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ON THE BEHAVIOR OF SOLUTIONS OF ONE 4 -TH ORDER DIFFERENTIAL EQUATION ELLIPTIC TYPE

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Abstract

An a priori energy estimate analogous to the inequalities expressing St. Venant's principle in elasticity theory is obtained for the solution of one 4-th order differential equation elliptic type with the conditions of the first boundary-value problem in an n-dimensional domain. These estimates are used to study the behavior of the solution and its derivatives near irregular boundary points and as a consequence of the geometric properties of the boundary in a neighborhood of these points.

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In this article we consider the solution of one 4-th order differential equation elliptic type with Dirichlet boundary conditions in an n-dimensional domain. We obtain a priori estimates analogous to the energy inequalities expressing St. Venant's principle in elasticity theory (see, for example, [1], [4]). On the basis of these estimates we investigate the behavior of solutions of one 4-th order differential equation and their derivatives near irregular boundary points.

A priori estimates analogous to the energy inequalities expressing St. Venant's principle in elasticity theory of one 4-th order differential equation enable us in the case when coefficients dependent only one argument to prove theorems for the generalized solution of the Dirichlet problem in the plane domain.

Inequalities of St. Venant type differ from the usual energy estimates in that they estimate the Dirichlet integral over a domain $\Omega_1 \subset \Omega$ in terms of the Dirichlet integral over a larger domain $\Omega_2 \supset \Omega_1$, with a coefficient depending on the distance ρ from Ω_1 to the boundary of Ω_2, which does not belong to $\partial\Omega$. Here it is assumed that the Dirichlet boundary conditions on $\partial\Omega \cap \partial\Omega_2$ are homogeneous. The nature of the decrease in this coefficient with increasing ρ depends on the geometric properties of Ω_2. The coefficient can be determined by solving a certain ordinary differential equation or differential inequality – with special initial data.

In this article we consider some properties the solution of one 4-th order differential equation elliptic type

$$\Delta[a(x)\Delta u(x)] + b^{\alpha\beta}(x)u_{,\alpha\beta}(x) = f(x) \tag{1}$$

in the a domain Ω which lies in $\mathbb{R}^n_+ = \{(x_1, \dots, x_n),\ x_1 \geq 0\}$ with boundary conditions

$$u\big|_{\partial\Omega} = \varphi, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \varphi_2, \tag{2}$$

where Δ is the Laplas operator, Greek indeces α and β on the value 1 and n and summation over the repeated indices from 1 to n is assumed ν is the direction of external normal to the boundary $\partial\Omega$ and

$$u_{,\alpha} \equiv \frac{\partial u}{\partial x_\alpha}, \quad u_{,\alpha\beta} \equiv \frac{\partial^2 u}{\partial x_\alpha x_\beta}, \quad \alpha, \beta = 1, \ldots, n.$$

When an equation is studied for two arguments this equation is 4-th order elliptic differential equation, which is "Equation of now-homogeneous elastic plane body" – [1] and we have studied this equation in the works [2], [3] and [4].

But in the work [3] it is studied, when coefficient of equation depends on the only one argument.

The present paper covers the case, when the number of arguments is more then two and when coefficients of equation (1) depends only on argument x_1 ($a_{,i} \equiv \frac{\partial a}{\partial x_i} = 0$ and $b^{\alpha\beta}_{,i} \equiv \frac{\partial b^{\alpha\beta}}{\partial x_i} = 0$, $i = \overline{2,n}$).

Definition. Let Ω be a bounded domain in $\mathbb{R}^n_+$. A function $u(x)$ is called a generalized solution of equation (1) in Ω with $H_m(\Omega, j)$ and satisfies the integral identity:

$$\int\limits_\Omega \Big[a(x)u_{,\alpha\beta}v_{,\alpha\beta} + \sum_{j=2}^{n} a_{,11}u_{,jj}v\,dx + b^{\alpha\beta}u_{,\alpha\beta}v\Big]\,dx = \int\limits_\Omega f(x)v(x)\,dx \qquad (3)$$

for any function $v \in H_m(\Omega, \partial\Omega)$; here $f \in L_2(\Omega)$.

Let functions $\mu(t)$, M_i $(i = 0, 1)$ and $W(u)$ such that:

$$0 \le \mu(t) \le \inf_{u\in\Gamma} \Big\{ \int\limits_{S_t} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx_2 \Big| \int\limits_{S_t} a(x)B_2(u)\,dx_2 \Big|^{-1} \Big\}, \qquad (4)$$

$$0 \le M_0(t) \le \inf_{u\in\Gamma} \Big\{ \int\limits_{S_t} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx_2 \Big| \int\limits_{S_t} a(x)u^2\,dx_2 \Big|^{-1} \Big\}, \qquad (5)$$

$$0 \le M_1(t) \le \inf_{u\in\Gamma} \Big\{ \int\limits_{S_t} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx_2 \Big| \int\limits_{S_t} a(x)u_{,\alpha}u_{,\alpha}\,dx_2 \Big|^{-1} \Big\}, \qquad (6)$$

$$W(u) \equiv a(x)u_{,\alpha\beta}\cdot u_{,\alpha\beta}, \qquad (7)$$

where $S_t = \Omega \cap \{x : \ x_1 = t\}$ and consists of finitely many bounded domains whose boundaries belong to $\partial\Omega$; Γ is the set of 2 times continuously differentiable functions in a neighborhood of $\overline{S}_t$ such that $u(x) = u_{,1}(x) = u_{,2}(x) = 0$ on $\overline{S}_t \cap \partial\Omega$; $B_2(u) \equiv u_{,\alpha}u_{,\alpha} - u_{,11}u$ at all $i = \overline{0,n}$.

When coefficients are dependents on only from argument x_1, that why satisfied conditions correspondingly:

$$\begin{gathered} a(x) > 0, \quad |b^{\alpha\beta}(x)| \le \eta_0(x_1)a(x), \\ |a_{,1}(x)| \le \eta_1(x_1)a(x), \quad |a_{,11}(x)| \le \eta_2(x_1)a(x), \end{gathered} \qquad (8)$$

where $\eta_0(t)$, $\eta_1(t)$, $\eta_2(t)$, $t\in[0,T]$ arbitrary functions with conditions:

$$|\eta_0(t)|\le \min\left\{\frac{1}{4},\frac{M_0(t)}{32}\right\},$$

$$|\eta_1(t)|\le \min\left\{\frac{1}{4},\frac{M_1(t)}{16}\right\},\quad |\eta_2(t)|\le \min\left\{\frac{M_1(t)}{16},\frac{M_0(t)}{16}\right\}.$$

Theorem 1. *Let a bounded domain Ω be studied in the half-space $\mathbb{R}^n_+$. The set S_t is nonempty for all $t\in(0,T]$, $T=\text{const}>0$, and coefficients $a(x)$ and $b^{\alpha\beta}(x)$ satisfies conditions* (8). *$f(x)=0$ on $\Omega\cap\Omega_T$. Then, for the generalized solution $u(x)$ of equation* (1) *in the domain Ω_T, with the boundary conditions $u=\partial u/\partial\nu=0$ on $\partial\Omega\cap\partial\Omega_T$ (if it exists), the following estimates are valid:*

$$\int\limits_{\Omega_T} a(x)u^2(x)M_0(x_1)\Phi(x_1,T,\varepsilon)\,dx$$
$$\le k_1\int\limits_{\Omega_T} a(x)u_{,\alpha\beta}\cdot u_{,\alpha\beta}\Phi(x_1,T,\varepsilon)\,dx\le \frac{1}{\varepsilon}\int\limits_{\Omega_T} W(u)\,dx, \tag{9}$$

$$\int\limits_{\Omega_T} u_{,\alpha}u_{,\alpha}M_1(x_1)\Phi(x_1,T,\varepsilon)\,dx$$
$$\le k_2\int\limits_{\Omega_T} a(x)u_{,\alpha\beta}\cdot u_{,\alpha\beta}\Phi(x_1,T,\varepsilon)\,dx, \tag{10}$$

where $\varepsilon=const\in(0,1/2)$, k_1 and k_2 are positive constants, which depends from geometrical structure of domain Ω_T and coefficients of equation (1).

The function $\Phi(x_1,T,\varepsilon)$ is a solution of the following Cauchy problem:

$$2|\Phi_{,11}(x_1,T,\varepsilon)|=(1-\varepsilon)\mu(x_1)\Phi(x_1,T,\varepsilon) \tag{11}$$

for $0\le x_1\le T$ with the initial conditions:

$$\Phi(T,T,\varepsilon)=1,\quad \Phi_{,1}(T,T,\varepsilon)\equiv\frac{d}{dx_1}\Phi(T,T,0)=0. \tag{12}$$

Proof. Let us construct a function $\psi(x_1,\delta)$, assuming that

$$\psi(x_1,\delta)\equiv\begin{cases}\Phi(x_1,T,\varepsilon) & \text{for}\quad 0<\delta\le x_1\le T,\\ (x_1-\delta)\Phi_{x_1}(\delta,T,\varepsilon)+\Phi(\delta,T,\varepsilon) & \text{for}\quad 0\le x_1\le\delta.\end{cases}$$

It is easy to see that: $v(x)\equiv u(x)(\psi(x_1,\delta)-1)\in H_m(\Omega_T,\partial\Omega_T)$. Substituting the function $v(x)=u(\psi-1)$ into the integral identity (3) for Ω_T, we obtain

$$\int\limits_{\Omega_T}\Big[a(x)u_{,\alpha\beta}u_{,\alpha\beta}(\psi-1)a(x)u_{,\alpha\beta}u_{,\alpha}\psi_{,\beta}$$

$$+a(x)u_{,\alpha\beta}u_{,\beta}\psi_{,\alpha}+a(x)u_{,\alpha\beta}u\psi_{,\alpha\beta}$$

$$+\sum_{j=2}^{n}a_{,11}u_{,jj}u(\psi-1)+b^{\alpha\beta}(x)u_{,\alpha\beta}(x)u(\psi-1)\Big]\,dx=0.$$

In the derivation of the last equality we have used integration by parts, which can easily be justified if we approximate $u(x)$ by functions of class $C^2(\overline{\Omega}_T)$ equal to zero in the neighborhood of $\partial\Omega\cap\partial\Omega_T$, and use the fact that $\psi_{,\alpha}=0$ if $\alpha\neq 1$ for $x_1=T$. Taking into account that ψ is independent of x_j, $j=\overline{2,n}$, we find that:

$$\int\limits_{\Omega_T}[a(x)u_{,\alpha\beta}u_{,\alpha\beta}(\psi-1)]\,dx$$

$$=\int\limits_{\Omega_T}a_{,1}u_{,\beta}u_{,\beta}\Phi_{,1}\,dx-\int\limits_{\Omega_T}b^{\alpha\beta}u_{,\alpha\beta}u(\psi-1)\,dx$$

$$+\int\limits_{\Omega_T\setminus\Omega_\delta}(a(x)u_{,\alpha}u_{,\alpha}-a(x)u_{,11}u)\psi_{,11}\,dx$$

$$-\int\limits_{\Omega_T}a_{,11}u_{,jj}u(\psi-1)\,dx. \tag{13}$$

We put $p(u)\equiv a(x)u_{,\alpha}u_{,\alpha}-a(x)u_{,11}u$.

Let u_n be a sequence of functions twice continuously differentiable in $\overline{\Omega}_T$, which are equal to zero in the neighborhood of the set $\partial\Omega\cap\partial\Omega_T$, converging to $u(x)$ in the norm as $n\to\infty$. It is easy to see that

$$\int\limits_{\Omega_T\setminus\Omega_\delta}p(u)\psi_{,11}\,dx=\int\limits_{\Omega_T\setminus\Omega_\delta}p(u_n)\psi_{,11}\,dx+\varepsilon_n,$$

where $\varepsilon_n\to 0$ as $n\to\infty$. From the definition (4) of the function $\mu(t)$ and the equation for $\Phi(x_1,,T,\varepsilon)$, it follows that

$$\Big|\int\limits_{\Omega_T\setminus\Omega_\delta}p(u_n)\Phi_{,11}\,dx\Big|\leq\int\limits_{0}^{T}\Phi_{,11}(x_1,T,\varepsilon)\Big|\int\limits_{S_t}p(u_n)\,dx_2\Big|\,dx_1$$

$$\leq\int\limits_{\delta}^{T}\frac{\Phi_{,11}(x_1,T,\varepsilon)}{\mu(x_1)}\Big|\int\limits_{S_t}W(u_n)\,dx_2\Big|\,dx_1$$

$$=(1-\varepsilon)\int\limits_{\Omega_T\setminus\Omega_\delta}W(u_n)\Phi(x_1)\,dx.$$

Letting n in this inequality go to ∞, we obtain

$$\Big|\int\limits_{\Omega_T\setminus\Omega_\delta} p(u)\Phi_{,11}\,dx\Big| \le (1-\varepsilon)\int\limits_{\Omega_T\setminus\Omega_\delta} W(u)\Phi(x_1,T,\varepsilon)\,dx. \tag{14}$$

From this and (13) we conclude that

$$\int\limits_{\Omega_T} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\psi(x_1,\delta)\,dx$$

$$\le \int\limits_{\Omega_T} W(u)\,dx + (1-\varepsilon)\int\limits_{\Omega_T\setminus\Omega_\delta} W(u)\Phi(x_1,T,\varepsilon)\,dx$$

$$+\frac{1}{4}\int\limits_{\Omega_T} W(u)(\psi-1)\,dx + \frac{1}{4}\int\limits_{\Omega_T} a(x)u^2(\psi-1)\,dx.$$

Letting $\delta\to 0$

$$k_1\int W(u)\Phi(x_1,T,\varepsilon)\,dx \le \int W(u)\,dx.$$

The remaining inequalities (9), (10) for the functions u_n follow immediately from the definitions of M_0 and M_1. Further, passing to the limit as $n\to\infty$, we obtain the desired inequalities for u. The Theorem 1 is proved.

Theorem 2. (*Analogue of Saint–Venant's principle*) *Under the conditions of Theorem* 1 *when coefficients* $a(x)$ *and* $b^{\alpha\beta}(x)$ *satisfies conditions* (8) *for any* $0<t_0\le t_1\le T$

$$\begin{aligned}&\int\limits_{\Omega_{t_0}} a(x)(1-k(x_1))u_{,\alpha\beta}u_{,\alpha\beta}\,dx\\ &\le \frac{1}{\Phi(t_0,t_1)}\int\limits_{\Omega_{t_1}} a(x)(1-k(x_1))u_{,\alpha\beta}u_{,\alpha\beta}\,dx,\end{aligned} \tag{15}$$

where function $\Phi(x_1,t_1)$ *satisfies for* $t_0\le x_1\le t_1$, *the ordinary differential equation*

$$\begin{aligned}2|\Phi_{,11}(x_1,t_1)| &= (1-k(x_1))\mu(x_1)\Phi(x_1,t_1),\\ \Phi_{,1}(x_1,t_1)&\le 0,\quad \Phi(x_1,t_1)\ge 1\end{aligned} \tag{16}$$

and conditions:

$$\Phi(t_1,t_1)=1,\quad \Phi_{,1}(t_1,t_1)=0. \tag{17}$$

Now we proved Theorem 3 and Theorem 4, when $\Omega_T\subset\mathbb{R}^2_+$.

Theorem 3. *Under the conditions of Theorem* 1, *when coefficients* $a(x)$ *and* $b^{\alpha\beta}(x)$ *satisfies conditions* (8) *hold estimate:*

$$\begin{aligned}&a(x)u^2(x)M_0^{1/4}(x_1)M_1^{1/2}(x_1)\Phi(x_1,T,\varepsilon)\\ &\le (3(1+\eta_1)\varepsilon^{-1}+\eta\varepsilon^{-1})\int\limits_{\Omega_T} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx,\end{aligned} \tag{18}$$

where the functions Φ*,* M_0 *and* M_1 *are defined in Theorem* 1*.*

Moreover, it is assumed that M_0 *and* M_1 *are nonincreasing functions continuously differentiable for* $0 < x_1 \leq T$*.*

Proof. Since, by definition, $u(x)$ belongs to $H_2(\Omega, j)$, where $j = \partial\Omega \cap \partial\Omega_T$ there exists a sequence of functions u_n such that $u_n \to u$ as $n \to \infty$ and $u_n = 0$ in a neighborhood of j. We define the functions u_n outside the set Ω_T by assining the value zero. We define functions Φ_δ, $M_{0\delta}$ and $M_{1\delta}$ in such a way that $\Phi_\delta = \Phi$, $M_{0\delta} = M_0$ and $M_{1\delta} = M_1$ for $x_1 > \delta$; Φ_δ, $M_{0\delta}$ and $M_{1\delta}$ are bounded, monotone and continuously differentiable with respect to x_1 for $0 < x_1 \leq T$, and $\Phi_\delta \leq \Phi$, $M_{0\delta} \leq M_0$ and $M_{1\delta} \leq M_1$. We estimate

$$a(x)u_n^2(x)\Phi_\delta(x_1, T, \varepsilon)M_{0\delta}^{1/4}(x_1)M_{1\delta}^{1/2}(x_1) \equiv a(x)u_n^2(x)\varphi_\delta(x_1).$$

We note that for a certain $\sigma = \sigma(n)$ the function $u_n(x)$ is equal to zero in Ω_σ. Hence we may write

$$a(x)u_n^2(x)\varphi_\delta(x_1) = \int_0^{x_1} \frac{\partial}{\partial x_1}(a(x)u_n^2(x)\varphi_\delta(x_1))\,dx_1$$

$$= \int_0^{x_2} 2u_n u_{n1} a(x)\varphi_\delta(x_1)\,dx_1 + \int_0^{x_1} u_n^2 a(x)\varphi_{\delta,1}\,dx_1 + \int_0^{x_1} a_1(x)u_n^2\varphi_\delta\,dx_1.$$

Since $\varphi_{\delta,1} \leq 0$,

$$a(x)u_n^2(x)\varphi_\delta(x_1)$$

$$\leq \int_0^T (u_{n,1})^2 a(x)M_{1\delta}^{1/2}\Phi_\delta\,dx_1 + \int_0^T u_n^2 a(x)M_{1\delta}^{1/2}M_{0\delta}^{1/2}\Phi_\delta\,dx_1$$

$$+\eta_1 \int_0^T a(x)u_n^2\varphi_\delta\,dx_1.$$

It is easy to see that:

$$\int_0^T (u_{n,1})^2 a(x)M_{1\delta}^{1/2}\Phi_\delta\,dx_1$$

$$\leq \int_{\Omega_T} (u_{n,1})^2 a(x)M_{1\delta}\Phi_\delta\,dx + \int_{\Omega_T} a(x)(u_{n,12})\Phi_\delta\,dx.$$

Analogously we have

$$\int_0^T u_n^2 a(x)M_{1\delta}^{1/2}M_{0\delta}^{1/2}\Phi_\delta\,dx_1$$

$$\leq \int_{\Omega_T} a(x)u_n^2\Phi_\delta M_{0\delta}\,dx + \int_{\Omega_T} a(x)(u_{n,2})^2\Phi_\delta M_{1\delta}\,dx.$$

From these inequalities it follows that

$$a(x)u_n^2(x)\varphi_\delta(x_1)$$

$$\leq \int\limits_{\Omega_T} a(x)(u_{n,12})^2\Phi_\delta\,dx + \int\limits_{\Omega_T} a(x)(u_{n,1})^2\Phi_\delta M_{1\delta}\,dx$$

$$+\int\limits_{\Omega_T} a(x)u_n^2\Phi_\delta M_{1\delta}\,dx + \int\limits_{\Omega_T} a(x)(u_{n,2})^2\Phi_\delta M_{1\delta}\,dx$$

$$+\eta_1\int\limits_{\Omega_T} a(x)(u_{n,2})^2\Phi_\delta M_{1\delta}\,dx + \eta_1\int\limits_{\Omega_T} a(x)u_n^2\Phi_\delta M_{1\delta}\,dx$$

$$+\eta_1\int\limits_{\Omega_T} a(x)u_n^2\Phi_\delta\,dx,$$

where

$$\eta_1\int\limits_0^T a(x)u_n^2\varphi_\delta\,dx_1 \leq \eta_1\int\limits_0^T a(x)(u_{n,2})^2\Phi_\delta M_{1\delta}\,dx$$

$$+\eta_1\int\limits_{\Omega_T} a(x)u_n^2\Phi_\delta\,dx + \eta_1\int\limits_{\Omega_T} a(x)u_n^2\varphi_\delta M_{1\delta}\,dx.$$

Thus, for $x \in \Omega_T$,

$$a(x)u_n^2(x)\varphi_\delta(x_1)$$

$$\leq \int\limits_{\Omega_T} W(u_n)\Phi_\delta(x)\,dx + (1+\eta_1)\int\limits_{\Omega_T} a(x)(u_{n,\alpha})^2\Phi_\delta M_{1\delta}\,dx$$

$$+(1+\eta_1)\int\limits_{\Omega_T} a(x)u_n^2\Phi_\delta M_{1\delta}\,dx + \eta_1\int\limits_{\Omega_T} a(x)u_n^2\Phi_\delta\,dx.$$

Using the definition of $M_0(x_1)$ and $M_1(x_1)$, we obtain

$$a(x)u_n^2(x)\varphi_\delta(x_1)$$

$$\leq 3(1+\eta_1)\int\limits_{\Omega_T} W(u_n)\Phi(x_1)\,dx + \int\limits_{\Omega_T} \Phi_\delta(x)W(u_n)\,dx$$

$$+\int\limits_{\Omega_T} a(x)(u_{,\alpha})^2\Phi_\delta(x_1)(1+\eta_1)M_{1\delta}\,dx + (1+\eta_1)\int\limits_{\Omega_T} a(x)u^2\Phi_\delta M_{1\delta}\,dx$$

$$+\eta_1\int\limits_{\Omega_T} a(x)u^2\Phi(x_1)\,dx.$$

We pass to the limit as $n \to \infty$. For any fixed $x \in \Omega_T$ and $\delta < x_1$, we find that:

$$\int\limits_{\Omega_\delta} a(x)\Phi_\delta W(u)\,dx + \int\limits_{\Omega_\delta} a(x)(u_{,\alpha})^2\Phi_\delta(x_1)(1+\eta_1)M_{1\delta}\,dx$$

$$+(1+\eta_1)\int\limits_{\Omega_\delta} a(x)u^2\Phi_\delta M_{1\delta}\,dx$$

$$\le \int\limits_{\Omega_\delta} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\Phi\,dx + (1+\eta_1)\int\limits_{\Omega_\delta} a(x)(u_{,\alpha})^2\Phi M_1\,dx$$

$$+(1+\eta_1)\int\limits_{\Omega_\delta} a(x)u^2\Phi(x_1)M_0(x_1)\,dx.$$

By virtue of (9), the right-hand side of this inequality tends to 0 as $\delta \to 0$. We have

$$a(x)u^2(x)\varphi(x_1)$$

$$\le 3(1+\eta_1)\varepsilon^{-1}\int\limits_{\Omega_T} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx + \eta_1\int\limits_{\Omega_T} a(x)u^2\Phi(x_1)\,dx$$

$$\le (3(1+\eta_1)\varepsilon^{-1} + \eta\varepsilon^{-1})\int\limits_{\Omega_T} W(u)\,dx,$$

where $\eta \equiv \eta_1 \cdot 1/\{\max M_0\}$.

The theorem is proved.

Theorem 4. *Let a bounded domain Ω be situated in the helfs-plane $\mathbb{R}^2_+$. The set S_t is nonempty for all $t \in [0,T]$, $T = \text{const} > 0$, and coefficients $a(x)$ and $b^{\alpha\beta}(x)$ satisfies conditions (8). Let $u(x)$ be the generalized solution of equation (1) in Ω_T, with boundary conditions $u = \partial u/\partial\nu = 0$ on $\partial\Omega \cap \partial\Omega_T$ and $f = 0$ in Ω_T. Then for any $0 < t_0 < t_1 < T$*

$$a(x)\max_{\Omega_{t_0}}|u|^2 \le \frac{p(t_0)}{\Phi(t_0,t_1)}\int\limits_{\Omega_{t_1}} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx,$$

where function $\Phi(x_1,t_1)$ satisfies, for $t_0 \le x_1 \le t_1$, the ordinary differential equation

$$\Phi_{,11}(x_1,t_1) - \mu(x_1)\Phi(x_1,t_1) = 0$$

and the initial conditions:

$$\Phi(t_1,t_1) = 1, \quad \Phi_{,1}(t_1,t_1) = 0$$

and

$$p(t_0) \equiv \Big(1+(1+\eta_1)\sup_{0<x_1\le t_0}(M_0(x_1))^{-1} + (1+\eta_1)\sup_{0<x_1\le t_0}(M_1(x_1))^{-1}\Big).$$

Proof. Let u_n be a sequence of functions twice continuously differentiable in $\overline{\Omega}_T$, which are equal to zero in the neighborhood of the set $\partial\Omega \cap \partial\Omega_T$. We extend $u_n(x)$ by zero outside Ω_{t_1}. For $x \in \Omega_{t_0}$ we have

$$\begin{aligned}
a(x)u_n^2(x) &= \int_0^{x_1} 2a(x)u_n u_{n1}\, dx_1 + \int_0^{x_1} a_1 u_n^2\, dx \\
&\leq \int_0^{t_1} a(x)u_n^2\, dx_1 + \int_0^{t_1} a(x)u_{n,1}^2\, dx + \eta_1 \int_0^{t_1} a(x)u_n^2\, dx_1, \\
a(x)(u_{n,1})^2 &\leq \int_{-\infty}^{\infty} a(x)u_{n,1}\, dx_2 + \int_{-\infty}^{\infty} a(x)u_{n,12}\, dx_2. \qquad (19)
\end{aligned}$$

Hence ($a_{,2} = 0$),

$$\int_0^{t_0} a(x)(u_{n,1})^2\, dx_1 \leq \int_{\Omega_{t_0}} a(x)(u_{n,1})^2\, dx + \int_{\Omega_{t_0}} a(x)(u_{n,12})^2\, dx.$$

In exactly the same way we obtain:

$$\int_0^{t_0} a(x)(u_n)^2\, dx_1 \leq \int_{\Omega_{t_0}} a(x)(u_n)^2\, dx + \int_{\Omega_{t_0}} a(x)(u_{n,2})^2\, dx.$$

Therefore, from (19) we conclude that

$$\begin{aligned}
a(x)u_n^2(x) &\leq \int_{\Omega_{t_0}} a(x)(u_n)^2\, dx + \int_{\Omega_{t_0}} a(x)(u_{n,2})^2\, dx \\
&+ \int_{\Omega_{t_0}} a(x)(u_{n,1})^2\, dx + \int_{\Omega_{t_0}} a(x)(u_{n,12})^2\, dx \\
&+ \eta_1 \int_{\Omega_{t_0}} a(x)(u_n)^2\, dx + \eta_1 \int_{\Omega_{t_0}} a(x)(u_{n,2})^2\, dx.
\end{aligned}$$

From the definitions (5) and (6) of the functions $M_0(x_1)$ and $M_1(x_1)$ we have

$$a(x)u_n^2(x)$$

$$\leq \sup_{0<x_1\leq t_0} (M_0(x_1))^{-1} \Big[\int_{\Omega_{t_0}} a(x)M_0(x_1)u_n^2(x)\, dx + \eta_1 \int_{\Omega_{t_0}} a(x)M_0(x_1)u_n^2\, dx \Big]$$

$$+ \sup_{0<x_1\leq t_0} (M_1(x_1))^{-1} \Big[\eta_1 \int_{\Omega_{t_0}} a(x)M_1(x_1)(u_{n,2})^2(x)\, dx$$

$$+\int\limits_{\Omega_{t_0}} a(x)M_1(x_1)(u_{n,\alpha})^2\,dx\Big] + \int\limits_{\Omega_{t_0}} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx$$

$$\leq \int\limits_{\Omega_{t_0}} W(u_n)\Big[1+\sup(M_0(x_1))^{-1}(1+\eta_1)+\sup(M_1(x_1))^{-1}(1+\eta_1)\Big]\,dx$$

$$\equiv p(t_0)\int\limits_{\Omega_{t_0}} W(u_n)\,dx.$$

Passing to the limit as $n \to \infty$, we obtain

$$a(x)u^2(x) \leq p(t_0)\int\limits_{\Omega_{t_0}} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx.$$

The theorem is proved.

Now we conside unbounded domain.

Definition. Let Ω be an unbounded domain in $\mathbb{R}^n_+$. A function $u(x)$ is called a generalized solution of equation (1) in Ω with boundary conditions $j = \partial\Omega \cap \partial\Omega_T^+$, $T = const > 0$, $\forall\, t > T$ $u(x) \in H_m(\Omega(t,T))$, $\partial\Omega(t,T) \cap \partial\Omega$ and satisfies the integral identity (3) for an function $v \in H_m(\Omega(t,T), \partial\Omega(t,T))$.

Theorem 5. *Let $u(x)$ be the generalized solution of equation* (1), (2) *in* Ω, *with boundary conditions* $j = \partial\Omega \cap \partial\Omega_T^+$, $T = \text{const} > 0$, $u(x) \in H_m(\Omega_T^+, \partial\Omega_T^+ \cap \partial\Omega)$ *and* $\dfrac{\partial a(x)}{\partial x_1} \leq 0$. *Then, for any* $\forall\, t > T$

$$\int\limits_{\Omega_T}^{+} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx \leq \frac{k^*}{\psi(x,T)}\int\limits_{\Omega_T^+} W(u)\,dx,$$

where function ψ is a solution of the following Cauchy problem.

Proof. Let $\Phi(x) \in C^2[T,\infty)$, $\Phi(T) = 0$, $\Phi_1(T) = 0$, $\Phi(x_1) \geq 0$ ($\Phi(x_1) = ax + b$, $a, b = const$, when $x_1 > T + dl$). It is easy to see that:

$$v(x) \equiv u(x)\Phi(x_1) \in H_m(\Omega(t,T), \partial\Omega(t,T))$$

into the integral identity (3) for Ω_T^+, we obtain:

$$\int\limits_{\Omega_T^+} \Big[a(x)u_{,\alpha\beta}u_{,\alpha\beta}\Phi(x_1) + a(x)u_{,\alpha\beta}u_{,\alpha}\Phi_{,\beta} + a(x)u_{,\alpha\beta}u_{,\beta}\Phi_{,\alpha}$$

$$+a(x)u_{,\alpha\beta}u\Phi_{,\alpha\beta} + b^{\alpha\beta}u_{,\alpha\beta}u\Phi + \sum a_{,ii}u_{,jj}\Phi u - 2\sum a_{,ij}u_{,ij}u\Phi\Big]\,dx$$

$$-\int\limits_{\Omega_T^+} a_1(x)u_{,\alpha}u_{,\alpha}\Phi_{,1}\,dx = 0.$$

In the derivation of the last equality we have used integration by parts, we find that:

$$\int\limits_{\Omega_T^+}\Big[a(x)u_{,\alpha\beta}u_{,\alpha\beta}\Phi(x_1)+\sum a_{,ii}u_{,jj}\Phi u-2\sum a_{,ij}u_{,ij}u\Phi+b^{\alpha\beta}u_{,\alpha\beta}u\Phi\Big]\,dx$$

$$-\int\limits_{\Omega(T,T+\delta)}(a(x)u_{,\alpha}u_{,\alpha}-a(x)u_{,11}u)\Phi_{,11}\,dx$$

$$-\int\limits_{\Omega_T^+}a_1(x)u_{,\alpha}u_{,\alpha}\Phi_{,1}\,dx$$

or

$$\int\limits_{\Omega_T^+}a(x)u_{,\alpha\beta}u_{,\alpha\beta}\Phi(x_1)\,dx\le\int\limits_{\Omega(T,T+\delta)}p(u)\Phi_{,11}\,dx$$

$$-\int\limits_{\Omega_T^+}\Big(b^{\alpha\beta}u_{,\alpha\beta}u\Phi+\sum a_{,ii}u_{,jj}u\Phi-2\sum a_{,ij}u_{,ij}u\Phi\Big]\,dx.$$

The function's make more precise

$$\Phi^\delta\equiv\begin{cases}\Phi(x_1) & \text{for}\;\; \mathrm{T}<\mathrm{x}_1<\delta\\ a\sin(x_1-\delta)+a\delta+b & \text{for}\;\; \delta<\mathrm{x}_1<\delta+\pi/2\\ a(1+\delta)x_1+b & \text{for}\;\; \delta+\pi/2<\mathrm{x}_1<\infty\end{cases}$$

and following $v(x)=u(x)\Phi^\delta(x_1)$. We have

$$\int\limits_{\Omega_T^+}a(x)u_{,\alpha\beta}u_{,\alpha\beta}\Phi^\delta(x_1)\,dx$$

$$\le\int\limits_{\Omega(T,T+\delta)}p(u)\Phi_{,11}\,dx+\int\limits_{\Omega(\delta,\delta+\frac{\pi}{2})}p(u)\Phi_{,11}\,dx$$

$$-\int\limits_{\Omega_T^+}\Big[b^{\alpha\beta}u_{,\alpha\beta}u\Phi+\sum a_{,ii}u_{,jj}u\Phi-2\sum a_{,ij}u_{,ij}u\Phi\Big]\,dx,$$

where

$$\int\limits_{\Omega_T^+}\Big(b^{\alpha\beta}u_{,\alpha\beta}u\Phi+\sum a_{,ii}u_{,jj}u\Phi-2\sum a_{,ij}u_{,ij}u\Phi\Big)\,dx$$

$$\le\frac{1}{2}\int\limits_{\Omega_T}\eta_0 a(x)u^2\Phi(x_1)\,dx$$

$$+\frac{1}{2}\int\limits_{\Omega_T}\eta_0 u_{,\alpha\beta}u_{,\alpha\beta}a(x)\Phi(x_1)\,dx+\int\limits_{\Omega_T}\sum a(x)\eta_{ij}u_{ij}^2\Phi\,dx$$

$$+\int_{\Omega_T}\Big(\sum a(x)\eta_{ij}u^2\Phi+\frac{1}{2}\sum a(x)\eta_{ii}u_{jj}\Phi+\frac{1}{2}\sum \eta_{ii}a(x)u^2\Phi\Big)\,dx$$

$$\leq \int_{\Omega_T^+}\Big(\frac{1}{2}\,\eta_0+(n-1)^2\eta_{\max}\Big)a(x)u^2\Phi\,dx$$

$$+\int_{\Omega_T^+}\Big(\frac{1}{2}\,\eta_0+\eta_{\max}\Big)a(x)u_{,\alpha\beta}u_{,\alpha\beta}\Phi\,dx.$$

Now, let $\delta=t$ and

$$\Phi(x_1)=\begin{cases}\psi(x_1,T)-1 & \text{for } \mathrm{T}\leq \mathrm{x}_1\leq \mathrm{t}\\ \psi_{,1}(t,T)(x_1-t)+\psi(t_1,T)-1 & \text{for } \mathrm{t}<\mathrm{x}_1<\infty\end{cases}$$

$$\int_{\Omega_T^+}a(x)u_{,\alpha\beta}u_{,\alpha\beta}\Phi(x_1)\,dx\leq \int_{\Omega(T,t)}p(u)\psi_{,11}(x_1,T)\,dx$$

$$+\int_{\Omega(T,t)}\Big(\frac{1}{2}\,\eta_0+(n-1)^2\eta_{\max}\Big)a(x)u^2(\psi-1)\,dx$$

$$-\int_{\Omega_t^+}\Big(\frac{1}{2}\,\eta_0+(n-1)^2\eta_{\max}\Big)a(x)u^2\,dx$$

$$+\int_{\Omega_t^+}\Big(\frac{1}{2}\,\eta_0+(n-1)^2\eta_{\max}\Big)a(x)u^2(\Phi(x_1)+1)\,dx$$

$$+\int_{\Omega(T,t)}\Big(\frac{1}{2}\,\eta_0+\eta_{\max}\Big)a(x)u_{,\alpha\beta}u_{,\alpha\beta}\Phi\,dx$$

$$-\int_{\Omega_t^+}\Big(\frac{1}{2}\,\eta_0+\eta_{\max}\Big)a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx$$

$$+\int_{\Omega_t^+}\Big(\frac{1}{2}\,\eta_0+\eta_{\max}\Big)a(x)u_{,\alpha\beta}u_{,\alpha\beta}(\Phi+1)\,dx.$$

Let u_n be a sequence of functions twice continuously differentiable in $\Omega(T,t)$, which are equal to zero in the neighborhood of the set $\partial\Omega(T,t)\cap\partial\Omega$, converging to $u(x)$ in the norm. It is easy to see that

$$\int_{\Omega(T,t)}p(u)\psi_{,11}(x_1,T)\,dx=\int_{\Omega(T,t)}p(u_n)\psi_{,11}(x_1,T)\,dx+\varepsilon_n$$

from (14), we have

$$\Big| \int\limits_{\Omega(T,t)} p(u)\psi_{,11}(x_1,T)\,dx \Big| \le \int\limits_{\Omega(T,t)} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\psi(x_1,T)\,dx$$

the following estimates are valid:

$$\int\limits_{\Omega_t^+} W(u)\,dx \le \frac{k^*}{\psi(t,T)} \int\limits_{\Omega_t^+} W(u)\,dx.$$

The theorem is proved.

Remark 1. Under the conditions of Theorem 5, when $\dfrac{\partial a}{\partial x_1} \le 0$ holds estimates:

$$\begin{aligned}
\int\limits_{\Omega_t^+} a(x)\Phi(x_1,T,\varepsilon)a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx &\le \varepsilon^{-1}\int\limits_{\Omega_t^+} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx,\\
\int\limits_{\Omega_t^+} a(x)\Phi(x_1,T,\varepsilon)M_0(x_1)u^2\,dx &\le \varepsilon^{-1}\int\limits_{\Omega_t^+} a(x)u_{,\alpha\beta}u_{,\alpha\beta}\,dx,\\
\int\limits_{\Omega_t^+} a(x)u_{,\alpha}u_{,\alpha}\Phi(x_1,T,\varepsilon)M_1\,dx &\le \varepsilon^{-1}\int\limits_{\Omega_t^+} W(u),dx,
\end{aligned}$$

where the function Φ is defined in Theorem 1.

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