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ON A NON-LINEAR VERSION OF GOURSAT PROBLEM WITH PARTIALLY FREE CHARACTERISTIC SUPPORT

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Abstract. On the plane is given one quasi-linear second order hyperbolic equation whose order and type degenerate on some set of points. For this equation we consider the Goursat problem with the partially free characteristic support. There are established some conditions which are sufficient to define the unknown characteristic line and to reduce the Goursat problem with the partially free characteristic support to known nonlinear Goursat problem (see [1]), when there are given the both of characteristics coming from common point and in this point the value of unknown solution is also given.

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The following second order hyperbolic equation is given:

$$y[(q^2-q)u_{xx}-(2pq-q-p-1)u_{xy}+(p^2+p)u_{yy}]=-u_x(u_x+1)(u_x-u_y+1). \quad (1)$$

This equation is hyperbolic along all those functions which fulfill the following condition

$$u_x-u_y+1\neq 0.$$

The parabolic degeneracy of the equation takes place where this condition is not fulfilled. In [1] we have considered one non-linear variant of Goursat problem (see also[2]): we had to find the solution of the equation (1) when there were given two curves coming from the common point. It was known the value of unknown solution in the common point and one curve was the characteristic of the family λ_1 while the second curve was the characteristic family of λ_2. In the work [1] it was shown the existence of the integral of the problem from which we could find the unknown solution $u(x,y)$.

In the present work for the given equation (1) we consider the Goursat problem with the partially free characteristic support: let us assume that the characteristic line γ of the equation (1) is given,which is defined in explicit form by relations: $y=\varphi(x), \varphi\in C^2[x_0,x_1]$. This line belongs to the λ_1 family of the characteristics. From the point x_0 comes out another unknown characteristic line which belongs to another λ_2 family of characteristics. Let us find the solution of the equation (1) if in the point $(x_0,\varphi(x_0))$ it equals to u_0 and if on an unknown characteristic the following condition is fulfilled

$$\alpha(x)\,u_x+\beta(x)\,u_y=\theta(x), \quad x\in[x_0,x_2],$$
$$\alpha,\beta,\theta\in C^1[x_0,x_2]. \quad (2)$$

Theorem. *If the following conditions are fulfilled*

$$\beta(x_0) \neq \alpha(x_0)\,\varphi'(x_0), \tag{3}$$

$$\Big(\frac{\beta y}{\alpha+\beta}\Big)' \neq 0, \tag{4}$$

$$\Lambda(\beta-\theta)(\theta+\alpha) < 0, \tag{5}$$

$$\Lambda\big[\alpha'(\beta-\theta) - \alpha(\beta'-\theta')\big] + \beta(\beta-\theta-\alpha) = 0, \tag{6}$$

where

$$\Lambda \equiv \frac{\varphi'(x_0)\,\varphi(x_0)}{\varphi'(x_0)+1},$$

then the unknown characteristic line $y = \psi(x)$ is defined uniquely:

$$y(x) = \Bigg[-\int_{x_0}^{x} 2\Lambda \frac{(\beta-\theta)(\theta+\alpha)}{(\beta-\theta-\alpha)^2}\,dx + \Big(\varphi(x_0) - \frac{\beta(x_0)-\theta(x_0)}{\beta(x_0)-\theta(x_0)-\alpha(x_0)}\,\Lambda\Big)^2\Bigg]^{1/2} + \frac{\beta(x)-\theta(x)}{\beta(x)-\theta(x)-\alpha(x)}\,\Lambda.$$

Proof. As it is known [1], the expression $u+x$ which is a characteristic invariant of the family λ_1, remains constant along the curve γ. The constant value of this invariant is

$$u(x,\varphi(x)) + x = u_0 + x_0. \tag{7}$$

Furthermore, by conditions (2), (7) we can define the values of first order derivatives u_x, u_y in the point $(x_0,\varphi(x_0))$. For this purpose let us differentiate the expression (7) and consider together with condition (2) in the point $(x_0,\varphi(x_0))$. Obtained by this way algebraic system is solvable due to condition (3)

$$u_x(x_0,\varphi(x_0)) = -\frac{\beta(x_0)+\varphi'(x_0)\,\theta(x_0)}{\beta(x_0)-\alpha(x_0)\,\varphi'(x_0)} \equiv p_0, \tag{8}$$

$$u_y(x_0,\varphi(x_0)) = \frac{\theta(x_0)+\alpha(x_0)}{\beta(x_0)-\alpha(x_0)\,\varphi'(x_0)} \equiv q_0. \tag{9}$$

By values p_0, q_0 we can define the constant value of invariant $\xi = y(u_x+1)(u_x-u_y+1)^{-1}$ along the unknown characteristic δ

$$\frac{(u_x+1)y}{u_x-u_y+1}\bigg|_{\delta} = \frac{\varphi'(x_0)\,\varphi(x_0)}{\varphi'(x_0)+1} = \Lambda,$$

The same is

$$[\psi(x)-\Lambda]u_x(x,\psi(x)) + \Lambda u_y(x,\psi(x)) = \Lambda - \psi(x). \tag{10}$$

Due to condition (4) the system (2)-(10) provides the dependance of derivatives u_x, u_y on function $\psi(x)$ along δ:

$$u_x(x,\psi(x)) = \frac{\beta(x)(\Lambda-\psi(x)))-\theta(x)\cdot\Lambda}{\beta(x)(\psi(x)-\Lambda)-\alpha(x)\cdot\Lambda} \tag{11}$$

$$u_y(x,\psi(x)) = \frac{(\psi(x)-\Lambda)(\theta(x)-\alpha(x))}{\beta(x)(\psi(x)-\Lambda)-\alpha(x)\cdot\Lambda} \tag{12}$$

Indeed, from the condition (4) follows, that expression $\frac{\beta y}{\alpha+\beta}$ does not remain constant along δ and it excludes the possibility for equations (2), (10) to be equal to each other.

By substituting of expressions (11), (12) in the expression of characteristic direction

$$\frac{dy}{dx} = \frac{p}{1-q}$$

we obtain ordinary differential equation for unknown function $y=\psi(x)$

$$\frac{dy}{dx} = -\frac{\beta(x)y-(\beta(x)-\theta(x))\Lambda}{(\beta(x)-\theta(x)-\alpha(x))y-(\beta(x)-\theta(x))\Lambda}\,. \tag{13}$$

We should mention, that as we have $\frac{\varphi'(x_0)\varphi(x_0)}{\varphi'(x_0)+1}\equiv\Lambda$, from the equation (13) we conclude, that if we take the characteristic curve $\varphi(x)$ for which $\varphi'(x)=0$, we can define function $y=\psi(x)$ in unique way:

$$y = -\int_{x_0}^{x}\frac{\beta}{\beta-\theta-\alpha}dx + y_0.$$

Let's give to the equation (13) the following form

$$y'\Big[y-\frac{\beta-\theta}{\beta-\theta-\alpha}\Lambda\Big] = -\frac{\beta y}{\beta-\theta-\alpha}+\frac{\beta-\theta}{\beta-\theta-\alpha}\Lambda. \tag{14}$$

This is Abel equation of the second type [3]. By the following substitution

$$y-\frac{\beta-\theta}{\beta-\theta-\alpha}\Lambda \equiv v(x) \tag{15}$$

and after simple transformations we give to the equation (14) the following form

$$\begin{aligned} v\,v' + \frac{v}{(\beta-\theta-\alpha)^2}\big[\Lambda(\alpha'(\beta-\theta)-\alpha(\beta'-\theta')+\beta(\beta-\theta-\alpha)\big] \\ +\Lambda\frac{(\beta-\theta)(\theta+\alpha)}{(\beta-\theta-\alpha)^2} = 0. \end{aligned}$$

When (6)is fulfilled we obtain

$$(v^2)' = -2\Lambda\frac{(\beta-\theta)(\theta+\alpha)}{(\beta-\theta-\alpha)^2}\,,$$

and finally we get the expression

$$v(x)=\pm\Big[-\int_{x_0}^{x}2\Lambda\frac{(\beta-\theta)(\theta+\alpha)}{(\beta-\theta-\alpha)^2}\,dx+v^2(x_0)\Big]^{1/2},$$

which is a real function due to the condition (5) of the theorem. The sign before the square root we select by the condition

$$v(x_0)=y_0-\frac{\beta(x_0)-\theta(x_0)}{\beta(x_0)-\theta(x_0)-\alpha(x_0)}\Lambda.$$

For the function $y(x)$, by taking into consideration of condition (15), finally we obtain

$$\begin{aligned}y(x)=\Big[&-\int_{x_0}^{x}2\Lambda\frac{(\beta-\theta)(\theta+\alpha)}{(\beta-\theta-\alpha)^2}\,dx\\&+\Big(\varphi(x_0)-\frac{\beta(x_0)-\theta(x_0)}{\beta(x_0)-\theta(x_0)-\alpha(x_0)}\,\Lambda\Big)^2\Big]^{1/2}\\&+\frac{\beta(x)-\theta(x)}{\beta(x)-\theta(x)-\alpha(x)}\,\Lambda,\end{aligned}$$

and proving of the theorem is finished.

We should note, that it is possible to find the explicit form of unknown characteristic in other concrete cases. For example, if the equation has the form (14) and the following equality is fulfilled

$$\beta(\beta-\theta-\alpha)=-\Lambda\Big(\frac{\theta-\beta}{\alpha}\Big)',$$

then unknown function $y=\psi(x)$ will be defined as follows:

$$y(x)=\frac{(\beta-\theta)\Lambda}{\beta-\theta-\alpha}\Big[-\int_{x_0}^{x}2\Lambda\frac{(\beta-\theta)}{(\beta-\theta-\alpha)}\Big(1+\frac{\Lambda}{(\beta-\theta-\alpha)^2}\Big)\Big(\frac{\theta-\beta}{\alpha}\Big)'dx\Big]^{1/2}.$$

The case when the equality $\alpha+\theta=0$ is fulfilled, should be mentioned. From the equation (13), if $y\neq\frac{\beta(x)-\theta(x)}{\beta(x)}\Lambda$, we obtain

$$\frac{dy}{dx}=-1$$

and thus, unknown function $y(x)$ is defined as follows

$$y=-x+1,$$

i.e., the unknown characteristic appears to be the straight line, which belongs to the family of singular lines of the equation (1).

Thus, there are obtained some sufficient conditions for existence of the function $\psi(x)$, which defines the curve δ in explicit form in the class of curves uniquely projected on the axis Ox. We can get rid of such conditions (which are really restrictions in some way), if we try to find the curve in the class of curves uniquely projected on the axis Oy. In this case we have to assume α, β, θ functions to be dependent on the argument $y \in [y_0, y_2]$, where y_2 is some number $y_2 \neq y_1$ and let us assume the curve γ to be represented in the following way $x = \varphi(y), x_0 = \varphi(y_0)$. Thus, instead of condition (2) we have

$$\alpha(y)\, u_x + \beta(y)\, u_y = \theta(y), \quad y \in [y_0, y_1], \quad y_0 = \varphi(x_0), \tag{16}$$
$$\alpha, \beta, \theta \in C^1[y_0, y_1].$$

In this case we can use the similar considerations to define the function ψ. Namely, on the curve γ we have

$$u(\varphi(y), y) + \varphi(y) = u_0 + x_0.$$

Let us differentiate this equality and consider obtained by this way equality together with the condition (16)in the point $(\varphi(y_0), y_0)$ or what is the same in the point $(\psi(y_0), y_0)$. Thus we obtain

$$\begin{cases} u_x(x_0, \varphi(x_0)) + u_y(x, \varphi(x_0))\varphi'(x_0) + 1 = 0, \\ \alpha(y_0)u_x(\psi(y_0), y_0) + \beta(y_0)u_y(\psi(y_0), y_0) = \theta(y_0). \end{cases}$$

From these equalities we can define derivatives u_x, u_y in the point $(\psi(y_0), y_0)$ if $\beta(y_0) - \varphi'(x_0)\alpha(y_0) \neq 0$. We obtain

$$\begin{aligned} u_x(x_0, \varphi(x_0)) &= \frac{-\beta(y_0) - \varphi'(x_0)\theta(y_0)}{\beta(y_0) - \varphi'(x_0)\alpha(y_0)}\,, \\ u_y(x_0, \varphi(x_0)) &= \frac{\theta(y_0) + \alpha(y_0)}{\beta(y_0) - \varphi'(x_0)\alpha(y_0)}\,. \end{aligned} \tag{17}$$

By these expressions, in the similar way as we did in the previous case,we can define the constant value to which is equal the invariant $\xi = y(u_x + 1)(u_x - u_y + 1)^{-1}$ along the unknown characteristic δ:

$$\left.\frac{(u_x + 1)y}{u_x - u_y + 1}\right|_\delta = \frac{\varphi'(x_0) \cdot y_0}{\varphi'(x_0) + 1} = \Lambda,$$

what is the same as

$$[y - \Lambda]u_x(\psi(y), y) + \Lambda u_y(\psi(y), y) = \Lambda - y. \tag{18}$$

If we consider the equation (18) together with the condition (16) as linear algebraic system, we can define the first order derivatives u_x, u_y along the unknown curve δ

$$u_x = \frac{\beta(y)(\Lambda - y) - \Lambda\theta(y)}{(y - \Lambda)\beta(y) - \Lambda\alpha(y)}, \tag{19}$$

$$u_y = \frac{(y - \Lambda)(\theta(y) + \alpha(y))}{(y - \Lambda)\beta(y) - \Lambda\alpha(y)}, \tag{20}$$

if equality $(y - \Lambda)\beta(y) \neq \Lambda\alpha(y)$ is fulfilled. By substitution of expressions (19), (20) in the following equality of characteristic direction

$$\frac{dx}{dy} = \frac{1 - u_y}{u_x}$$

we obtain the ordinary differential equation with separated variables

$$\frac{dx}{dy} = \frac{(y-\Lambda)(\beta(y)-\theta(y)) - y\alpha(y)}{\beta(y)(\Lambda - y) - \Lambda\theta(y)}\,.$$

The curve itself will be represented by the formula

$$\psi(y) = \int_{y_0}^{y} \frac{(t-\Lambda)(\beta(t)-\theta(t)) - t\alpha(t)}{\beta(t)(\Lambda - t) - \Lambda\theta(t)} dt + x_0.$$

Thus, in every case considered above in this work, the function ψ is defined on the given segment and our problem is reduced to above-mentioned nonlinear Goursat problem [1], when there are given two characteristics γ and δ coming from the common point, in this point the value of unknown solution is also given and it equals to u_0.

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