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THE THERMO-ELASTICITY PROBLEM OF DEFORMATION OF FLEXIBLE MULTILAYERED SHELLS OF REVOLUTION WITH LAYERS OF VARIABLE THICKNESS IN A REFINED SETTING

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Abstract. A version of a refined theory of deformation of flexible multilayered shells of revolution with layers of variable thickness is considered which takes into account non-homogenity of deformation of lateral shear strains. Using the approach for shells of revolution we get a non-linear boundary value problem for the system of ordinary differential equations. The solution of this problem is obtained using the methods of linearization and discrete orthogonalization. Based on the given approach we investigate the concrete examples of the stress-strain state of shells under the action of temperature field. Some numerical results are also discussed.

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In the present paper we consider stress-strain state of flexible orthotropic layered shells, taking into account strain nonhomogeneity of lateral displacement with respect to the thickness of shells. In the paper [1] it is considered the same situation by means of refined theory for the class of problems of stress-strain state of flexible layered shells of revolution with layers with thickness variable along the meridian, influenced by forced interaction. In this paper using the approach for shells of revolution we consider strain of flexible layered shells of rotating with layers with thickness variable along the meridian, taking into account temperature field.

Now let us consider stress-strain state of flexible orthotropic layered shells, which are under influence of force tension and temperature field. We assume that strain of shell is elastic, i.e. connection between stress and strain for each i-th layer is described by the Hooke's law taking into account Duhamel-Neumann hypothesis in the following way

$$\sigma_\alpha^i = B_{11}^i \varepsilon_{\alpha\alpha}^{(\gamma)} + B_{12}^i \varepsilon_{\beta\beta}^{(\gamma)} - \beta_1^i T;$$

$$\sigma_\beta^i = B_{21}^i \varepsilon_{\alpha\alpha}^{(\gamma)} + B_{22}^i \varepsilon_{\beta\beta}^{(\gamma)} - \beta_2^i T; \qquad (1)$$

$$\tau_{\beta\gamma}^i = B_{44}^i \varepsilon_{\beta\gamma}^{(\gamma)}; \quad \tau_{\alpha\gamma}^i = B_{55}^i \varepsilon_{\alpha\gamma}^{(\gamma)}; \quad \tau_{\alpha\beta}^i = B_{66}^i \varepsilon_{\alpha\beta}^{(\gamma)},$$

where $T(\alpha, \beta, \gamma)$ is temperature field .

Let us present the basic relations of the refined theory of flexible layered orthotropic shells [3,5,6].

In particular, for the tangential displacements we have

$$u_\alpha^{(\gamma)} = u + a_1^{(i)}\gamma_\alpha^{(0)} + \gamma(\psi_\alpha + a_2^{(i)}\gamma_\alpha^{(0)});$$
$$u_\beta^{(\gamma)} = v + b_1^{(i)}\gamma_\beta^{(0)} + \gamma(\psi_\beta + b_2^{(i)}\gamma_\beta^{(0)}); \tag{2}$$

where u and v are the tangential displacements of the coordinate surface, ψ_α and ψ_β are the complete angles of rotation of the normal. $\gamma_\alpha^{(0)}$ and $\gamma_\beta^{(0)}$ are the lateral shears in the layer containing the coordinate surface, and α and β are the orthogonal coordinates on the datum surface. The quantities $a_1^{(i)}$, $a_2^{(i)}$, $b_1^{(i)}$, and $b_2^{(i)}$ are determined in [4,6]. Taking (2) into account, we present the strain components as

$$\varepsilon_{\alpha\alpha}^{(\gamma)} = \varepsilon_{\alpha\alpha}^{(i)} + \gamma\varkappa_{\alpha\alpha}^{(i)}; \quad \varepsilon_{\alpha\beta}^{(\gamma)} = \varepsilon_{\alpha\beta}^{(i)} + \gamma 2\varkappa_{\alpha\beta}^{(i)}; \quad \varepsilon_{\beta\beta}^{(\gamma)} = \varepsilon_{\beta\beta}^{(i)} + \gamma\varkappa_{\beta\beta}^{(i)};$$
$$\varepsilon_{\alpha\gamma}^{(\gamma)} = \gamma_\alpha^{(i)}; \quad \varepsilon_{\beta\gamma}^{(\gamma)} = \gamma_\beta^{(i)}; \quad \varepsilon_{\gamma\gamma}^{(\gamma)} = 0. \tag{3}$$

In (3) the components $\varepsilon_{\alpha\alpha}^{(i)}$, $\varepsilon_{\alpha\beta}^{(i)}$, $\varepsilon_{\beta\beta}^{(i)}$, $\varkappa_{\alpha\alpha}^{(i)}$, $\varkappa_{\alpha\beta}^{(i)}$ and $\varkappa_{\beta\beta}^{(i)}$ are the same as in [4]. The quantities specifying the strain of the coordinate surface are

$$\varepsilon_{\alpha\alpha} = \varepsilon_\alpha + \frac{1}{2}\theta_\alpha^2; \quad \varepsilon_{\beta\beta} = \varepsilon_\beta + \frac{1}{2}\theta_\beta^2; \quad \varepsilon_{\alpha\beta}^* = \varepsilon_{\alpha\beta} + \theta_\alpha\theta_\beta;$$
$$\theta_\alpha = -\frac{1}{A}\frac{\partial w}{\partial\alpha} + k_1 u; \quad \theta_\beta = -\frac{1}{B}\frac{\partial w}{\partial\beta} + k_2 v; \tag{4}$$
$$\gamma_\alpha^{(0)} = \psi_\alpha - \theta_\alpha; \quad \gamma_\beta^{(0)} = \psi_\beta - \theta_\beta,$$

where ε_α, $\varepsilon_{\alpha,\beta}$ and ε_β are given in [4,6].

Based on Hooke's law (1), we obtain the classic relations

$$N_\alpha = C_{11}\varepsilon_{\alpha\alpha} + C_{12}\varepsilon_{\beta\beta} + K_{11}\varkappa_\alpha + K_{12}\varkappa_\beta + A_{11}\frac{\partial\gamma_\alpha^{(0)}}{\partial\alpha} + A_{12}\gamma_\alpha^{(0)}$$
$$+B_{11}\frac{\partial\gamma_\beta^{(0)}}{\partial\beta} + B_{12}\gamma_\beta^{(0)} - N_{\alpha T};$$
$$N_\beta = C_{12}\varepsilon_{\alpha\alpha} + C_{22}\varepsilon_{\beta\beta} + K_{12}\varkappa_\alpha + K_{22}\varkappa_\beta + A_{21}\frac{\partial\gamma_\alpha^{(0)}}{\partial\alpha} + A_{22}\gamma_\alpha^{(0)}$$
$$+B_{21}\frac{\partial\gamma_\beta^{(0)}}{\partial\beta} + B_{22}\gamma_\beta^{(0)} - N_{\beta T};$$
$$N_{\alpha\beta} = C_{66}\varepsilon_{\alpha\beta}^* + 2K_{66}\varkappa_{\alpha\beta} + k_2(K_{66}\varepsilon_{\alpha\beta}^* + 2D_{66}\varkappa_{\alpha\beta})$$
$$(A_{16} + k_2E_{16})\frac{\partial\gamma_\alpha^{(0)}}{\partial\beta} + (A_{26} + k_2E_{26})\gamma_\alpha^{(0)}$$
$$+(B_{16} + k_2F_{16})\frac{\partial\gamma_\beta^{(0)}}{\partial\alpha} + (B_{26} + k_2F_{26})\gamma_\beta^{(0)}; \tag{5}$$

$$N_{\beta\alpha} = C_{66}\varepsilon^*_{\alpha\beta} + 2K_{66}\varkappa_{\alpha\beta} + k_1(K_{66}\varepsilon^*_{\alpha\beta} + 2D_{66}\varkappa_{\alpha\beta})$$
$$+(A_{16} + k_1E_{16})\frac{\partial\gamma_\alpha^{(0)}}{\partial\beta} + (A_{26} + k_1E_{26})\gamma_\alpha^{(0)}$$
$$+(B_{16} + k_1F_{16})\frac{\partial\gamma_\beta^{(0)}}{\partial\alpha} + (B_{26} + k_1F_{26})\gamma_\beta^{(0)};$$

$$M_\alpha = K_{11}\varepsilon_{\alpha\alpha} + K_{12}\varepsilon_{\beta\beta} + D_{11}\varkappa_\alpha + D_{12}\varkappa_\beta + E_{11}\frac{\partial\gamma_\alpha^{(0)}}{\partial\alpha} + E_{12}\gamma_\alpha^{(0)}$$
$$+F_{11}\frac{\partial\gamma_\beta^{(0)}}{\partial\beta} + F_{12}\gamma_\beta^{(0)} - M_{\alpha T};$$
$$M_\beta = K_{12}\varepsilon_{\alpha\alpha} + K_{22}\varepsilon_{\beta\beta} + D_{12}\varkappa_\alpha + D_{22}\varkappa_\beta + E_{21}\frac{\partial\gamma_\alpha^{(0)}}{\partial\alpha} + E_{22}\gamma_\alpha^{(0)}$$
$$+E_{21}\frac{\partial\gamma_\beta^{(0)}}{\partial\beta} + F_{22}\gamma_\beta^{(0)} - M_{\beta T};$$
$$M_{\alpha\beta} = M_{\beta\alpha} = K_{66}\varepsilon^*_{\alpha\beta} + 2D_{66}\varkappa_{\alpha\beta} + E_{16}\frac{\partial\gamma_\alpha^{(0)}}{\partial\beta} + E_{26}\gamma_\alpha^{(0)}$$
$$+F_{16}\frac{\partial\gamma_\beta^{(0)}}{\partial\alpha} + F_{26}\gamma_\beta^{(0)}$$
$$Q_\alpha = K_1\gamma_\alpha^{(0)}; \quad Q_\beta = K_2\gamma_\beta^{(0)};$$

where N_α, N_β, $N_{\alpha\beta}$ and $N_{\beta\alpha}$ are the tangential forces. Q_α and Q_β are the shearing forces. M_α and M_β are the bending moments. $M_{\alpha\beta}$ and $M_{\beta\alpha}$ are the torques. C_{ij}, K_{ij}, D_{ij}, K_1 and K_2 are the rigidity characteristics determined in terms of the elastic parameters of the layers and their thicknesses. $A_{11}, A_{12}, \ldots, F_{26}$ are quantities depending on the geometric and mechanical parameters of the layers, and k_1 and k_2 are the curvatures [4,6].

The equilibrium equations for an element of the shell are

$$\begin{aligned}
&\frac{\partial BN_\alpha}{\partial\alpha} + \frac{\partial AN_{\beta\alpha}}{\partial\beta} + \frac{\partial A}{\partial\beta}N_{\alpha\beta} - \frac{\partial B}{\partial\alpha}N_\beta + ABk_1Q^*_\alpha + ABq_1 = 0;\\
&\frac{\partial AN_\beta}{\partial\beta} + \frac{\partial BN_{\alpha\beta}}{\partial\alpha} + \frac{\partial B}{\partial\alpha}N_{\beta\alpha} - \frac{\partial A}{\partial\beta}N_\alpha + ABk_2Q^*_\alpha + ABq_2 = 0;\\
&\frac{\partial BQ^*_\alpha}{\partial\alpha} + \frac{\partial AQ^*_\beta}{\partial\beta} - ABk_1N_\alpha - ABk_2N_\beta + ABq_3 = 0;\\
&\frac{\partial BM_\alpha}{\partial\alpha} + \frac{\partial AM_{\beta\alpha}}{\partial\beta} + \frac{\partial A}{\partial\beta}M_{\alpha\beta} - \frac{\partial B}{\partial\alpha}M_\beta - ABQ_\alpha = 0;\\
&\frac{\partial AM_\beta}{\partial\beta} + \frac{\partial BM_{\alpha\beta}}{\partial\alpha} + \frac{\partial B}{\partial\alpha}M_{\beta\alpha} - \frac{\partial A}{\partial\beta}M_\alpha - ABQ_\beta = 0,
\end{aligned} \tag{6}$$

where

$$Q^*_\alpha = Q_\alpha - (N_\alpha + k_1 M_\alpha)\theta_\alpha - (N_{\alpha\beta} + k_1 M_{\alpha\beta})\theta_\beta;$$
$$Q^*_\beta = Q_\beta - (N_{\beta\alpha} + k_2 M_{\beta\alpha})\theta_\alpha - (N_\beta + k_2 M_\beta)\theta_\beta, \tag{7}$$

In (6) q_1, q_2 and q_3 are the projections of the surface load onto the coordinate axes α, β and γ, respectively.

Supplementing Eqs. (2)-(7) with respective boundary conditions, we obtain a non-linear boundary-value problem. The static boundary conditions are specified in terms of forces and moments in an integral form and the kinematic boundary conditions are specified at discrete number of points of the periphery.

We dwell on problems on the stress-strain state of layered shell of revolution with rigidity variable along the meridian. Assuming that $\alpha = s$ is the are length of the meridian and $\beta = \theta$ is the central angle in the parallel circle, from the general equations (2)-(7) for axisymmetric deformation of layered shell of revolution with rigidity variable along the meridian, under influence of force tension and temperature field we obtain the resolving system of differential equations

$$\frac{d\overline{Y}}{ds} = A^*(s)\overline{Y} + \overline{F}(s,\overline{Y}) + \overline{f}(s) + \overline{F}_T(s,\overline{Y}) + \overline{f}_T(s), \tag{8}$$

$$\overline{Y} = \{N_s, Q^*_s, M_s, u, w, \psi_s\}^T,$$

where the elements a^*_{ij} of the matrix $A^*(s)$, the components of the vector-function $\overline{F}(s,\overline{Y})$ and the components of the vector $\overline{f}(s)$ defines in the same way as in [1], the components of the vector function $\overline{F}_T(s,\overline{Y})$ and the vector $\overline{f}_T(s)$ have the following form

$$\begin{aligned}
&F_{1T} = d_1\Phi_T; \qquad F_{2T} = d_2\Phi_T; \qquad F_{3T} = d_3\Phi_T;\\
&F_{4T} = d_4\Phi_T; \qquad F_{5T} = 0; \qquad F_{6T} = d_6\Phi_T;\\
&f_{1T} = d_{11}N_{sT} + d_{12}M_{sT} + d_{13}N_{\theta T};\\
&f_{2T} = d_{21}N_{sT} + d_{22}M_{sT} + d_{23}N_{\theta T};\\
&f_{3T} = d_{31}N_{sT} + d_{32}M_{sT} + d_{33}N_{\theta T} + d_{34}M_{\theta T};\\
&f_{4T} = d_{41}N_{sT} + d_{42}M_{sT} + d_{43}N_{\theta T}; \qquad f_{5T} = 0;\\
&f_{6T} = d_{61}N_{sT} + d_{62}M_{sT} + d_{63}N_{\theta T},
\end{aligned} \tag{9}$$

where

$$\Phi_T = \frac{1}{c_0 - c_1 N_s - c_2 M_s - c_3\psi_s}\left\{\frac{k_2}{c_0}(c_1 N_s + c_2 M_s + c_3\psi_s)(b_{31}N_{sT}\right.$$
$$+ b_{32}M_{sT} - N_{\theta T}) + \frac{\cos\varphi}{r}[(b_{31} + k_1 b_{42})M_{sT} - N_{\theta T} - k_1 M_{\theta T}]\psi_s \tag{10}$$
$$\left. + (N_s + k_1 M_s)(b_1 N_{sT} + b_3 M_{sT})\right\}.$$

Here $r = r(s)$ is the radius of the parallel circle and $\varphi = \varphi(s)$ is the angle between the normal and the axis of revolution. The values $d_i, d_{ij}, c_i, b_i, b_{ij}$ are defined as in [11].

To solve the nonlinear boundary-value problem for the system of equation (8) describing the axisymmetric deformation of shells of revolution with rigidity variable along the meridian, the linearization method and the stable numerical discrete-orthogonalization method [3] are applied.

Based on the refined theory in question, let us consider, as an example, the deformation of a three-layer orthotropic toroidal shell with an elliptic cross section and layers of thickness variable along the meridian under a boundary force P and normal extremal pressure q_3 and the temperatural field T. In solving the problem, we assume that the shell is uniformly warm, that is $T = T_0 = const$, and the coordinate surface formed by revolution the elliptic are about the axis of revolution passes through the middle layer of the shell.

The parametric equation of the meridian of the coordinate surface has the form

$$r = R + a\cos t; \quad z = b\sin t \ \left(-\frac{\pi}{3} \le t \le 0\right).$$

The geometric characteristic of the shell are

$$\sin\varphi = \frac{b\cos t}{\gamma(t)}; \quad \cos\varphi = \frac{a\sin t}{\gamma(t)},$$

where

$$\gamma(t) = \sqrt{a^2\sin^2 t + b^2\cos^2 t}.$$

The left contour of the shell is subjected to the force P parallel to the axis of revolution and the right contour is rigidly clamped, i.e., the following conditions are satisfied:

$$N_s\sin\varphi - Q_s\cos\varphi = -P; \quad N_s\cos\varphi + Q_s\sin\varphi = 0; \quad t_0 = -\frac{\pi}{3};$$
$$u = w = \psi_s = 0, \quad t_N = 0.$$

Let h_1, h_2 and h_3 be the thicknesses of the outer, middle, and inner layers respectively. E_1^i and E_2^i be the elastic module in the coordinate directions s and θ, respectively, ν_{12}^i and ν_{21}^i be Poisson's ratios, and G_{13}^i be the shear modulus in the $\theta = const$, where $i = 1, 2, 3$ is the layer number. The following values are adopted: $R = 180$, $a = 75$, $b = 25$, $E_1^1 = 1,5\cdot 10^6$, $E_2^1 = 3\cdot 10^6$, $E_1^2 = 2\cdot 10^2$, $E_2^2 = 3\cdot 10^2$, $E_1^3 = 1,2\cdot 10^4$, $E_2^3 = 2,5\cdot 10^4$, $\nu_{12}^1 = 0,2$, $\nu_{21}^1 = 0,34$, $\nu_{12}^2 = 0,1$, $\nu_{21}^2 = 0,15$, $\nu_{12}^3 = 0,14$, $\nu_{21}^3 = 0,17$, $G_{13}^1 = 0,15\cdot 10^6$, $G_{13}^2 = 0,55\cdot 10^2$, $G_{13}^3 = 0,35\cdot 10^4$. The layer thicknesses vary along the meridian in the following fashion:

$$h_1(t) = 0,2\left(1 + \frac{t - t_0}{t_N - t_0}\right); \quad h_2(t) = 0,6\left(1 + \frac{1}{3}\cdot\frac{t - t_0}{t_N - t_0}\right);$$
$$h_3(t) = 0,3\left(1 + \frac{t - t_0}{t_N - t_0}\right).$$

TABLE

T_0	w			
	1	2	1	2
	$P = 8; \quad q_3 = 1,25$			
	$t = -\frac{\pi}{3}$		$t = -\frac{4\pi}{15}$	
0	$-0,0545$	$-0,2775$	$0,3861$	$0,5907$
50	$-0,0689$	$-0,2903$	$0,4214$	$0,6832$
100	$-0,0815$	$-0,3405$	$0,5625$	$0,7553$
150	$-0,0984$	$-0,5322$	$0,6234$	$0,9705$

The table contains the solutions of this problem for the deflection w at the points $t = -\frac{\pi}{3}, -\frac{4\pi}{15}$ when $P = 8$, $q_3 = 1,25$ at the various values of temperature T_0. The problem was solved in nonlinear formulation under both classical (1) and refined theory (2).

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