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ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES WITH RESPECT TO COMPLETE ORTHONORMAL SYSTEM

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Abstract. In the paper the sufficient and in some special cases, necessary conditions are given for the absolute convergence of Fourier series with respect to orthonormal complete systems.

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Let $f \in L_2(I)$, $I = [0,1]$; assume that

$$\{\varphi_k(x)\}_{k=0}^{\infty} \tag{1}$$

is an orthonormal complete in $L_2(I)$ system, and let

$$\sum_{k=0}^{\infty} c_k(f)\varphi_k(x)$$

be the Fourier series of the function f with respect to system (1).

It will be said (see [1]) that the sequence of non-negative numbers (γ_k) belongs to the class A_α, $\alpha \geq 1$, if there exists $\varkappa_\alpha > 0$ such that for any natural n

$$\left(\sum_{k=2^n+1}^{2^{n+1}} \gamma_k^\alpha\right)^{\frac{1}{\alpha}} \leq \varkappa_\alpha 2^{n\,\frac{1-\alpha}{\alpha}} \sum_{k=2^{n-1}+1}^{2^n} \gamma_k.$$

If

$$\max_{2^n<k\leq 2^{n+1}} \gamma_k \leq \varkappa\, 2^{-n} \sum_{k=2^{n-1}+1}^{2^n} \gamma_k,$$

then the sequence (γ_k) will be said to belong to the class A_∞. It is easily seen that if $\alpha_1 > \alpha_2$, then $A_\infty \subset A_{\alpha_1} \subset A_{\alpha_2}$. Note that $\overline{A} \subset A_\infty$, where $\overline{A}$ is the class of sequences, introduced by P.L. Ul'yanov [2] in the following way: $(\gamma_k) \in \overline{A}$ if there exists $\varkappa \geq 1$ such that

$$\max_{2^n<k\leq 2^{n+1}} \gamma_k \leq \varkappa \min_{2^{n-1}<\varkappa\leq 2^n} \gamma_k.$$

It is clear that if the sequence (γ_k) is monotone decreasing, then $(\gamma_k) \in \overline{A}$.

The paper deals with convergence of the series

$$\sum_{k=0}^{\infty} |c_k(f)|^r \gamma_k. \tag{2}$$

We will now introduce the theorem in which the sufficient condition for series (2) to be convergent is given.

Theorem 1. *Let* $f \in L_2(I)$, $r \in (0,2]$, $\{\gamma_k\} \in A_{\frac{2}{2-r}}$. *If the series*

$$\sum_{k=1}^{\infty} E_k^r(f) k^{-\frac{r}{2}} \gamma_k \tag{3}$$

is convergent, where $E_k(f)$ *is the best approximation of the function* f *by polynomials of order* $\leq k$ *with respect to system* (1) *in the norm of the space* $L_2(I)$, *then series* (2) *will be convergent.*

This result when $r = 1$, $\gamma_k = 1$, was obtained by S. Stechkin [3], and when $\gamma_k = k^{\beta}$ for the trigonometric system, by A.A. Konyushkov [4].

In some special cases the inverse theorem of Theorem 1 is also valid, i.e. the necessary conditions for the convergence of series (2) are found.

Theorem 2. *Let* $f \in L_2(I)$, $r \in (0,2]$, $(|c_k(f)|^r) \in A_{\frac{2}{r}}$. *If the series*

$$\sum_{k=1}^{\infty} |c_k(f)|^r$$

is convergent, then the series

$$\sum_{k=1}^{\infty} E_k^r(f) k^{-\frac{r}{2}}$$

will also be convergent.

When $r = 1$ and the sequence $(|c_k(f)|)$ is monotone decreasing, Theorem 2 was obtained by S.B. Stechkin [3].

Let

$$\Gamma_k = \sum_{n=2^k+1}^{2^{k+1}} \gamma_n.$$

Theorem 3. *Let* $f \in L_2(I)$, $r \in (0,2]$, $(|c_k(f)|^r) \in A_\infty$ *and for some* $\beta > r$ *the sequence* $\Gamma_k 2^{-\beta \frac{n}{2}}$ *is non-increasing.*

If series (2) *is convergent, then series* (3) *will also be convergent.*

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