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A CUSPED KIRCHHOFF-LOVE PLATE UNDER THE ACTION OF CONCENTRATED LOADINGS

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Abstract. The paper deals with a cusped Kirchhoff-Love plate under the action of concentrated bending moment M and concentrated generalized shearing force Q. In the case when the projection of the plate is a half-plane the problem is solved in the explicit form.

Keywords and phrases: Cusped plate, Kirchhoff-Love bending model, concentrated bending moment, concentrated shearing force.

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The paper deals with a cusped Kirchhoff-Love plate [1-3] with the flexural rigidity

$$D = D_0 x_2^{\kappa} = D_0 r^{\kappa} \sin^{\kappa} \psi, \quad x_2 \geq 0, \ \ D_0, \ \kappa = \text{const}, \ \ D_0 > 0, \ \ \kappa \geq 0.$$

Let at the edge of the plate a concentrated bending moment M and concentrated generalized shearing force Q are applied:

$$M_2 (x_1, 0) = -f_0 (x_1) = -M\delta (x_1), \tag{1}$$

$$Q_2^* (x_1, 0) = Q_2 (x_1, 0) - \frac{\partial M_{12} (x_1, 0)}{\partial x_1} = -f_1 (x_1) = -Q\delta (x_1), \tag{2}$$

where M_2 and M_{12} denote bending and twisting moments, respectively, Q_2^* and Q_2 denote shearing and generalized shearing forces, correspondingly, $\delta_{x_1}(\varphi)$, $\delta := \delta_0$ is the Dirac Function, i. e.,

$$\delta_{x_1}(\varphi) = \varphi (x_1) = \int\limits_{-\infty}^{+\infty} \varphi(\xi)\delta (\xi - x_1) \, d\xi.$$

As it is well-known generalized derivative of the Heaviside function

$$Y(\xi) = \begin{cases} +1, \ \xi > 0. \\ \ \ 0, \ \xi \leq 0, \end{cases}$$

is the Dirac function. Consequently, for the antiderivative $(Q\delta(x_1))^{(-1)}$ we have

$$(Q\delta (x_1))^{(-1)} = QY(\xi) = \begin{cases} Q, \ \xi > 0. \\ 0, \ \xi \leq 0. \end{cases}$$

Let

$$r^2 := x_1^2 + x_2^2, \ \ \psi := \text{arcctg} \frac{x_1}{x_2};$$

$$\rho^2 := (x_1-\xi)^2 + x_2^2, \quad \theta := \operatorname{arcctg}\frac{x_1-\xi}{x_2},$$

i.e.,

$$\xi = x_1 - x_2 \operatorname{ctg}\theta, \quad d\xi = x_2\frac{d\theta}{\sin^2\theta};$$

$$A(c,b) := \int_0^\pi \cos(c\theta)\sin^{-b}\theta d\theta, \ b<1; \quad B(c,b) := \int_0^\pi \sin(c\theta)\sin^{-b}\theta d\theta, \ b<1;$$

$$K := cD_0\left(b^2-c^2\right)\left[A^2(c,-b)+B^2(c,-b)\right];$$

$$\Lambda(c,b) := \int_0^\pi e^{a\theta}\sin^{-b}\theta d\theta, \ b<1;$$

$$a := \sqrt{(1-\kappa\sigma)(\kappa-1)}, \quad c := \sqrt{(\kappa\sigma-1)(\kappa-1)}, \quad b := \kappa-1.$$

Applying formally well known (see [1] and [2], p. 112) representation of the solution of the bending of the cusped plate under the action of a bending moment and generalized shearing force, when the projection of the plate is a half-plane, for the deflection w we get

$$\begin{aligned}
w(x_1,x_2) &= \frac{1}{K}\int_{-\infty}^{+\infty}\left\{[cA(c,-b)M\delta(\xi)-bB(c,-b)QY(\xi)]\cos\left(c\operatorname{arcctg}\frac{x_1-\xi}{x_2}\right)\right.\\
&+[cB(c,-b)M\delta(\xi)-bA(c,-b)QY(\xi)]\sin\left.\left(c\operatorname{arcctg}\frac{x_1-\xi}{x_2}\right)\right\}\\
&\times\left[(x_1-\xi)^2+x_2\right]^{-\frac{b}{2}}d\xi = K^{-1}\left\{Mc\left[A(c,-b)\cos\left(c\operatorname{arcctg}\frac{x_1}{x_2}\right)\right.\right.\\
&+B(c,-b)\sin\left.\left(c\operatorname{arcctg}\frac{x_1}{x_2}\right)\right](x_1+x_2)^{-\frac{b}{2}}\\
&-Qb\int_0^{+\infty}\left[B(c,-b)\cos\left(c\operatorname{arcctg}\frac{x_1-\xi}{x_2}\right)+A(c,-b)\sin\left(c\operatorname{arcctg}\frac{x_1-\xi}{x_2}\right)\right]\\
&\times\left.\left[(x_1-\xi)+x_2^2\right]^{-\frac{b}{2}}d\xi\right\}\\
&= K^{-1}\left\{Mc\left[A(c,-b)\cos(c\psi)+B(c,-b)\cos(c\psi)\right]r^{-b}\right.\\
&-Qb\int_\psi^\pi\left[B(c,-b)\cos(c\theta)+A(c,-b)\sin(c\theta)\right]\left.\frac{x_2^{1-b}\sin^b\theta}{\sin^2\theta}d\theta\right\}\\
&= K^{-1}\left\{Mc\left[A(c,-b)\cos(c\psi)+B(c,-b)\cos(c\psi)\right]r^{-b}\right.\\
&\qquad -Qbr^{1-b}\sin^{1-b}\psi\int_\psi^\pi\left[B(c,-b)\cos(c\theta)+A(c,-b)\sin(c\theta)\right]\left.\sin^{b-2}\theta d\theta\right\}
\end{aligned} \tag{3}$$

for $\kappa > \dfrac{1}{\sigma}$

(since the Poisson's ratio $\sigma < \frac{1}{2}$ in the case under consideration we have $\kappa > 2$ and all the integrals in the previous expressions are convergent);

$$w(x_1,x_2) = \frac{1}{D_0 b\Lambda(0,-b)} \int\limits_{-\infty}^{+\infty} \left[b^{-1}M\delta(\xi) - \frac{\pi}{2}QY(\xi) + QY(\xi)\theta\right]\rho^{-b}d\xi$$

$$= \frac{M}{D_0 b^2\Lambda(0,-b)}r^{-b} - \frac{Q}{D_0 b\Lambda(0,-b)} \int\limits_{0}^{+\infty} \left(\frac{\pi}{2} - \theta\right)\rho^{-b}d\xi = \frac{M}{D_0 b^2\Lambda(0,-b)}r^{-b}$$

$$-\frac{Q}{D_0 b\Lambda(0,-b)}r^{1-b}\sin^{1-b}\psi \int\limits_{\psi}^{\pi} \left(\frac{\pi}{2} - \theta\right)\sin^{b-2}\theta d\theta \ \text{ for } \ \kappa = \frac{1}{\sigma} \tag{4}$$

(in the case under consideration we have $\kappa > 2$ and all the integrals in the previous expressions are convergent);

$$w(x_1,x_2) = \frac{1}{2a(a^2+b^2)D_0\Lambda(a,-b)}$$

$$\times \int\limits_{-\infty}^{+\infty} \left\{\left[aM\delta(\xi) - bQY(\xi)\right]e^{a(\pi-\theta)} + \left[aM\delta(\xi) + bQY(\xi)\right]e^{a\theta}\right\}\rho^{-b}d\xi$$

$$= \frac{M}{2a(a^2+b^2)D_0\Lambda(a,-b)}\left[e^{a(\pi-\psi)} + e^{a\psi}\right]r^{-b}$$

$$-\frac{Qb}{2a(a^2+b^2)D_0\Lambda(a,-b)} \int\limits_{0}^{+\infty} \left[e^{a(\pi-\theta)} - e^{a\theta}\right]\rho^{-b}d\xi$$

$$= \frac{M}{2a(a^2+b^2)D_0\Lambda(a,-b)}\left[e^{a(\pi-\psi)} + e^{a\psi}\right]r^{-b}$$

$$-\frac{Qb}{2a(a^2+b^2)D_0\Lambda(a,-b)}\sin^{1-b}\psi\cdot r^{1-b} \int\limits_{\psi}^{\pi} \left[e^{a(\pi-\theta)} - e^{a\theta}\right]\sin^{b-2}\theta d\theta \quad \text{for } 1 < \kappa < \frac{1}{\sigma} \tag{5}$$

(in the case under consideration the integral exists only if $b > 1$, i.e., $\kappa > 2$, while by $Q = 0$ there remains only the first summand which has a physical sense for $\kappa > 1$).

Using (3)-(5) it is not difficult to get expressions for bending moments and shearing forces and directly to verify that they with (3)-(5) represent the explicit solution of the problem under consideration.

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