overset{k}{v}_+ = \overset{k}{v}_1 + i\overset{k}{v}_2, \quad z = x^1 + ix^2,$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right)\Bigg),$$

where $\overset{k}{\varphi}(z)$,$\overset{k}{\psi}(z)$,$\overset{k}{f}(z)$ and $\overset{k}{g}(z)$ are any analytic functions of z, $\overset{k}{\chi}(z,\bar z)$ and $\overset{k}{w}(z,\bar z)$ are the general solutions of the following Helmholtz's equations, respectively:

$$\begin{aligned}
\Delta \overset{k}{\chi} &- \eta^2 \overset{k}{\chi} = 0 \quad \left(\eta^2 = \frac{12(\lambda+\mu)}{\lambda+2\mu}\right),\\
\Delta \overset{k}{w} &- \gamma^2 \overset{k}{w} = 0 \quad \left(\gamma^2 = 3\right).
\end{aligned}$$

Here $\widehat{\overset{k}{u}}_+$, $\widehat{\overset{k}{v}}_3$ and $\widehat{\overset{k}{v}}_+$, $\widehat{\overset{k}{u}}_3$ are particular solutions of the non-homogeneous equations (1) and (2), respectively.

We solve the problem when the middle surface of the body after development on the plane, is the circle with the radius R. Let's consider the concrete problem, when the components of the external force are constant $X_1 = X_2 = 0, X_3 = q$. Boundary conditions are

$$u_r + iu_\vartheta = 0, \quad |\, z \,|= R, \qquad v_3 = 0 \quad |\, z \,|= R, \tag{3}$$

$$v_r + iv_\vartheta = 0, \quad |\, z \,|= R, \qquad u_3 = 0 \quad |\, z \,|= R, \tag{4}$$

This problem for the approximation $k = 1$ is a well known case in the theory of elasticity for which we have

$$\begin{aligned}
2\mu \overset{1}{u}_{+} &= \left(\frac{2(\lambda+2\mu)}{3\lambda+2\mu}a_1 + \frac{\lambda}{12(\lambda+\mu)}q\right) z - \frac{\lambda\eta}{12(\lambda+\mu)}\alpha_0 I_1(\eta r)e^{i\theta},\\
2\mu \overset{1}{v}_{3} &= \alpha_0 I_0(\eta r) - \frac{\lambda+2\mu}{6(\lambda+\mu)}q - \frac{4\lambda}{3\lambda+2\mu}a_1,\\
2\mu \overset{1}{v}_{+} &= -\frac{3\mu R^2}{8(\lambda+2\mu)}qz + \frac{3\mu R^2}{8(\lambda+2\mu)}qz^2\bar{z},\\
2\mu \overset{1}{u}_{3} &= -\left(1+\frac{3\mu R^2}{16(\lambda+2\mu)}\right)\frac{R^2 q}{2} + \left(1+\frac{3\mu R^2}{8(\lambda+2\mu)}\right)\frac{qz\bar{z}}{2} - \frac{3\mu}{32(\lambda+2\mu)}qz^2\bar{z}^2,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{-\dfrac{\lambda R}{12(\lambda+2\mu)} + \dfrac{\lambda(\lambda+2\mu)\eta I_1(\eta R)}{72(\lambda+\mu)^2 I_0(\eta R)}}{\dfrac{2(\lambda+2\mu)R}{3\lambda+2\mu} - \dfrac{\lambda^2\eta I_1(\eta R)}{3(\lambda+\mu)(3\lambda+2\mu)I_0(\eta R)}}q,\\
\alpha_0 &= \left[\frac{\lambda+2\mu}{6(\lambda+\mu)} - \frac{-\dfrac{\lambda^2 R}{3(\lambda+2\mu)} + \dfrac{\lambda^2(\lambda+2\mu)\eta I_1(\eta R)}{18(\lambda+\mu)^2 I_0(\eta R)}}{2(\lambda+2\mu)R - \dfrac{\lambda^2\eta I_1(\eta R)}{3(\lambda+\mu)}}\right]\frac{q}{I_0(\eta R)}.
\end{aligned}$$

The system of equilibrium equations, for the approximation $k = 2$, are:

$$\begin{aligned}
\mu\Delta \overset{k}{v}_{+} + 2(\lambda+\mu)\partial_{\bar{z}} \overset{k}{\Theta} - 3\mu(2\partial_{\bar{z}} \overset{2}{u}_3 + \overset{2}{v}_{+}) &= A_1 + A_2 z\bar{z} + A_3 z^2\bar{z}^2 \\
&\quad + A_4(z+\bar{z}) + A_5(I_1(\eta r)e^{i\vartheta} + I_{-1}(\eta r)e^{-i\vartheta}) \qquad (5)\\
\mu\Delta \overset{2}{u}_3 + \mu \overset{2}{\Theta} &= B_1 + B_2 z\bar{z} + B_3 z^2\bar{z}^2 + B_4(z^2+\bar{z}^2) \qquad (6)\\
&\quad + B_5(z^3\bar{z} + \bar{z}^3 z).
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= -\frac{3\lambda}{2\mu}\left(1+\frac{3\mu R^2}{16(\lambda+2\mu)}\right)\frac{R^2 q}{2}, \qquad A_2 = \frac{3\lambda}{4\mu}\left(1+\frac{3\mu R^2}{8(\lambda+2\mu)}\right)q,\\
A_3 &= -\frac{9\lambda q}{64(\lambda+2\mu)}, \quad A_4 = \frac{3\mu R^2 q}{8(\lambda+2\mu)}, \quad A_5 = -\frac{3\mu q}{8(\lambda+2\mu)} - \frac{\lambda+2\mu}{2\mu}\alpha_0,\\
B_1 &= \frac{9\mu R^2 q^2}{128(\lambda+2\mu)} + \frac{3\mu R^2 q}{8(\lambda+2\mu)} + \frac{3\lambda}{4\mu}\left(1+\frac{3\mu R^2}{16(\lambda+2\mu)}\right)R^2 q,\\
B_2 &= \left(1+\frac{3\mu R^2}{8(\lambda+2\mu)}\right)\left(\frac{3\lambda q}{2\mu} - \frac{3(3\lambda+10\mu)q^2}{8(\lambda+2\mu)}\right) - \frac{27\mu^2 R^2 q^2}{128(\lambda+2\mu)} + \frac{3\mu q}{4(\lambda+2\mu)},\\
B_3 &= \frac{9\mu^2 q^2}{16(\lambda+2\mu)^2} - \frac{27\mu(\lambda+\mu)q^2}{128(\lambda+2\mu)^2} - \frac{9\lambda q}{64(\lambda+2\mu)^2},\\
B_4 &= -\frac{9q^2}{32}\left(1+\frac{3\mu R^2 q}{8(\lambda+2\mu)}\right) - \frac{9\mu^2 R^2 q^2}{128(\lambda+2\mu)^2}, \qquad B_5 = -\frac{9\mu q}{128(\lambda+2\mu)}.
\end{aligned}$$

The general solutions of systems (5) end (6) are written in the following form

$$
\begin{aligned}
2\mu \overset{2}{v}_+ &= \frac{4(\lambda+2\mu)}{3\mu}\overline{\overset{2}{f}{}''(z)} + z\overline{\overset{2}{f}{}'(z)} + \overset{2}{f}(z) - 2\overline{\overset{2}{g}{}'(z)} + i\frac{\partial \overset{2}{w}(z,\bar{z})}{\partial \bar{z}} + N_0 + N_1 z + N_2\bar{z}\\
&+ N_3 z^2 + N_4\bar{z}^2 + N_5 z\bar{z} + N_6 z^2\bar{z} + N_7 z^3\bar{z}^2 + N_8 I_0(\eta r)\\
&+ N_9 I_{-1}(\eta r)e^{-i\vartheta} + N_{10} I_3(\eta r)e^{3i\vartheta},\\
2\mu \overset{2}{u}_3 &= -\frac{1}{2}\left(\bar{z}\,\overset{2}{f}(z) + z\overline{\overset{2}{f}(z)}\right) + \overset{2}{g}(z) + \overline{\overset{2}{g}(z)} + M_0(z^2\bar{z} + \bar{z}^2 z) + M_1(z^3\bar{z} + \bar{z}^3 z)\\
&+ M_2 z^2\bar{z}^2 + M_3 z^3\bar{z}^3 + M_4 z^4\bar{z}^4 + M_5 I_0(\eta r) + M_6(I_2(\eta r)e^{2i\vartheta} + I_{-2}(\eta r)e^{-2i\vartheta}),
\end{aligned}
$$

where

$$
\begin{aligned}
M_0 &= -\frac{\mu A_1}{16(\lambda+2\mu)}, \qquad M_1 = \frac{\mu}{24(\lambda+2\mu)}\left(\frac{2(\lambda+\mu)}{\mu}B_4 - \frac{A_4}{2}\right),\\
M_2 &= \frac{\mu}{16(\lambda+2\mu)}\left(\frac{2(\lambda+\mu)}{\mu}B_2 + \frac{3B_1}{2} - A_4\right),\\
M_3 &= \frac{\mu}{72(\lambda+2\mu)}\left(\frac{4(\lambda+\mu)}{\mu}B_3 - \frac{B_2}{2}\right),\\
M_4 &= -\frac{\mu B_3}{384(\lambda+2\mu)}, \qquad M_5 = -\frac{\mu A_5}{12(\lambda+\mu)}, \qquad M_6 = -\frac{\mu A_5}{24(\lambda+\mu)},\\
N_0 &= -\frac{A_1}{3}, \qquad N_1 = B_1, \qquad N_2 = \frac{A_4}{3} - \frac{4(\lambda+\mu)}{3\mu}B_2 + \frac{64}{9}B_5,\\
N_3 &= -\frac{A_2}{6} - \frac{8A_3}{9} - 2M_0, \qquad N_4 = \frac{4B_5}{9},\\
N_5 &= -\frac{A_2}{3} - \frac{16A_3}{9} - 4M_0, \qquad N_6 = \frac{B_2}{2} - 4M_4, \qquad N_7 = \frac{B_3}{3} - 6M_6,\\
N_8 &= \frac{(\lambda+2\mu)\eta A_5}{6(3\lambda+2\mu)}, \qquad N_9 = \frac{(\lambda+2\mu)\eta A_5}{6(3\lambda+2\mu)} - 2M_6,\\
N_{10} &= \frac{(\lambda+2\mu)\eta A_5}{6(\lambda+2\mu)} - 2M_6.
\end{aligned}
$$

Boundary conditions are

$$
\overset{2}{v}_r + i\,\overset{2}{v}_\vartheta = 0, \qquad \overset{2}{v}_3 = 0, \quad |\, z \,|= R. \tag{7}
$$

Let us introduce the functions $\overset{2}{f}(z)$, $\overset{2}{g}(z)$and $\overset{2}{w}(z,\bar{z})$ by the series

$$
\overset{2}{f}(z) = \sum_{n=1}^{\infty} c_n z^n, \quad \overset{2}{g}(z) = \sum_{n=0}^{\infty} d_n z^n, \quad \overset{2}{w}(z,\bar{z}) = \sum_{-\infty}^{\infty} \beta_n I_n(\eta r)e^{in\theta}. \tag{8}
$$

where $I_n(\eta r)$ are Bessel's modifications functions.

By substituting (8) into (7) we obtain

$$
\begin{aligned}
c_1 &= -N_1 - N_6 R^2 - N_7 R^4,\\
c_2 &= -\frac{N_0 + N_5 R^2 + N_8 I_0(\eta R) + \frac{I_0(\gamma R)}{I_2(\gamma R)} N_3 R^2 + 2M_0 R^2}{\left(\frac{I_0(\gamma R)}{I_2(\gamma R)} + 1\right) R^2 + \frac{8(\lambda+2\mu)}{\mu}},\\
c_3 &= -\frac{N_2 R + N_9 I_{-1}(\eta R) + \frac{I_1(\gamma R)}{I_3(\gamma R)} N_{10} I_3(\eta R) + 4M_1 R^3}{\left(\frac{I_1(\gamma R)}{I_3(\gamma R)} + 1\right) R^3 + \frac{8(\lambda+2\mu)}{\mu} R},\\
c_4 &= \frac{N_4}{\left(\frac{I_2(\gamma R)}{I_4(\gamma R)} + 1\right) R^2 + \frac{8(\lambda+2\mu)}{\mu}},\\
d_0 &= -\frac{1}{4}\left(N_1 R^2 + N_6 R^4 + N_7 R^6\right) - M_2 R^4 - M_3 R^5 - M_4 R^8 - M_5 I_0(\eta R),\\
d_1 &= \frac{R^2}{2} c_2 - M_0 R^2, \quad d_2 = \frac{R^2}{2} c_3 - \frac{1}{R^2}\left(M_1 R^4 + M_6 I_3(\eta R)\right), \quad d_3 = \frac{R^2}{2} c_4,\\
\beta_1 &= \frac{2i}{\gamma I_3(\gamma R)}\left(N_3 R^2 - R^2 c_2\right),\\
\beta_2 &= \frac{2i}{\gamma I_3(\gamma R)}\left(N_{10} I_3(\eta R) - R^3 c_3\right).
\end{aligned}
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Received: 20.06.2008; revised: 16.10.2008; accepted: 20.11.2008.