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EXPLICIT ESTIMATES FOR ERROR OF APPROXIMATE SOLUTION OF THE SYMMETRICAL DECOMPOSITION SCHEME FOR ABSTRACT PARABOLIC EQUATION

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Abstract. G. Baker and T. Oliphant symmetrical differential and difference decomposition schemes for approximate solution of Cauchy problem for non-homogeneous abstract parabolic equation have been considered. Explicit a priori estimates for approximate solution error have been obtained on a base of semigroup approximation.

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1. Differential Schemes

Let us consider Cauchy problem in Banach space X:

$$u'(t) + Au(t) = f(t), \quad u(0) = \varphi, \quad t > 0, \tag{1.1}$$

where A is linear densely defined closed operator in X, represented in the following form: $A = A_1 + A_2$; A_1 and A_2 are also densely defined closed operators in X.

We will consider the approximate solution of the problem (1.1) by the G. Baker and T.Oliphant symmetrical decomposition scheme. Our aim is to obtain explicit estimates for approximate solution error. Under the explicit estimates we implicate such a priori estimates for solution approximation, where constants on the right-hand side do not depend on the solution of the initial continuous problem, i.e. they represent absolute constants.

Different types of decomposition schemes are examined in G.I. Marchuk's well-known book (see [1] and extensive bibliography added to it).

G. Baker and T. Oliphant symmetrical decomposition differential scheme for approximate solution of (1.1) problems have the form (see [2],[3]):

$$\begin{aligned}
&\frac{dv_k^{(1)}(t)}{dt} + \frac{1}{2}A_1 v_k^{(1)}(t) = \sigma_0 f(t),\ v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}),\ u_0(0) = \varphi,\\
&\frac{dv_k^{(2)}(t)}{dt} + A_2 v_k^{(2)}(t) = (1 - (\sigma_0 + \sigma_1))f(t),\ v_k^{(2)}(t_{k-1}) = v_k^{(1)}(t_k),\\
&\frac{du_k(t)}{dt} + \frac{1}{2}A_1 u_k(t) = \sigma_1 f(t),\ u_k(t_{k-1}) = v_k^{(2)}(t_k),\quad t \in [t_{k-1}, t_k],
\end{aligned} \tag{1.2}$$

where $k = 1, 2, \ldots, t_k = k \cdot \tau, \tau > 0$ is a time step.

Approximate value of exact solution of (1.1) problem at point $t = t_k = k\tau$ is $u(t_k)$, $u(t_k) \approx u_k(t_k)$.

Theorem 1.1. *Assume the following conditions are fulfilled:*

(a) There exists such $\omega_0 > 0$, that for any $\lambda > \omega_0$, operator $A + \lambda I$ is invertible and the estimate is valid:

$$\|(A+\lambda I)^{-k}\| \le \frac{M}{(\lambda-\omega_0)^k}, \quad M = const > 0, \quad k = 1, 2, \cdots \;;$$

(b) There exists such $\omega_1 > 0$, that for any $\xi > \omega_1$, operators $A_i + \xi I$, $i = 1, 2$ are invertible and the following estimates are valid:

$$\|(A_i+\xi I)^{-1}\| \le \frac{1}{\xi-\omega_1} \;;$$

(c) $D(A^m) \subset D(A_i^m)$, $m = 1, 2, 3$ $(i = 1, 2)$; A_i operators reflect $D(A^m)$, $m = 2, 3$ in $D(A^{m-1})$,$(A_i : D(A^m) \to D(A^{m-1}))$ and the following inequalities are valid:

$$\|A_i^2 u\| + \|A_i A_{3-i} u\| \le c\|A_0^2 u\|, \quad u \in D(A^2),$$

$$\|A_i^3 u\| + \|A_i^2 A_{3-i} u\| + \|A_1 A_2 A_1 u\| \le c\|A_0^3 u\|, \quad u \in D(A^3),$$

where $A_0 = A - \lambda_0 I, \lambda_0$ is regular point operator of A; $c = const > 0$;

(d) $f(t)$ is continuously differentiable function and $f'(t)$ satisfies Lipschitz condition; for each fixed t from $[0; +\infty[$, $f(t) \in D(A^3), f'(t) \in D(A)$ and $\varphi \in D(A^3)$; Than, if $\sigma_0 = \sigma_1$, for error of the scheme (1.2) the following estimate will be valid:

$$\|u(t_k) - u_k(t_k)\| \le c\tau^2 \big[e^{\omega t_k} \big(t_k \|A_0^3 \varphi\| + \int_0^{t_k} \|A_0 f'(t)\| dt$$

$$+\tau \sum_{i=1}^{k} \big(\|A_0^2 f(t_{i-\frac{1}{2}})\| + \|A_0 f(t_{i-\frac{1}{2}})\| \big) + t_k \big) + \int_0^{t_k} (t_k - s) e^{\omega(t_k - s)} \|A_0^3 f(s)\| ds \big], \quad (1.3)$$

where $\omega = max(\omega_0, \omega_1)$, $c = const > 0$.

To prove the theorem, we need two lemmas. First of them deals with approximation of the semigroup by means of the Trotter-type formula and the other - with approximation of the integral containing semigroup by means of quadratic formula, in particular, the formula of central rectangle.

Lemma 1.2. *If operators A_1, A_2 and A satisfy conditions of Theorem 1.1, than for any natural n the following estimation is valid:*

$$\big\| \big[U(t) - \big(V\big(\frac{t}{n}\big)\big)^n \big] \varphi \big\| \le \frac{ct^3}{n^2} e^{\omega t} \|A_0^3 \varphi\|, \quad \varphi \in D(A^3), \quad (1.4)$$

$$V(t) = U_1\Big(\frac{t}{2}\Big) U_2(t) U_1\Big(\frac{t}{2}\Big),$$

where $U(t) = exp(-tA)$ and $U_i(t) = exp(-tA_i)$ are strongly continuous semigroups generated by operators A and A_i $(i = 1, 2)$ respectively.

Proof. According to the property of semigroup:

$$\big[U(t_n) - \big(V\big(\frac{t_n}{n}\big)\big)^n\big]\varphi = [(U(\tau))^n - (V(\tau))^n]\varphi, \tag{1.5}$$

where $\tau = t/n$.

The following identity is valid:

$$(U(\tau))^n - (V(\tau))^n = \sum_{i=1}^{n}(V(\tau))^{n-i}[U(\tau) - V(\tau)]U(t_{i-1}). \tag{1.6}$$

Let us evaluate the difference $U(\tau) - V(\tau)$.

It is easy to prove that for the semigroup $U(t)$ the following expansion is valid:

$$U(t) = \sum_{k=0}^{n}(-1)^k\frac{t^k}{k!}A^k + R^{(n+1)}(t), \tag{1.7}$$

where

$$R^{(n+1)}(t) = (-A)^{n+1}\int_0^t\int_0^{s_1}\int_0^{s_2}\cdots\int_0^{s_n}U(s)ds ds_n ds_{n-1}\cdots ds_1.$$

Applying formula (1.7), we will obtain:

$$V(\tau) = U_1(\frac{\tau}{2})U_2(\tau)U_1(\frac{\tau}{2}) = U_1(\frac{\tau}{2})U_2(\tau)[I - \frac{\tau}{2}A_1 + \frac{\tau^2}{8}A_1^2 + R_1^{(3)}(\frac{\tau}{2})]$$

$$= U_1(\frac{\tau}{2})[U_2(\tau) - \frac{\tau}{2}U_2(\tau)A_1 + \frac{\tau^2}{8}U_2(\tau)A_1^2 + U_2(\tau)R_1^{(3)}(\frac{\tau}{2})]$$

$$= U_1(\frac{\tau}{2})[I - \tau A_2 + \frac{\tau^2}{2}A_2^2 + R_2^{(3)}(\tau) - \frac{\tau}{2}(I - \tau A_2 + +R_2^{(2)}(\tau))A_1$$

$$+\frac{\tau^2}{8}(I + R_2^{(1)}(\tau))A_1^2 + U_2(\tau)R_1^{(3)}(\frac{\tau}{2})]$$

$$= U_1(\frac{\tau}{2})[I - \frac{\tau}{2}(2A_2 + A_1) + \frac{\tau^2}{8}(4A_2^2 + 4A_2A_1 + A_1^2)$$

$$+R_2^{(3)}(\tau) - \frac{\tau}{2}R_2^{(2)}(\tau)A_1 + \frac{\tau^2}{8}R_2^{(1)}(\tau))A_1^2 + U_2(\tau)R_1^{(3)}(\frac{\tau}{2})]$$

$$= U_1(\frac{\tau}{2}) - \frac{\tau}{2}U_1(\frac{\tau}{2})(2A_2 + A_1) + \frac{\tau^2}{8}U_1(\frac{\tau}{2})(4A_2^2 + 4A_2A_1 + A_1^2)$$

$$+U_1(\frac{\tau}{2})(R_2^{(3)}(\tau) - \frac{\tau}{2}R_2^{(2)}(\tau)A_1 + \frac{\tau^2}{8}R_2^{(1)}(\tau))A_1^2 + U_2(\tau)R_1^{(3)}(\frac{\tau}{2}))$$

$$= I - \frac{\tau}{2}A_1 + \frac{\tau^2}{8}A_1^2 + R_1^{(3)} - \frac{\tau}{2}(I - \frac{\tau}{2}A_1 + R_1^{(2)})(2A_2 + A_1)$$

$$+\frac{\tau^2}{8}(I + R_1^{(1)}(\frac{\tau}{2})(4A_2^2 + 4A_2A_1 + A_1^2)$$

$$+U_1(\frac{\tau}{2})(R_2^{(3)}(\tau) - \frac{\tau}{2}R_2^{(2)}(\tau)A_1 + \frac{\tau^2}{8}R_2^{(1)}(\tau))A_1^2 + U_2(\tau)R_1^{(3)}(\frac{\tau}{2}))$$
$$= I - \tau(A_1 + A_2) + \frac{\tau^2}{2}(A_1^2 + A_1A_2 + A_2A_1 + A_2^2) + R_3(\tau),$$

where

$$R_3(\tau) = \sum_{j=1}^{7} R_{3,j}(\tau), \tag{1.8}$$

$$R_{3,1}(\tau) = R_1^{(3)}(\frac{\tau}{2}), \quad R_{3,2}(\tau) = -\frac{\tau}{2}R_1^{(2)}(\frac{\tau}{2})(2A_2 + A_1), \quad R_{3,3}(\tau) = U_1(\frac{\tau}{2})(R_2^{(3)}(\tau),$$
$$R_{3,4}(\tau) = -\frac{\tau}{2}R_2^{(2)}(\tau)A_1, \quad R_{3,5}(\tau) = \frac{\tau^2}{8}R_1^{(1)}(\frac{\tau}{2})(4A_2^2 + 4A_2A_1 + A_1^2),$$
$$R_{3,6}(\tau) = \frac{\tau^2}{8}R_2^{(1)}(\tau)A_1^2, \quad R_{3,7}(\tau) = U_2(\tau)R_1^{(3)}(\frac{\tau}{2})).$$

As $A = A_1 + A_2$ and

$$A^2 = A_1^2 + A_1A_2 + A_2A_1 + A_2^2.$$

$V(\tau)$ will be:

$$V(\tau) = I - \tau A + \frac{\tau^2}{2}A^2 + R_3(\tau). \tag{1.9}$$

According to the formulas (1.7) and (1.9):

$$U(\tau) - V(\tau) = R^{(3)}(\tau) - R_3(\tau). \tag{1.10}$$

If, in (1.6) we move to the norms and take (1.10) into consideration, we shall obtain:

$$\|[(U(\tau))^n - (V(\tau))^n]\varphi\| \leq \sum_{i=1}^{k} \|(V(\tau))^{k-i}\| \|[R^{(3)}(\tau) - R_3(\tau)]U(t_{i-1})\varphi\|. \tag{1.11}$$

It is clear that inequality:

$$\|[R^{(3)}(\tau) - R_3(\tau)]U(t_{i-1})\varphi\| \leq \sum_{j=1}^{7} \|R_{3,j}(\tau)U(t_{i-1})\varphi\| + \|R^{(3)}(\tau)U(t_{i-1})\varphi\|. \tag{1.12}$$

According to conditions (a) and (b) of Theorem 1.1 we have (see [4]):

$$\|U(t)\| \leq Me^{\omega_0 t}, \tag{1.13}$$

$$\|U_i(t)\| \leq e^{\omega_1 t}. \tag{1.14}$$

According to estimates (1.13) - (1.14) and condition (c) of Theorem 1.1, the following is obtained:

$$\|R_{3,1}(\tau)U(t_{i-1})\varphi\| = \|A_1^3 \int_0^{\frac{\tau}{2}} \int_0^{s_1} \int_0^{s_2} U_1(s)U(t_{i-1})\varphi ds ds_2 ds_1\|$$

$$\leq \int_0^{\frac{\tau}{2}} \int_0^{s_1} \int_0^{s_2} \|U_1(s)\| \|U(t_{i-1})\| \|A_1^3\varphi\| ds ds_2 ds_1 \leq c\tau^3 e^{\omega t_i} \|A_0^3\varphi\|,\ \varphi \in D(A^3). \qquad (1.15)$$

Similarly the following estimates are obtained:

$$\|R_{3,j}(\tau)U(t_{i-1})\varphi\| \leq c\tau^3 e^{\omega t_i} \|A_0^3\varphi\|,\ \varphi \in D(A^3),\ j = 2, 3, \cdots, 7, \qquad (1.16)$$

$$\|R^{(3)}(\tau)U(t_{i-1})\varphi\| \leq c\tau^3 e^{\omega_0 t_i} \|A^3\varphi\|, \quad \varphi \in D(A^3).$$

From the following representation

$$A^3(A - \lambda_0 I)^{-3} = (A(A - \lambda_0 I)^{-1})^3 = (I + \lambda_0(A - \lambda_0 I)^{-1})^3,$$

we have:

$$\|R^{(3)}(\tau)U(t_{i-1})\varphi\| \leq c\tau^3 e^{\omega_0 t_i} \|A_0^3\varphi\|, \ \ \varphi \in D(A^3). \qquad (1.17)$$

From (1.12) taking into account (1.15), (1.16) and (1.17) evaluations, the following is obtained:

$$\|[R^{(3)}(\tau) - R_3(\tau)]U(t_{i-1})\varphi\| \leq c\tau^3 e^{\omega_0 t_i} \|A_0^3\varphi\|, \ \ \varphi \in D(A^3). \qquad (1.18)$$

From (1.11) taking into account (1.14) and (1.18) evaluations, the following evaluation is obtained:

$$\|[(U(\tau))^n - (V(\tau))^n]\varphi\| \leq c\tau^3 \|A_0^3\varphi\| \sum_{i=1}^{n} \left(\|U_1(\frac{\tau}{2})\| U_2(\tau)\| \|U_1(\frac{\tau}{2})\|\right)^{n-i} e^{\omega t_i}$$

$$\leq c\|A_0^3\varphi\| \sum_{i=1}^{n} e^{2\omega_1 t_{n-i}} e^{\omega t_i} \leq c\tau^2 t_n e^{\omega t_n} \|A_0^3\varphi\|. \ \square \qquad (1.19)$$

The following lemma, required for proving of the main theorem, deals with error estimate of approximation by means of quadrature formula of the integral containing semigroup.

Lemma 1.3. *Assume operator A and function $f(t)$) comply with the conditions of Theorem 1.1. Then the following estimate is valid:*

$$\Big\| \int_{t_{i-1}}^{t_i} U(t_i - s) f(s) ds - \tau U(\frac{\tau}{2}) f(t_{i-\frac{1}{2}}) \Big\|$$

$$\leq c\tau^2 e^{\omega_0 \tau} \Big(\int_{t_{i-1}}^{t_i} \|Af'(t)\| dt + \tau(\|A^2 f(t_{i-\frac{1}{2}})\| + 1) \Big), \quad c = const > 0, \qquad (1.20)$$

where $U(t) = exp(-tA)$) is strongly continuous semigroup generated by operator A , $\tau = t_i - t_{i-1}$ ($t_i \geq t_{i-1} \geq 0$).

Proof. It is clear that the inequality:

$$\Big\| \int\limits_{t_{i-1}}^{t_i} U(t_i - s) f(s) ds - \tau U(\frac{\tau}{2}) f(t_{i-\frac{1}{2}}) \Big\| \leq \|I_0\| + \|I_1\|, \qquad (1.21)$$

where

$$I_0 = \int\limits_{t_{i-1}}^{t_i} U(t_i - t_{i-\frac{1}{2}}) f(s) ds - \int\limits_{t_{i-1}}^{t_i} U(t_i - s) f(s) ds,$$

$$I_1 = U(\frac{\tau}{2}) \Big(\int\limits_{t_{i-1}}^{t_i} f(s) ds - \tau f(t_{i-\frac{1}{2}}) \Big).$$

Let us evaluate I_0. Certainly we have:

$$I_0 = I_{0,1} + I_{0,2}, \qquad (1.22)$$

Where

$$I_{0,1} = \int\limits_{t_{i-1}}^{t_i} [U(t_i - t_{i-\frac{1}{2}}) - U(t_i - s)][f(s) - f(t_{i-\frac{1}{2}})] ds,$$

$$I_{0,2} = \int\limits_{t_{i-1}}^{t_i} [U(t_i - t_{i-\frac{1}{2}}) - U(t_i - s)] f(t_{i-\frac{1}{2}}) ds.$$

For member $I_{0,1}$, the following expression is obtained:

$$I_{0,1} = \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} [U(t_i - t_{i-\frac{1}{2}}) - U(t_i - s)][f(s) - f(t_{i-\frac{1}{2}})] ds$$

$$+ \int\limits_{t_{i-\frac{1}{2}}}^{t_i} [U(t_i - t_{i-\frac{1}{2}}) - U(t_i - s)][f(s) - f(t_{i-\frac{1}{2}})] ds$$

$$= -A \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} \Big(\int\limits_{\frac{\tau}{2}}^{t_i - s} U(t) dt \int\limits_{s}^{t_{i-\frac{1}{2}}} f'(t) dt \Big) ds - A \int\limits_{t_{i-\frac{1}{2}}}^{t_i} \Big(\int\limits_{t_i - s}^{\frac{\tau}{2}} U(t) dt \int\limits_{t_{i-\frac{1}{2}}}^{s} f'(t) dt \Big) ds. \qquad (1.23)$$

Here we have applied the following formula (see [4]):

$$A \int\limits_{r}^{t} U(s) ds = U(r) - U(t), \quad 0 \leq r \leq t.$$

If, in (1.23), we move to the norms and take (1.13) into consideration, we shall obtain:

$$\|I_{0,1}\| \leq \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} \Big(\int\limits_{\frac{\tau}{2}}^{t_i-s} \|U(t)\|dt \int\limits_{s}^{t_{i-\frac{1}{2}}} \|Af'(t)\|dt\Big) ds + \int\limits_{t_{i-\frac{1}{2}}}^{t_i} \Big(\int\limits_{t_i-s}^{\frac{\tau}{2}} \|U(t)\|dt \int\limits_{t_{i-\frac{1}{2}}}^{s} \|Af'(t)\|dt\Big) ds$$

$$\leq M \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} \Big(\int\limits_{\frac{\tau}{2}}^{t_i-s} e^{\omega_0 t}dt \int\limits_{s}^{t_{i-\frac{1}{2}}} \|Af'(t)\|dt\Big) ds + M \int\limits_{t_{i-\frac{1}{2}}}^{t_i} \Big(\int\limits_{t_i-s}^{\frac{\tau}{2}} e^{\omega_0 t}dt \int\limits_{t_{i-\frac{1}{2}}}^{s} \|Af'(t)\|dt\Big) ds$$

$$\leq M \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} \|Af'(t)\|dt \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} (t_{i-\frac{1}{2}} - s) e^{\omega_0 (t_i - s)} ds + M \int\limits_{t_{i-\frac{1}{2}}}^{t_i} \|Af'(t)\|dt \int\limits_{t_{i-\frac{1}{2}}}^{t_i} (s - t_{i-\frac{1}{2}}) e^{\omega_0 \frac{\tau}{2}} ds$$

$$\leq \frac{1}{4} M e^{\omega_0 \frac{\tau}{2}} \tau^2 \int\limits_{t_{i-1}}^{t_i} \|Af'(t)\|dt.$$

Thus, we obtain estimate:

$$\|I_{0,1}\| \leq c e^{\omega_0 \frac{\tau}{2}} \tau^2 \int\limits_{t_{i-1}}^{t_i} \|Af'(t)\|dt. \tag{1.24}$$

For $I_{0,2}$, the following expression takes place:

$$I_{0,2} = \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} [U(t_i - t_{i-\frac{1}{2}}) - U(t_i - s)] f(t_{i-\frac{1}{2}}) ds + \int\limits_{t_{i-\frac{1}{2}}}^{t_i} [U(t_i - t_{i-\frac{1}{2}}) - U(t_i - s)] f(t_{i-\frac{1}{2}}) ds$$

$$= A \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} \int\limits_{s}^{t_{i-\frac{1}{2}}} [U(t_i - t_{i-\frac{1}{2}}) - U(t_i - t)] f(t_{i-\frac{1}{2}}) dt ds$$

$$+ A \int\limits_{t_{i-\frac{1}{2}}}^{t_i} \int\limits_{t_{i-\frac{1}{2}}}^{s} [U(t_i - t) - U(t_i - t_{i-\frac{1}{2}})] f(t_{i-\frac{1}{2}}) dt ds$$

$$= \int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}} \int\limits_{s}^{t_{i-\frac{1}{2}}} \int\limits_{\frac{\tau}{2}}^{t_i - t} U(\xi) A^2 f(t_{i-\frac{1}{2}}) d\xi dt ds + \int\limits_{t_{i-\frac{1}{2}}}^{t_i} \int\limits_{t_{i-\frac{1}{2}}}^{s} \int\limits_{t_i - t}^{\frac{\tau}{2}} U(\xi) A^2 f(t_{i-\frac{1}{2}}) d\xi dt ds.$$

If in this equality, we move to norms and take (1.13) into consideration, we receive the inequality:

$$\|I_{0,2}\| \le M\|A^2 f(t_{i-\frac{1}{2}})\|\Big(\int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}}\int\limits_{s}^{t_{i-\frac{1}{2}}}\int\limits_{\frac{\tau}{2}}^{t_i-t} e^{\omega_0\xi}d\xi dtds + \int\limits_{t_{i-\frac{1}{2}}}^{t_i}\int\limits_{t_{i-\frac{1}{2}}}^{s}\int\limits_{t_i-t}^{\frac{\tau}{2}} e^{\omega_0\xi}d\xi dtds\Big)$$

$$\le \frac{1}{48}Me^{\omega_0\tau}\tau^3\|A^2 f(t_{i-\frac{1}{2}})\|. \tag{1.25}$$

From (1.22), taking (1.24) and (1.25) inequalities into account, the following evaluation is obtained:

$$\|I_0\| \le \|\int\limits_{t_{i-1}}^{t_i} U(\frac{\tau}{2})f(s)ds - \int\limits_{t_{i-1}}^{t_i} U(t_i - s)f(s)ds\|$$

$$\le ce^{\omega_0\tau}\tau^2\Big(\int\limits_{t_{i-1}}^{t_i} \|Af'(t)\|dt + \tau\|A^2 f(t_{i-\frac{1}{2}})\|\Big). \tag{1.26}$$

Let's evaluate I_1. Certainly we have:

$$I_1 = U(\frac{\tau}{2})\Big(\int\limits_{t_{i-1}}^{t_i} f(s)ds + \tau f(t_{i-\frac{1}{2}})\Big)$$

$$= U(\frac{\tau}{2})\Big(\int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}}\int\limits_{s}^{t_{i-\frac{1}{2}}} (f'(t_{i-\frac{1}{2}}) - f'(t))dtds - \int\limits_{t_{i-\frac{1}{2}}}^{t_i}\int\limits_{t_{i-\frac{1}{2}}}^{s} (f'(t) - f'(t_{i-\frac{1}{2}}))dtds\Big).$$

If in this equality we move to the norms and take into consideration that $f'(t)$ satisfies Lipschitz condition, we obtain:

$$\|U(\frac{\tau}{2})\|\Big\|\int\limits_{t_{i-1}}^{t_i} f(s)ds - \tau f(t_{i-\frac{1}{2}})\Big\|$$

$$\le ce^{\omega_0\frac{\tau}{2}}\Big(\int\limits_{t_{i-1}}^{t_{i-\frac{1}{2}}}\int\limits_{s}^{t_{i-\frac{1}{2}}} (t_{i-\frac{1}{2}} - t)dtds + \int\limits_{t_{i-\frac{1}{2}}}^{t_i}\int\limits_{t_{i-\frac{1}{2}}}^{s} (t - t_{i-\frac{1}{2}})dtds\Big) \le \frac{1}{24}c\tau^3 e^{\omega_0\frac{\tau}{2}}. \tag{1.27}$$

From (1.21), taking (1.26) and (1.27) inequalities into account, estimate (1.20) is obtained. □

Proof of Theorem 1.1. As it is known, solution of problem (1.1), by means of semigroup $U(t) = exp(-tA)$, is expressed by the following formula (see for example [4],[5]):

$$u(t) = U(t)\varphi + \int\limits_0^t U(t-s)f(s)ds. \tag{1.28}$$

According to this formula, for the first equation of (1.2) scheme, we obtain:

$$v_k^{(1)}(t) = U_1(\frac{1}{2}(t-t_{k-1}))u_{k-1}(t_{k-1}) + \sigma_0 \int\limits_{t_{k-1}}^{t_k} U_1(\frac{1}{2}(t-s))f(s)ds.$$

In $t = t_k$ point we have:

$$v_k^{(1)}(t_k) = U_1(\frac{\tau}{2})u_{k-1}(t_{k-1}) + \sigma_0 \int\limits_{t_{k-1}}^{t_k} U_1(\frac{1}{2}(t_k-s))f(s)ds. \tag{1.29}$$

From the second and third equations of (1.2) system, in accordance with (1.28), the following is obtained:

$$v_k^{(2)}(t_k) = U_2(\tau)v_k^{(1)} + (1-\sigma_0-\sigma_1) \int\limits_{t_{k-1}}^{t_k} U_2(t_k-s)f(s)ds, \tag{1.30}$$

$$u_k(t_k) = U_1(\frac{\tau}{2})v_k^{(2)}(t_k) + \sigma_1 \int\limits_{t_{k-1}}^{t_k} U_1(\frac{1}{2}(t_k-s))f(s)ds. \tag{1.31}$$

From (1.31) and (1.29), taking into consideration (1.30), the following is obtained:

$$u_k(t_k) = U_1(\frac{\tau}{2})U_2(\tau)U_1(\frac{\tau}{2})u_{k-1}(t_{k-1}) + \sigma_0 \int\limits_{t_{k-1}}^{t_k} U_1(\frac{\tau}{2})U_2(\tau)U_1(\frac{1}{2}(t_k-s))f(s)ds$$

$$+(1-\sigma_0-\sigma_1) \int\limits_{t_{k-1}}^{t_k} U_1(\frac{\tau}{2})U_2(t_k-s)f(s)ds + \sigma_1 \int\limits_{t_{k-1}}^{t_k} U_1(\frac{1}{2}(t_k-s))f(s)ds. \tag{1.32}$$

If we assume the designations:

$$V(\tau) = U_1(\frac{\tau}{2})U_2(\tau)U_1(\frac{\tau}{2}),$$

$$V_0(\tau,t) = \sigma_0 U_1(\frac{\tau}{2})U_2(\tau)U_1(\frac{t}{2}) + (1-\sigma_0-\sigma_1)U_1(\frac{\tau}{2})U_2(t) + \sigma_1 U_1(\frac{t}{2}),$$

(1.32) could be written as:

$$u_k(t_k) = V(\tau)u_{k-1}(t_{k-1}) + \int\limits_{t_{k-1}}^{t_k} V_0(\tau, t_k - s))f(s)ds.$$

Hence the following is obtained:

$$u_k(t_k) = (V(\tau))^k\varphi + \sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} (V(\tau))^{k-i}V_0(\tau, t_i - s))f(s)ds. \tag{1.33}$$

According to formula (1.28) we obtain:

$$u(t_k) = U(t_k)\varphi + \int\limits_0^{t_k} U(t_k - s)f(s)ds.$$

Hence the following is obtained:

$$u(t_k) = (U(\tau))^k\varphi + \sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} U(t_k - s))f(s)ds. \tag{1.34}$$

Taking into consideration the identity

$$U(t_k - s) = U(t_{k-i})U(t_i - s) = (U(\tau))^{k-i}U(t_i - s),$$

then (1.34) could be written as:

$$u(t_k) = (U(\tau))^k\varphi + \sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} (U(\tau))^{k-i}U(t_i - s))f(s)ds. \tag{1.35}$$

According to formulas (1.33) and (1.35) the following is obtained:

$$u(t_k) - u_k(t_k) = [(U(\tau))^k - (V(\tau))^k]\varphi$$

$$+\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} [(U(\tau))^{k-i} - (V(\tau))^{k-i}]U(t_i - s)f(s)ds$$

$$+\sum_{i=1}^{k}(V(\tau))^{k-i}\int\limits_{t_{i-1}}^{t_i} [U(t_i - s) - V_0(\tau, t_i - s)]f(s)ds. \tag{1.36}$$

According to Lemma 1.2 the following estimate is valid:

$$\|[(U(\tau))^{k-i} - (V(\tau))^{k-i}]\varphi\| \leq c\tau^2 t_{k-i}e^{\omega t_{k-i}}\|A_0^3\varphi\|.$$

Hence:

$$\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} \|[(U(\tau))^{k-i}-(V(\tau))^{k-i}]U(t_i-s)f(s)\|ds$$

$$\leq c\tau^2\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} t_{k-i}e^{\omega t_{k-i}}\|A^3U(t_i-s)f(s)\|ds$$

$$\leq c\tau^2\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} t_{k-i}e^{\omega t_{k-i}}e^{\omega(t_i-s)}\|A^3f(s)\|ds = c\tau^2\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} t_{k-i}e^{\omega(t_k-s)}\|A^3f(s)\|ds$$

$$\leq c\tau^2\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} (t_k-s)e^{\omega(t_k-s)}\|A^3f(s)\|ds = c\tau^2\int\limits_{0}^{t_k}(t_k-s)e^{\omega(t_k-s)}\|A^3f(s)\|ds.$$

Thus, we have:

$$\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i} \|[(U(\tau))^{k-i}-(V(\tau))^{k-i}]U(t_i-s)f(s)\|ds$$

$$\leq c\tau^2\int\limits_{0}^{t_k}(t_k-s)e^{\omega(t_k-s)}\|A^3f(s)\|ds. \tag{1.37}$$

Let's rewrite the integral in the second sum of equation (1.36) as follows:

$$\int\limits_{t_{i-1}}^{t_i}[U(t_i-s)-V_0(\tau,t_i-s)]f(s)ds = J_1+J_2+J_3,$$

where

$$J_1=\int\limits_{t_{i-1}}^{t_i}U(t_i-s)f(s)ds-\tau U(t_i-t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}}),$$

$$J_2=\tau[U(t_i-t_{i-\frac{1}{2}})-V_0(\tau,t_i-t_{i-\frac{1}{2}})]f(t_{i-\frac{1}{2}}),$$

$$J_3=\tau V_0(\tau,t_i-t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}})-\int\limits_{t_{i-1}}^{t_i}V_0(\tau,t_i-s)f(s)ds.$$

It is clear that for J_3 we have the following expression:

$$J_3=-[\sigma_0J_{3,1}+(1-\sigma_0-\sigma_1)J_{3,2}+\sigma_1J_{3,3}],$$

where

$$J_{3,1} = U_1(\frac{\tau}{2})U_2(\tau)\big[\int\limits_{t_{i-1}}^{t_i} U_1(\frac{1}{2}(t_i - s))f(s)ds - \tau U_1(\frac{1}{2}(t_i - t_{i-\frac{1}{2}}))f(t_{i-\frac{1}{2}})\big],$$

$$J_{3,2} = U_1(\frac{\tau}{2})\big[\int\limits_{t_{i-1}}^{t_i} U_2(t_i - s)f(s)ds - \tau U_2(t_i - t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}})\big],$$

$$J_{3,3} = \int\limits_{t_{i-1}}^{t_i} U_1(\frac{1}{2}(t_i - s))f(s)ds - \tau U_1(\frac{1}{2}(t_i - t_{i-\frac{1}{2}}))f(t_{i-\frac{1}{2}}).$$

In accordance with Lemma 1.3 the following estimates are valid:

$$\|J_{3,1}\| \le ce^{2\omega_1\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_1 f'(t)\|dt + \tau(\|A_1^2 f(t_{i-\frac{1}{2}})\| + 1)\big],$$

$$\|J_{3,2}\| \le ce^{2\omega_1\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_2 f'(t)\|dt + \tau(\|A_2^2 f(t_{i-\frac{1}{2}})\| + 1)\big],$$

$$\|J_{3,3}\| \le ce^{\omega_1\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_1 f'(t)\|dt + \tau(\|A_1^2 f(t_{i-\frac{1}{2}})\| + 1)\big],$$

$$\|J_1\| \le ce^{\omega_0\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A f'(t)\|dt + \tau(\|A^2 f(t_{i-\frac{1}{2}})\| + 1)\big].$$

These inequalities, according to condition (c) of Theorem 1.1 provide the following inequalities:

$$\|J_{3,j}\| \le ce^{\omega\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_0 f'(t)\|dt + \tau(\|A_0^2 f(t_{i-\frac{1}{2}})\| + 1)\big],\ j = 1, 2, 3, \tag{1.38}$$

$$\|J_1\| \le ce^{\omega\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_0 f'(t)\|dt + \tau(\|A_0^2 f(t_{i-\frac{1}{2}})\| + 1)\big]. \tag{1.39}$$

According to estimate (1.38), we have for J_3:

$$\|J_3\| \le ce^{\omega\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_0 f'(t)\|dt + \tau(\|A_0^2 f(t_{i-\frac{1}{2}})\| + 1)\big]. \tag{1.40}$$

Let's evaluate norm of J_2. According to formula (1.7) we have:

$$V_0(\tau,\frac{\tau}{2})=\sigma_0U_1(\frac{\tau}{2})U_2(\tau)U_1(\frac{\tau}{4})+(1-(\sigma_0+\sigma_1))U_1(\frac{\tau}{2})U_2(\frac{\tau}{2})+\sigma_1U_1(\frac{\tau}{4})$$

$$=\sigma_0U_1(\frac{\tau}{2})U_2(\tau)\big(I-\frac{\tau}{4}A_1+R_1^{(2)}(\frac{\tau}{4})\big)+(1-(\sigma_0+\sigma_1))U_1(\frac{\tau}{2})U_2(\frac{\tau}{2})+\sigma_1U_1(\frac{\tau}{4})$$

$$=\sigma_0U_1(\frac{\tau}{2})\big(U_2(\tau)-\frac{\tau}{4}U_2(\tau)A_1+U_2(\tau)R_1^{(2)}(\frac{\tau}{4})\big)$$

$$+(1-(\sigma_0+\sigma_1))U_1(\frac{\tau}{2})U_2(\frac{\tau}{2})+\sigma_1U_1(\frac{\tau}{4})$$

$$=\sigma_0U_1(\frac{\tau}{2})\big[I-\tau A_2+R_2^{(2)}(\tau)-\frac{\tau}{4}(I+R_2^{(1)}(\tau))A_1+U_2(\tau)R_1^{(2)}(\frac{\tau}{4})\big]$$

$$+(1-(\sigma_0+\sigma_1))U_1(\frac{\tau}{2})\big(I-\frac{\tau}{2}A_2+R_2^{(2)}(\frac{\tau}{2})\big)+\sigma_1U_1(\frac{\tau}{4})$$

$$=\sigma_0\big[U_1(\frac{\tau}{2})-\tau U_1(\frac{\tau}{2})A_2+U_1(\frac{\tau}{2})R_2^{(2)}(\tau)$$

$$-\frac{\tau}{4}U_1(\frac{\tau}{2})(I+R_2^{(1)}(\tau))A_1+U_1(\frac{\tau}{2})U_2(\tau)R_1^{(2)}(\frac{\tau}{4})\big]$$

$$+(1-(\sigma_0+\sigma_1))\big(U_1(\frac{\tau}{2})-\frac{\tau}{2}U_1(\frac{\tau}{2})A_2+U_1(\frac{\tau}{2})R_2^{(2)}(\frac{\tau}{2})\big)+\sigma_1U_1(\frac{\tau}{4})$$

$$=\sigma_0\big[I-\frac{\tau}{2}A_1+R_1^{(2)}(\frac{\tau}{2})-\tau(I+R_1^{(1)}(\frac{\tau}{2}))A_2-\frac{\tau}{4}(I+R_1^{(2)}(\frac{\tau}{2}))A_1$$

$$-\frac{\tau}{4}U_1(\frac{\tau}{2})(I+R_2^{(1)}(\tau))A_1+U_1(\frac{\tau}{2})R_2^{(2)}(\tau)+U_1(\frac{\tau}{2})U_2(\tau)R_1(\frac{\tau}{4})\big]$$

$$+(1-(\sigma_0+\sigma_1))\big(I-\frac{\tau}{2}A_1+R_1^{(2)}(\frac{\tau}{2})-\frac{\tau}{2}(I+R_1^{(1)}(\frac{\tau}{2}))A_2+U_1(\frac{\tau}{2})R_2(\frac{\tau}{2})\big)$$

$$+\sigma_1\big(I-\frac{\tau}{4}A_1+R_1^{(2)}(\frac{\tau}{4})\big)$$

$$=I-\tau\big(\frac{1}{4}\sigma_0-\frac{1}{4}\sigma_1+\frac{1}{2}\big)A_1-\tau\big(\frac{1}{2}\sigma_0-\frac{1}{2}\sigma_1+\frac{1}{2}\big)A_2+R_2(\tau),$$

where

$$R_2(\tau)=\sum_{j=1}^{5}R_{2,j}(\tau), \tag{1.41}$$

$$R_{2,1}(\tau)=(\sigma_0U_1(\frac{\tau}{2})U_2(\tau)+\sigma_1I)R_1^{(2)}(\frac{\tau}{4}),\quad R_{2,2}(\tau)=-\sigma_0\frac{\tau}{4}U_1(\frac{\tau}{2})R_2^{(1)}(\tau)A_1,$$

$$R_{2,3}=-\tau(3-\sigma_0-\sigma_1)R_1^{(1)}(\frac{\tau}{2}))A_2,\quad R_{2,4}=(1-\sigma_1)R_1^{(2)}(\frac{\tau}{2}),$$

$$R_{2,5}(\tau)=(1-\sigma_1)U_1(\frac{\tau}{2})R_2^{(2)}(\tau).$$

Thus, we obtained:

$$V_0(\tau,\frac{\tau}{2})==I-\tau\big(\frac{1}{4}\sigma_0-\frac{1}{4}\sigma_1+\frac{1}{2}\big)A_1-\tau\big(\frac{1}{2}\sigma_0-\frac{1}{2}\sigma_1+\frac{1}{2}\big)A_2+R_2(\tau). \tag{1.42}$$

For $U(\frac{\tau}{2})$, in accordance with formula (1.7) we have:

$$U(\frac{\tau}{2}) = I - \frac{\tau}{2}(A_1 + A_2) + R^{(2)}(\frac{\tau}{2}). \tag{1.43}$$

On the basis of expressions (1.42) and (1.43) we make conclusion: if parameters σ_0 and σ_1 satisfy the following system

$$\frac{1}{4}\sigma_0 - \frac{1}{4}\sigma_1 + \frac{1}{2} = \frac{1}{2},$$
$$\frac{1}{2}\sigma_0 - \frac{1}{2}\sigma_1 + \frac{1}{2} = \frac{1}{2},$$

then difference $U(\frac{\tau}{2}) - V_0(\tau, \frac{\tau}{2})$ will be of the same order as $\mathrm{O}(\tau^2)$. Hence $\sigma_0 = \sigma_1$.

Thus, when $\sigma_0 = \sigma_1$, we have:

$$U(\frac{\tau}{2}) - V_0(\tau, \frac{\tau}{2}) = R_2(\tau) - R^{(2)}(\frac{\tau}{2}),$$

where $R_2(\tau)$ and $R^{(2)}(\frac{\tau}{2})$, respectively, are calculated by formulas (1.41) and (1.7). It is clear that inequality:

$$\|(V_0(\tau, \frac{\tau}{2}) - U(\frac{\tau}{2}))\varphi\| \leq \sum_{j=1}^{4} \|R_{2,j}(\tau)\varphi\| + \|R^{(2)}(\frac{\tau}{2})\varphi\|. \tag{1.44}$$

According to estimates (1.13) and (1.14) and condition (c) of Theorem 1.1 we have:

$$\|R_{2,1}(\tau)\varphi\| = \|(\sigma_0 U_1(\frac{\tau}{2})U_2(\tau) + \sigma_1 I)R_1^{(2)}(\frac{\tau}{4})\varphi\|$$

$$\leq ce^{\omega_1 \frac{\tau}{2}} e^{\omega_1 \tau} \int_0^{\frac{\tau}{4}} \int_0^{s_1} \|U_1(s)\| \|A_1^2 \varphi\| ds ds_1 \leq c\tau^2 e^{2\omega_1 \tau} \|A_0^2 \varphi\|, \ \varphi \in D(A^2). \tag{1.45}$$

Similarly the following estimates are obtained:

$$\|R_{2,j}(\tau)\varphi\| \leq c\tau^2 e^{2\omega_1 \tau} \|A_0^2 \varphi\|, \ \varphi \in D(A^3), \ j = 2, 3, 4, 5, \tag{1.46}$$

$$\|R^{(2)}(\frac{\tau}{2})\varphi\| \leq \|A^2 \int_0^{\frac{\tau}{2}} \int_0^{s_1} U(s)\varphi ds ds_1\| \leq c\tau^2 e^{\omega_0 \frac{\tau}{2}} \|A^2 \varphi\|$$

$$\leq c\tau^2 e^{\omega_0 \frac{\tau}{2}} \|A_0^2 \varphi\|, \quad \varphi \in D(A^2). \tag{1.47}$$

From (1.44), taking into account estimates (1.46) and (1.47), the following is obtained:

$$\|(V_0(\tau, \frac{\tau}{2}) - U(\frac{\tau}{2}))\varphi\| \leq c\tau^2 e^{\omega\tau} \|A_0^2 \varphi\|, \quad \varphi \in D(A^2). \tag{1.48}$$

From (1.36), taking into account estimates (1.19), (1.37), (1.39), (1.40), (1.48). and (1.14), estimate (1.3) is obtained. □

2. Difference analogue

To find approximate solution of problem (1.1) we apply the difference analogue of differential decomposition scheme (1.2):

$$\frac{u_k^{(1)}-u_{k-1}^{(1)}}{\tau}+\frac{1}{2}A_1\frac{u_k^{(1)}+u_{k-1}^{(1)}}{2}=\sigma_0 f(t_{k-\frac{1}{2}}),\ u_{k-1}^{(1)}=u_{k-1},\quad u_0=\varphi,$$
$$\frac{u_k^{(2)}-u_{k-1}^{(2)}}{\tau}+A_2\frac{u_k^{(2)}+u_{k-1}^{(2)}}{2}=(1-(\sigma_0+\sigma_1))f(t_{k-\frac{1}{2}})\quad u_{k-1}^{(2)}=u_k^{(1)},$$
$$\frac{u_k^{(3)}-u_{k-1}^{(3)}}{\tau}+\frac{1}{2}A_1\frac{u_k^{(3)}+u_{k-1}^{(3)}}{2}=\sigma_1 f(t_{k-\frac{1}{2}}),\qquad u_{k-1}^{(3)}=u_k^{(2)}. \tag{2.1}$$

We state as approximate solution at $t=t_k=k\tau$ point:

$$u_k=u_k^{(3)}.$$

Scheme (2.1) is the analogue of difference of (1.2) decomposition differential scheme. Our goal is to obtain explicit a priori estimate for scheme (2.1).

The following theorem takes place:

Theorem 2.1. *Let us assume that the conditions (a),(c),(d) of Theorem 1.1 and additionally the following condition are fulfilled:*

For any $\tau>0$, operators $I+\tau A_i,\ i=1,2$ is invertible and the following inequalities are valid:

$$\|(I-\tau A_i)(I+\tau A_i)^{-1}\|\le e^{\omega_1\tau},\quad \omega_1=const>0,$$
$$\|(I+\tau A_i)^{-1}\|\le ce^{\omega_1\tau},\quad c=const>0; \tag{2.2}$$

Then, if $\sigma_0=\sigma_1$, for error of the scheme (2.1) the following estimate will be valid:

$$\|u(t_k)-u_k\|\le c\tau^2\big[e^{\omega t_k}\big(t_k\|A_0^3\varphi\|+\int_0^{t_k}\|A_0f'(t)\|dt$$

$$+\tau\sum_{i=1}^{k}\big(\|A_0^2f(t_{i-\frac{1}{2}})\|+\|A_0f(t_{i-\frac{1}{2}})\|\big)+t_k\big)+\int_0^{t_k}(t_k-s)e^{\omega(t_k-s)}\|A_0^3f(s)\|ds\big], \tag{2.3}$$

where $\omega=max(\omega_0,2\omega_1)$, $\quad c=const>0$.

Proof. From (2.1) the following is obtained:

$$u_k^{(1)}=S_1(\frac{\tau}{2})u_{k-1}^{(1)}+\tau\sigma_0L_1(\frac{\tau}{2})f(t_{k-\frac{1}{2}}),$$

$$u_k^{(2)}=S_2(\tau)u_{k-1}^{(2)}+\tau(1-(\sigma_0+\sigma_1))L_2(\tau)f(t_{k-\frac{1}{2}}),$$

$$u_k^{(3)}=S_1(\frac{\tau}{2})u_{k-1}^{(3)}+\tau\sigma_1L_1(\frac{\tau}{2})f(t_{k-\frac{1}{2}}),$$

where

$$S_i(\tau) = (I - \frac{\tau}{2}A_i)(I + \frac{\tau}{2}A_i)^{-1}, \quad L_i(\tau) = (I + \frac{\tau}{2}A_i)^{-1}, \quad i = 1, 2.$$

Consequently:

$$u_k = V(\tau)u_{k-1} + \tau L(\tau) f(t_{k-\frac{1}{2}}), \tag{2.4}$$

where

$$V(\tau) = S_1(\frac{\tau}{2})S_2(\tau)S_1(\frac{\tau}{2})$$

and

$$L(\tau) = \sigma_0 S_1(\frac{\tau}{2})S_2(\tau)L_1(\frac{\tau}{2}) + (1 - (\sigma_0 + \sigma_1))S_1(\frac{\tau}{2})L_2(\tau) + \sigma_1 L_1(\frac{\tau}{2}).$$

From (2.4), through induction the following is obtained:

$$u_k = (V(\tau))^k \varphi + \tau \sum_{i=1}^{k} (V(\tau))^{k-i} L(\tau) f(t_{i-\frac{1}{2}}). \tag{2.5}$$

According to formulas (2.5) and (1.35) the following is received:

$$u(t_k) - u_k = [(U(\tau))^k - (V(\tau))^k]\varphi$$

$$+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (U(\tau))^{k-i} U(t_i - s)) f(s) ds - \sum_{i=1}^{k} (V(\tau))^{k-i} L(\tau) f(t_{i-\frac{1}{2}}) ds.$$

Let us write the right-hand side as:

$$u(t_k) - u_k = [(U(\tau))^k - (V(\tau))^k]\varphi + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} [(U(\tau))^{k-i} - (V(\tau))^{k-i}] U(t_i - s) f(s) ds$$

$$- \sum_{i=1}^{k} (V(\tau))^{k-i} \big[\tau(L(\tau) - U(\frac{\tau}{2})) f(t_{i-\frac{1}{2}})$$

$$+ (\tau U(\frac{\tau}{2}) f(t_{i-\frac{1}{2}}) - \int_{t_{i-1}}^{t_i} U(t_i - s) f(s) ds)\big]. \tag{2.6}$$

The following formula is valid:

$$[(U(\tau))^k - (V(\tau))^k]\varphi = \sum_{i=1}^{k} (V(\tau))^{k-i} [U(\tau) - V(\tau)] U(t_{i-1})\varphi. \tag{2.7}$$

According to formula (1.7), for $U(\tau)$ we have:

$$U(\tau) = I - \tau A + \frac{\tau^2}{2} A^2 + R^{(3)}(\tau), \tag{2.8}$$

where

$$R^{(3)}(\tau)=-A^3\int\limits_0^{\tau}\int\limits_0^{s_1}\int\limits_0^{s_2}U(s)dsds_2ds_1. \tag{2.9}$$

The following formulas are valid:

$$S_i(\tau)=I+R_i^{(1)}(\tau),\quad R_i^{(1)}(\tau)=-\frac{\tau}{2}A_i(I+S_i(\tau)), \tag{2.10}$$

$$S_i(\tau)=I-\tau A_i+R_i^{(2)}(\tau),\ \ R_i^{(2)}(\tau)=\frac{\tau^2}{4}A_i^2(I+S_i(\tau)), \tag{2.11}$$

$$S_i(\tau)=I-\tau A_i+\frac{\tau^2}{4}A_i^2+R_i^{(3)}(\tau),\ R_i^{(3)}(\tau)=-\frac{\tau^3}{8}A_i^3(I+S_i(\tau)). \tag{2.12}$$

By means of these formulas, the following is obtained:

$$\begin{aligned}
V(\tau)&=S_1(\frac{\tau}{2})S_2(\tau)S_1(\frac{\tau}{2})=S_1(\frac{\tau}{2})S_2(\tau)[I-\frac{\tau}{2}A_1+\frac{\tau^2}{8}A_1^2+R_1^{(3)}(\frac{\tau}{2})]\\
&=S_1(\frac{\tau}{2})[S_2(\tau)-\frac{\tau}{2}S_2(\tau)A_1+\frac{\tau^2}{8}S_2(\tau)A_1^2+S_2(\tau)R_1^{(3)}(\frac{\tau}{2})]\\
&=S_1(\frac{\tau}{2})[I-\tau A_2+\frac{\tau^2}{2}A_2^2+R_2^{(3)}(\tau)-\frac{\tau}{2}(I-\tau A_2+R_2^{(2)}(\tau))A_1\\
&\quad+\frac{\tau^2}{8}(I+R_2^{(1)}(\tau))A_1^2+S_2(\tau)R_1^{(3)}(\frac{\tau}{2})]\\
&=S_1(\frac{\tau}{2})[I-\frac{\tau}{2}(2A_2+A_1)+\frac{\tau^2}{8}(4A_2^2+4A_2A_1+A_1^2)\\
&\quad+R_2^{(3)}(\tau)-\frac{\tau}{2}R_2^{(2)}(\tau)A_1+\frac{\tau^2}{8}R_2^{(1)}(\tau))A_1^2+U_2(\tau)R_1^{(3)}(\frac{\tau}{2})]\\
&=S_1(\frac{\tau}{2})-\frac{\tau}{2}S_1(\frac{\tau}{2})(2A_2+A_1)+\frac{\tau^2}{8}S_1(\frac{\tau}{2})(4A_2^2+4A_2A_1+A_1^2)\\
&\quad+S_1(\frac{\tau}{2})(R_2^{(3)}(\tau)-\frac{\tau}{2}R_2^{(2)}(\tau)A_1+\frac{\tau^2}{8}R_2^{(1)}(\tau))A_1^2+S_2(\tau)R_1^{(3)}(\frac{\tau}{2}))\\
&=I-\frac{\tau}{2}A_1+\frac{\tau^2}{8}A_1^2+R_1^{(3)}(\frac{\tau}{2})-\frac{\tau}{2}(I-\frac{\tau}{2}A_1+R_1^{(2)}(\frac{\tau}{2}))(2A_2+A_1)\\
&\quad+\frac{\tau^2}{8}(I+R_1^{(1)}(\frac{\tau}{2})(4A_2^2+4A_2A_1+A_1^2)\\
&\quad+S_1(\frac{\tau}{2})(R_2^{(3)}(\tau)-\frac{\tau}{2}R_2^{(2)}(\tau)A_1+\frac{\tau^2}{8}R_2^{(1)}(\tau))A_1^2+U_2(\tau)R_1^{(3)}(\frac{\tau}{2}))\\
&=I-\tau(A_1+A_2)+\frac{\tau^2}{2}(A_1^2+A_1A_2+A_2A_1+A_2^2)+R_3(\tau),
\end{aligned}$$

where

$$R_3(\tau)=\sum_{j=1}^{5}R_{3,j}(\tau), \tag{2.13}$$

$$R_{3,1}(\tau) = -\frac{\tau^3}{8}(3I - S_1(\frac{\tau}{2})S_2(\tau))(I + S_1(\frac{\tau}{2})A_1^3,$$

$$R_{3,2}(\tau) = -\frac{\tau^3}{4}(I + S_1(\frac{\tau}{2}))(A_1^2A_2 + A_1A_2^2),$$

$$R_{3,3}(\tau) = -\frac{\tau^3}{16}S_1(\frac{\tau}{2})(I + S_2(\tau))(2A_2^2A_1 + A_2A_1^2),$$

$$R_{3,4}(\tau) = -\frac{\tau^3}{4}(S_1(\frac{\tau}{2}))A_1A_2A_1, \quad R_{3,5}(\tau) = -\frac{\tau^3}{8}S_1(\frac{\tau}{2})(I + S_2(\tau))A_2^3.$$

As $A = A_1 + A_2$ and

$$A^2 = A_1^2 + A_1A_2 + A_2A_1 + A_2^2,$$

therefore $V(\tau)$ will be expressed as:

$$V(\tau) = I - \tau A + \frac{\tau^2}{2}A^2 + R_3(\tau). \tag{2.14}$$

According to formulas (2.8) and (2.14) we have:

$$U(\tau) - V(\tau) = R^{(3)}(\tau) - R_3(\tau). \tag{2.15}$$

where $R^{(3)}(\tau)$ and $R_3(\tau)$, respectively, are calculated by formulas (2.9) and (2.13).

According to the conditions (a) of the Theorem 1.1 and (2.2), we have:

$$\|U(\tau)\| \le Me^{\omega_0\tau}, \tag{2.16}$$

$$\|V(\tau)\| = \|S_1(\frac{\tau}{2})S_2(\tau)S_1(\frac{\tau}{2})\| \le e^{\omega_1\frac{\tau}{2}}e^{\omega_1\tau}e^{\omega_1\frac{\tau}{2}} \le e^{2\omega_1\tau}. \tag{2.17}$$

From (2.7), taking into account (2.15), (2.16), (2.17), and the cinditions (c) of the Theorem 1.1, we obtain:

$$\|[(U(\tau))^k - (V(\tau))^k]\varphi\| \le c\tau^2 t_k e^{\omega t_k}\|A_0^3\varphi\|. \tag{2.18}$$

Let us evaluate difference $L(\tau) - U(\frac{\tau}{2})$.

The following formula is valid:

$$L_i(\tau) = I - \frac{\tau}{2}A_i + \frac{\tau^2}{4}A_i^2L_i(\tau),$$

According to this formula and formulas (2.10) - (2.12), we have:

$$L(\tau) = \sigma_0S_1(\frac{\tau}{2})S_2(\tau)L_1(\frac{\tau}{2}) + (1 - (\sigma_0 + \sigma_1))S_1(\frac{\tau}{2})L_2(\tau) + \sigma_1L_1(\frac{\tau}{2})$$

$$= \sigma_0S_1(\frac{\tau}{2})S_2(\tau)\big(I - \frac{\tau}{4}A_1 + \frac{\tau^2}{16}L_1(\frac{\tau}{2}){A_1}^2\big) + (1 - (\sigma_0 + \sigma_1))S_1(\frac{\tau}{2})L_2(\tau) + \sigma_1L_1(\frac{\tau}{2})$$

$$= \sigma_0S_1(\frac{\tau}{2})\big(S_2(\tau) - \frac{\tau}{4}S_2(\tau)A_1 + \frac{\tau^2}{16}S_2(\tau)L_1(\frac{\tau}{2}){A_1}^2\big)$$

$$+(1-(\sigma_0+\sigma_1))S_1(\frac{\tau}{2})L_2(\tau)+\sigma_1 L_1(\frac{\tau}{2})$$

$$=\sigma_0 S_1(\frac{\tau}{2})\big[I-\tau A_2+R_2^{(2)}(\tau)-\frac{\tau}{4}(I+R_2^{(1)}(\tau))A_1+\frac{\tau^2}{16}S_2(\tau)L_1(\frac{\tau}{2}){A_1}^2\big]$$

$$+(1-(\sigma_0+\sigma_1))S_1(\frac{\tau}{2})\big(I-\frac{\tau}{2}A_2+\frac{\tau^2}{4}L_2(\tau)A_2^2\big)+\sigma_1 L_1(\frac{\tau}{2})$$

$$=\sigma_0 S_1(\frac{\tau}{2})\big[S_1(\frac{\tau}{2})-\tau S_1(\frac{\tau}{2})A_2+S_1(\frac{\tau}{2})R_2^{(2)}(\tau)-\frac{\tau}{4}S_1(\frac{\tau}{2})(I+R_2^{(1)}(\tau))A_1$$

$$+\frac{\tau^2}{16}S_1(\frac{\tau}{2})S_2(\tau)L_1(\frac{\tau}{2}){A_1}^2\big]$$

$$+(1-(\sigma_0+\sigma_1))\big(S_1(\frac{\tau}{2})-\frac{\tau}{2}S_1(\frac{\tau}{2})A_2+\frac{\tau^2}{4}S_1(\frac{\tau}{2})L_2(\tau)A_2^2\big)+\sigma_1 L_1(\frac{\tau}{2})$$

$$=\sigma_0\big[I-\frac{\tau}{2}A_1+R_1^{(2)}(\frac{\tau}{2})-\tau(I+R_1^{(1)}(\frac{\tau}{2}))A_2-\frac{\tau}{4}(I+R_1^{(2)}(\frac{\tau}{2}))A_1$$

$$-\frac{\tau}{4}S_1(\frac{\tau}{2})(I+R_2^{(1)}(\tau))A_1+S_1(\frac{\tau}{2})R_2^{(2)}(\tau)\frac{\tau^2}{16}S_1(\frac{\tau}{2})S_2(\tau)L_1(\frac{\tau}{2}){A_1}^2\big]$$

$$+(1-(\sigma_0+\sigma_1))\big(I-\frac{\tau}{2}A_1+R_1^{(2)}(\frac{\tau}{2})-\frac{\tau}{2}(I+R_1^{(1)}(\frac{\tau}{2}))A_2+\frac{\tau^2}{4}S_1(\frac{\tau}{2})L_2(\tau)A_2^2\big)$$

$$+\sigma_1\big(I-\frac{\tau}{4}A_1+\frac{\tau^2}{16}L_1(\frac{\tau}{2})A_1^2\big)$$

$$=I-\tau\big(\frac{1}{4}\sigma_0-\frac{1}{4}\sigma_1+\frac{1}{2}\big)A_1-\tau\big(\frac{1}{2}\sigma_0-\frac{1}{2}\sigma_1+\frac{1}{2}\big)A_2+R_2(\tau),$$

where

$$R_2(\tau)=\sum_{j=1}^{4}R_{2,j}(\tau), \tag{2.19}$$

$$R_{2,1}(\tau)=\frac{\tau^2}{16}\big(3I+3S_1(\frac{\tau}{2})+S_1(\frac{\tau}{2})S_2(\tau)L_1(\frac{\tau}{2})+L_1(\frac{\tau}{2})\big)A_1^2,$$

$$R_{2,2}(\tau)=\frac{\tau^2}{4}S_1(\frac{\tau}{2})\big(I+S_1(\frac{\tau}{2})+L_2(\tau)\big)A_2^2,$$

$$R_{2,3}=\frac{3\tau^2}{8}(I+S_1(\frac{\tau}{2}))A_1A_2,\quad R_{2,4}=\frac{\tau^2}{8}S_1(\frac{\tau}{2})(I+S_2(\tau))A_2A_1.$$

Thus we obtained:

$$L(\tau)=I-\tau\big(\frac{1}{4}\sigma_0-\frac{1}{4}\sigma_1+\frac{1}{2}\big)A_1-\tau\big(\frac{1}{2}\sigma_0-\frac{1}{2}\sigma_1+\frac{1}{2}\big)A_2+R_2(\tau). \tag{2.20}$$

In accordance with formula (1.7), for $U(\frac{\tau}{2})$ we have:

$$U(\frac{\tau}{2})=I-\frac{\tau}{2}(A_1+A_2)+R^{(2)}(\frac{\tau}{2}), \tag{2.21}$$

$$R^{(2)}(\frac{\tau}{2})=A^2\int_0^{\frac{\tau}{2}}\int_0^{t}U(s)dsdt. \tag{2.22}$$

On the basis of expressions (2.20) and (2.21) we conclude: if parameters σ_0 and σ_1 satisfy the following system

$$\frac{1}{4}\sigma_0 - \frac{1}{4}\sigma_1 + \frac{1}{2} = \frac{1}{2},$$
$$\frac{1}{2}\sigma_0 - \frac{1}{2}\sigma_1 + \frac{1}{2} = \frac{1}{2},$$

then difference $L(\tau) - U(\frac{\tau}{2})$ will be of the same order as $\mathrm{O}(\tau^2)$. Hence $\sigma_0 = \sigma_1$.

Thus, when $\sigma_0 = \sigma_1$, we have:

$$L(\tau) - U(\frac{\tau}{2}) = R_2(\tau) - R^{(2)}(\frac{\tau}{2}),$$

where $R_2(\tau)$ and $R^{(2)}(\frac{\tau}{2})$, respectively, are calculated by formulas (2.19) and (2.22). It is obvious that inequality:

$$\|(L(\tau) - U(\frac{\tau}{2}))\varphi\| \leq \sum_{j=1}^{4} \|R_{2,j}(\tau)\varphi\| + \|R^{(2)}(\frac{\tau}{2})\varphi\|. \qquad (2.23)$$

From (2.23), taking into account evaluations (1.45) - (1.47), the following is obtained:

$$\|(L(\tau) - U(\frac{\tau}{2}))\varphi\| \leq c\tau^2 e^{\omega\tau} \|A_0^2\varphi\|, \quad \varphi \in D(A^2). \qquad (2.24)$$

From (2.6), taking into account evaluations (1.39), (1.13), (2.17), (2.18) and (2.24) evaluation (2.3) is obtained. □

The estimates, given in the Theorems 1.1 and 1.2, were made by the author has earlier and were published in the paper [6].

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