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EXPLICIT SOLUTION OF SECOND BVP OF THE ELASTIC MIXTURE FOR HALF-SPACE

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Abstract. In this paper we consider the second BVP of elastic mixture theory for a transversally-isotropic half-space. The solution of second BVP for the transversally-isotropic half-space is given in [1]. The present paper is an attempt to extend this result to BVP of elastic mixture theory for a transversally-isotropic elastic body. Using the potential method and the theory of integral equations, the uniqueness theorem is proved for half-space and the second BVP is solved effectively (in quadratures).

Keywords and phrases: Elastic mixture, uniqueness theorem, potential method.

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Second BVP and the uniqueness theorem for half-space

Let the plane Ox_1x_2 be the boundary of the half-space $x_3>0$. Let the upper half-space will be denoted by D and the boundary of D by S. Let the axis Ox_3 be directed vertically upwards and the normal is $n(0,0,1)$.

A basic equation of statics of transversally-isotropic elastic mixture theory can be written in the form [2]

$$C(\partial x)U=\begin{pmatrix} C^{(1)}(\partial x) & C^{(3)}(\partial x)\\ C^{(3)}(\partial x) & C^{(2)}(\partial x)\end{pmatrix}U=0, \tag{1}$$

where

$$C^{(j)}=(C^{(j)}_{pq})_{3x3},C^{(j)}_{pq}=C^{(j)}_{qp},j=1,2,3,$$
$$C^{(j)}_{11}(\partial x)=c^{(j)}_{11}\frac{\partial^2}{\partial x_1^2}+c^{(j)}_{66}\frac{\partial^2}{\partial x_2^2}+c^{(j)}_{44}\frac{\partial^2}{\partial x_3^2},C^{(j)}_{12}(\partial x)=(c^{(j)}_{11}-c^{(j)}_{66})\frac{\partial^2}{\partial x_1\partial x_2},$$
$$C^{(j)}_{k3}(\partial x)=(c^{(j)}_{13}+c^{(j)}_{44})\frac{\partial^2}{\partial x_k\partial x_3},k=1,2,C^{(j)}_{22}(\partial x)=c^{(j)}_{66}\frac{\partial^2}{\partial x_1^2}+c^{(j)}_{11}\frac{\partial^2}{\partial x_2^2}$$
$$+c^{(j)}_{44}\frac{\partial^2}{\partial x_3^2},C^{(j)}_{33}(\partial x)=c^{(j)}_{44}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+c^{(j)}_{33}\frac{\partial^2}{\partial x_3^2},$$

$c^{(k)}_{pq}$ are constants characterizing the physical properties of the mixture and satisfying certain inequalities caused by the positive definiteness of potential energy. $U^T(x)=(u',u'')$ is six-dimensional displacement vector-function, $u'(x)=(u'_1,u'_2,u'_3)$ and $u''(x)=(u''_1,u''_2,u''_3)$ are partial displacement vectors. Throughout this paper the superscript "T" denotes transposition.

Definition 1. A vector-function $U(x)$ defined in the domain D, is called regular if it has integrable continuous second derivatives in D and $U(x)$ itself and its first derivatives are continuously extendable at every point of the boundary of D, i.e., $U(x) \in C^2(D) \cap C^1(D)$ and satisfies the following conditions at infinite

$$U(x) = O(|x|^{-1}), \frac{\partial U}{\partial x_k} = O(|x|^{-2}), |x|^2 = x_1^2 + x_2^2 + x_3^2, j = 1,2,3; k = 1,2,3.$$

For the equation (1) we pose the following BVP. Find a regular function $U(x)$, satisfying in D the equation (1), if on the boundary S the stress vector is given in the form

$$[T(\partial x, n)U]^+ = f(z), z \in S, \tag{2}$$

where $(.)^+$ denotes the limiting value from D and f is a given vector. $T(\partial x, n)U$ is a stress vector

$$\begin{aligned}
(T(\partial x, n)U)_k &= c_{44}^{(1)}\left(\frac{\partial u_k'}{\partial x_3} + \frac{\partial u_3'}{\partial x_k}\right) + c_{44}^{(3)}\left(\frac{\partial u_k''}{\partial x_3} + \frac{\partial u_3''}{\partial x_k}\right), k = 1,2,\\
(T(\partial x, n)U)_3 &= c_{13}^{(1)}\left(\frac{\partial u_1'}{\partial x_1} + \frac{\partial u_2'}{\partial x_2}\right) + c_{13}^{(3)}\left(\frac{\partial u_1''}{\partial x_1} + \frac{\partial u_2''}{\partial x_2}\right) + c_{33}^{(1)}\frac{\partial u_3'}{\partial x_3} + c_{33}^{(3)}\frac{\partial u_3''}{\partial x_3},\\
(T(\partial x, n)U)_k &= c_{44}^{(3)}\left(\frac{\partial u_{k-3}'}{\partial x_3} + \frac{\partial u_3'}{\partial x_{k-3}}\right) + c_{44}^{(2)}\left(\frac{\partial u_{k-3}''}{\partial x_3} + + \frac{\partial u_3''}{\partial x_{k-3}}\right), k = 4,5,\\
(T(\partial x, n)U)_6 &= c_{13}^{(3)}\left(\frac{\partial u_1'}{\partial x_1} + \frac{\partial u_2'}{\partial x_2}\right) + c_{13}^{(2)}\left(\frac{\partial u_1''}{\partial x_1} + \frac{\partial u_2''}{\partial x_2}\right) + c_{33}^{(3)}\frac{\partial u_3'}{\partial x_3} + c_{33}^{(2)}\frac{\partial u_3''}{\partial x_3}.
\end{aligned} \tag{3}$$

The Uniqueness Theorem. Let us prove that the second homogeneous BVP has only trivial solution. Note that, if U is the regular solution of the equation (1) and satisfies the following conditions at infinite

$$U(x) = O(|x|^{-\alpha}), P(\partial x, n)U = O(|x|^{-1-\alpha}), \alpha > 0$$

we have the following formula

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(U,U)dy_1dy_2 = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U^-[TU]^-dy_1dy_2, \tag{4}$$

where $T(\partial y, n)U$ is a stress vector, $E(U,U) \geq 0$. If $[TU]^- = 0$,from (4) follows $U = a + [b, x]$, but $U(x) = O(|x|^{-\alpha})$,that $a = 0, b = 0$, and $U = 0, x \in D$. Therefore the homogeneous equation has only a trivial solution.Thus we shall formulate the following

Theorem. *The second BVP has at most one regular solution.*

The second BVP. The solution of the second BVP will be sought in the domain D in the form

$$U(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(x-y)g(y)dy_1dy_2, \tag{5}$$

where g is an unknown real vector. $M(x-y)$ is the following matrix

$$M(x-y) = \begin{pmatrix} \Gamma^{(1)} & \Gamma^{(2)} \\ \Gamma^{(3)} & \Gamma^{(4)} \end{pmatrix}, \tag{6}$$

where

$$\Gamma^{(j)}(x-y)=\sum_{k=1}^{6}\|\Gamma_{pq}^{j(k)}\|_{3x3}, j=1,2,3,$$

$$\Gamma_{pq}^{1(k)}=\left(\delta_{pq}\frac{A_{11}^{(k)}}{r_k}+A_{12}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_q}\right)A_k, \Gamma_{3p}^{1(k)}=A_{13}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_3}A_k, \Gamma_{p3}^{1(k)}=A_{13}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_3}B_k,$$
$$\Gamma_{33}^{1(k)}=\frac{A_{33}^{(k)}}{r_k}B_k, \Gamma_{pq}^{2(k)}=\left(\delta_{pq}\frac{A_{14}^{(k)}}{r_k}+A_{42}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_q}\right)C_k, \Gamma_{p3}^{2(k)}=A_{16}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_3}D_k,$$
$$\Gamma_{3p}^{2(k)}=A_{34}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_3}C_k, \Gamma_{33}^{2(k)}=\frac{A_{36}^{(k)}}{r_k}D_k, \Gamma_{pq}^{3(k)}=\left(\delta_{pq}\frac{A_{14}^{(k)}}{r_k}+A_{42}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_q}\right)A_k,$$
$$\Gamma_{p3}^{3(k)}=A_{16}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_3}A_k, \Gamma_{3p}^{3(k)}=A_{34}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_3}B_k, \Gamma_{33}^{3(k)}=\frac{A_{36}^{(k)}}{r_k}B_k, \Gamma_{pq}^{4(k)}=\Bigg(\delta_{pq}\frac{A_{44}^{(k)}}{r_k}$$
$$+A_{45}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_q}\Bigg)C_k, \Gamma_{p3}^{4(k)}=A_{46}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_3}D_k, \Gamma_{3p}^{4(k)}=A_{46}^{(k)}\frac{\partial^2\Phi_k}{\partial x_p\partial x_3}C_k, \Gamma_{33}^{4(k)}=\frac{A_{66}^{(k)}}{r_k}D_k,$$
$$\phi_k=x_3\ln(x_3+r_k)-r_k, r_k^2=a_k[x_1-y_1)^2+(x_2-y_2)^2]+x_3^2, q,p=1,2; \delta_{pq}=1,$$
$$p=q, \delta_{pq}=0, p\neq q, \tag{7}$$

The coefficients $A_{pq}^{(k)}$ are following

$$A_{11}^{(k)}=\frac{(-1)^k b_0(c_{44}^{(2)}-c_{66}^{(2)}a_k)}{r_0(a_1-a_2)}, A_{14}^{(k)}=-\frac{(-1)^k b_0(c_{44}^{(3)}-c_{66}^{(3)}a_k)}{r_0(a_1-a_2)}, A_{12}^{(k)}=\frac{A_{11}^{(k)}}{a_k},$$
$$A_{24}^{(k)}=\frac{A_{14}^{(k)}}{a_k}, A_{45}^{(k)}=\frac{A_{44}^{(k)}}{a_k}, A_{44}^{(k)}=\frac{(-1)^k b_0(c_{44}^{(1)}-c_{66}^{(1)}a_k)}{r_0(a_1-a_2)}, k=1,2,$$
$$A_{12}^{(k)}=\frac{\delta_k}{a_k}[-q_3c_{44}^{(2)}+a_kt_{12}-a_k^2t_{11}+c_{11}^{(2)}q_4a_k^3], A_{42}^{(k)}=\frac{\delta_k}{a_k}[q_3c_{44}^{(3)}+a_kt_{13}-a_k^2t_{22}-c_{11}^{(3)}q_4a_k^3],$$
$$A_{45}^{(k)}=\frac{\delta_k}{a_k}[-q_3c_{44}^{(1)}+a_kt_{23}-a_k^2t_{33}+c_{11}^{(1)}q_4a_k^3], A_{33}^{((k))}=\delta_k[q_4c_{33}^{(2)}-a_kt_{42}+a_k^2t_{44}-c_{44}^{(2)}q_1a_k^3],$$
$$A_{36}^{(k)}=\delta_k[-q_4c_{33}^{(3)}-a_kt_{62}+a_k^2t_{66}+c_{44}^{(3)}q_1a_k^3], A_{66}^{(k)}=\delta_k[q_4c_{33}^{(1)}-a_kt_{52}+a_k^2t_{55}-c_{44}^{(1)}q_1a_k^3],$$
$$A_{13}^{(k)}=\delta_k[v_{13}-v_{11}a_k+v_{12}a_k^2], A_{16}^{(k)}=\delta_k[w_{13}-w_{12}a_k+v_{11}a_k^2], k=3,..,6,$$
$$A_{34}^{(k)}=\delta_k[v_{23}-v_{21}a_k+v_{22}a_k^2], A_{46}^{(k)}=\delta_k[w_{34}-w_{14}a_k+w_{24}a_k^2], k=3,.,6,$$
$$\sum_{k=1}^{6}A_{13}^{(k)}=0, \sum_{k=1}^{6}A_{45}^{(k)}=0, \sum_{k=1}^{6}A_{34}^{(k)}=0, \sum_{k=1}^{6}A_{16}^{(k)}=0, \sum_{k=1}^{6}A_{46}^{(k)}=0, \sum_{k=1}^{6}A_{12}^{(k)}=0,$$
$$\delta_k=d_k(a_1-a_k)(a_2-a_k), k=3,..,6, \tag{8}$$

where a_k are the positive root of the characteristic equation

$$(r_0a^2-c_0a+q_4)(b_0a^4-b_1a^3+b_2a^2-b_3a+b_4)=0,$$

$$r_0=c_{66}^{(1)}c_{66}^{(2)}-c_{66}^{(3)2}, c_0=c_{66}^{(1)}c_{44}^{(2)}+c_{44}^{(1)}c_{66}^{(2)}-2c_{66}^{(3)}c_{44}^{(3)}.$$

The coefficients $d_k, b_k, v_{ij}, w_{ij}, t_{ij}$ are given in [3].

We can easily prove that every column of the matrix $M(x-y)$ is a solution of the system (1) with respect to the point x, if $x \neq y$.

From (5) for the stress vector we obtain

$$T(\partial x,n)U(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}T(\partial x,n)M(x-y)g(y)dy_1dy_2, \tag{9}$$

where

$$T(\partial x,n)M(x-y)=\sum_{k=1}^{6}\begin{pmatrix} M^{(1k)} & M^{(3k)} \\ M^{(4k)} & M^{(2k)} \end{pmatrix}, \tag{10}$$

and the elements of the matrix $M^{j(k)}(x-y), j=1,2,3,4$, can be written as

$$\begin{aligned}
&M^{(1k)}_{pq}=\left(\delta_{pq}\gamma^{(k)}_{11}\frac{\partial}{\partial x_3}\frac{1}{r_k}+\gamma^{(k)}_{12}\frac{\partial^3\Phi_k}{\partial x_p\partial x_q\partial x_3}\right)A_k, M^{(1k)}_{p3}=B_k\gamma^{(k)}_{13}\frac{\partial}{\partial x_p}\frac{1}{r_k},\\
&M^{(1k)}_{3p}=a_k\gamma^{(k)}_{12}\frac{\partial}{\partial x_p}\frac{1}{r_k}A_k, M^{(1k)}_{33}=a_k\gamma^{(k)}_{13}B_k\frac{\partial}{\partial x_3}\frac{1}{r_k}, M^{(3k)}_{pq}=\left(\delta_{pq}\gamma^{(k)}_{14}\frac{\partial}{\partial x_3}\frac{1}{r_k}\right.\\
&\left.+\gamma^{(k)}_{24}\frac{\partial^3\Phi_k}{\partial x_p\partial x_q\partial x_3}\right)C_k, M^{(3k)}_{p3}=D_k\gamma^{(k)}_{16}\frac{\partial}{\partial x_p}\frac{1}{r_k}, M^{(3k)}_{3p}=a_k\gamma^{(k)}_{24}\frac{\partial}{\partial x_p}\frac{1}{r_k}C_k,\\
&M^{(3k)}_{33}=D_ka_k\gamma^{(k)}_{16}B_k\frac{\partial}{\partial x_3}\frac{1}{r_k}, M^{(4k)}_{pq}=\left(\delta_{pq}\gamma^{(k)}_{41}\frac{\partial}{\partial x_3}\frac{1}{r_k}+\gamma^{(k)}_{42}\frac{\partial^3\Phi_k}{\partial x_p\partial x_q\partial x_3}\right)A_k,\\
&M^{(4k)}_{p3}=B_k\gamma^{(k)}_{43}\frac{\partial}{\partial x_p}\frac{1}{r_k}, M^{(4k)}_{3p}=a_kA_k\gamma^{(k)}_{42}\frac{\partial}{\partial x_p}\frac{1}{r_k}{}_k, M^{(4k)}_{33}=a_k\gamma^{(k)}_{43}B_k\frac{\partial}{\partial x_3}\frac{1}{r_k},\\
&M^{(2k)}_{pq}=\left(\delta_{pq}\mu^{(k)}_{44}\frac{\partial}{\partial x_3}\frac{1}{r_k}+\gamma^{(k)}_{45}\frac{\partial^3\Phi_k}{\partial x_p\partial x_q\partial x_3}\right)C_k, M^{(2k)}_{p3}=D_k\gamma^{(k)}_{46}\frac{\partial}{\partial x_p}\frac{1}{r_k},\\
&M^{(2k)}_{3p}=a_k\gamma^{(k)}_{45}\frac{\partial}{\partial x_p}\frac{1}{r_k}C_k, M^{(2k)}_{33}=D_ka_k\gamma^{(k)}_{46}B_k\frac{\partial}{\partial x_3}\frac{1}{r_k}, p,q=1,2,
\end{aligned}$$

where

$$\begin{aligned}
&\gamma^{(k)}_{11}=c^{(1)}_{44}A^{(k)}_{11}+c^{(3)}_{44}A^{(k)}_{41}, \gamma^{(k)}_{12}=c^{(1)}_{44}(A^{(k)}_{12}+A^{(k)}_{13})+c^{(3)}_{44}(A^{(k)}_{61}+A^{(k)}_{24}),\\
&\gamma^{(k)}_{13}=c^{(1)}_{44}(A^{(k)}_{13}+A^{(k)}_{33})+c^{(3)}_{44}(A^{(k)}_{43}+A^{(k)}_{36}), \gamma^{(k)}_{14}=c^{(1)}_{44}A^{(k)}_{41}+c^{(3)}_{44}A^{(k)}_{44},\\
&\gamma^{(k)}_{16}=c^{(1)}_{44}(A^{(k)}_{36}+A^{(k)}_{16})+c^{(3)}_{44}(A^{(k)}_{46}+A^{(k)}_{66}), \gamma^{(k)}_{45}=c^{(3)}_{44}(A^{(k)}_{42}+A^{(k)}_{43})+c^{(2)}_{44}(A^{(k)}_{45}+A^{(k)}_{46}),\\
&\gamma^{(k)}_{43}=c^{(3)}_{44}(A^{(k)}_{33}+A^{(k)}_{13})+c^{(2)}_{44}(A^{(k)}_{36}+A^{(k)}_{43}), \gamma^{(k)}_{46}=c^{(3)}_{44}(A^{(k)}_{36}+A^{(k)}_{16})+c^{(2)}_{44}(A^{(k)}_{66}+A^{(k)}_{46}),\\
&\gamma^{(k)}_{24}=c^{(1)}_{44}(A^{(k)}_{42}+A^{(k)}_{43})+c^{(3)}_{44}(A^{(k)}_{46}+A^{(k)}_{45}), \gamma^{(k)}_{42}=c^{(3)}_{44}(A^{(k)}_{12}+A^{(k)}_{13})+c^{(2)}_{44}(A^{(k)}_{42}+A^{(k)}_{16}),\\
&\mu^{(k)}_{44}=c^{(3)}_{44}A^{(k)}_{41}+c^{(2)}_{44}A^{(k)}_{44}, \gamma^{(k)}_{41}=c^{(3)}_{44}A^{(k)}_{11}+c^{(2)}_{41}A^{(k)}_{44}, k=3,..,6.
\end{aligned} \tag{11}$$

For the sought for unknown constants A_k, B_k, C_k, D_k we have the following equation

$$\begin{gathered}
\sum_{k=1}^{2}\frac{\gamma_{11}^{(k)}}{a_k}A_k=-1,\sum_{k=1}^{2}\frac{\gamma_{41}^{(k)}}{a_k}A_k=0,\sum_{k=1}^{2}\frac{\gamma_{14}^{(k)}}{a_k}C_k=0,\sum_{k=1}^{2}\frac{\mu_{44}^{(k)}}{a_k}C_k=-1,\\
\sum_{k=3}^{6}\gamma_{12}^{(k)}A_k=1,\sum_{k=3}^{6}\gamma_{12}^{(k)}\sqrt{a_k}A_k=0,\sum_{k=3}^{6}\gamma_{42}^{(k)}A_k=0,\sum_{k=3}^{6}\gamma_{42}^{(k)}\sqrt{a_k}A_k=0,\\
\sum_{k=3}^{6}\gamma_{24}^{(k)}C_k=0,\sum_{k=3}^{6}\gamma_{24}^{(k)}\sqrt{a_k}C_k=0,\sum_{k=3}^{6}\gamma_{45}^{(k)}C_k=1,\sum_{k=3}^{6}\gamma_{45}^{(k)}\sqrt{a_k}C_k=0,\\
\sum_{k=3}^{6}\frac{\gamma_{13}^{(k)}}{\sqrt{a_k}}B_k=0,\sum_{k=3}^{6}\gamma_{13}^{(k)}B_k=-1,\sum_{k=3}^{6}\frac{\gamma_{43}^{(k)}}{\sqrt{a_k}}B_k=0,\sum_{k=3}^{6}\gamma_{43}^{(k)}B_k=0,\\
\sum_{k=3}^{6}\frac{\gamma_{16}^{(k)}}{\sqrt{a_k}}D_k=0,\sum_{k=3}^{6}\gamma_{16}^{(k)}D_k=0,\sum_{k=3}^{6}\frac{\gamma_{46}^{(k)}}{\sqrt{a_k}}D_k=0,\sum_{k=3}^{6}\gamma_{46}^{(k)}D_k=-1.
\end{gathered}\tag{12}$$

By the uniqueness theorem,we conclude that the system (12) is solvable and we uniquely define A_k, B_k, C_k, D_k.

Taking into account the properties of the double layer potential and the boundary condition for determining g, from (9) we obtain the following Fredholm integral equation of second kind:

$$-g(z)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T(\partial z,n)M(x-y)g(y)dy_1dy_2=f(z),z\in S.\tag{13}$$

From the last equation we have $g=-f$ and (5) takes the form

$$U(x)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}M(x-y)f(y)dy_1dy_2,x\in D.\tag{14}$$

Thus we have obtained the Poisson formula for the solution of the second BVP for the half space. Note that (14) is valid if and only if $f\in C^{0,\alpha}(S)$ and satisfies the condition $f=\frac{A}{|x|^{1+\beta}}\alpha>0$ at infinite, where A is a constant vector and $\beta>0$.

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R E F E R E N C E S

1. Kupradze B.D., Gegelia T.G., Basheleishvili M.O., Burchuladze T.V. Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity, North-Holland Publ. Comp., Amsterdam, 1979.

2. Rushchitski Ya.Ya. Elements of Mixture Theory, (in Russian), Naukova Dumka, Kiev, 1991.

3. Bitsadze L. The basic BVPs of statics of the theory of elastic transversally isotropic mixtures for half-plane, Reports Seminar of I. Vekua Inst. of Appl. Math. **26-27**, 1-3 (2000-2001), 79-87.

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