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ON A ONE INVERSE PROBLEM OF THE BENDING OF A PLATE WITH VARIABLE FLEXURAL-RIGIDITY

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Abstract

The paper deals with the inverse problem of the cylindrical problem of the cusped plate with variable flexural-rigidity in case of a strip.

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In 1955 I. Vekua raised the problem of investigation of cusped plates (see [1], [2]), i.e., such ones whose thickness on the part of the plate boundary or on the whole one vanishes. Problems connected with the investigation of cusped plate have received much attention in the mathematical and engineering scientific literature (see brief survey, e.g., in [3], [4]). The methods developed for studying such kind of plates can be used for the plate with variable flexural-rigidity in case of smooth changeable thickness.

The paper deals with the inverse problem of the cylindrical problem of the plate with variable flexural-rigidity in case of an infinite strip.

Let us consider the plate with variable flexural-rigidity whose projection on the complex plane $z = x + iy$ is an infinite strip as follows

$$\Pi = \{-\infty < x < \infty; \;\; 0 \le y \le 1\},$$

and let the edges of the plate are freely supported.

We have to determine flexural-rigidity of the plate if the deflection of the middle plane is given by the following expression

$$w(x, y) = \varepsilon y(1 - y),$$

where ε is given positive constant.

It is a well known that bending equation of the plate with variable flexural-rigidity in case of smooth changeable thickness can be written as follows

$$\frac{\partial^2 M_1(x,y)}{\partial x^2} + 2\frac{\partial^2 M_{12}(x,y)}{\partial x \partial y} + \frac{\partial^2 M_2(x,y)}{\partial y^2} = -q(x) \tag{1}$$

where

$$M_1(x,y) = -D(x,y)\left[\frac{\partial^2 w(x,y)}{\partial x^2} + \sigma\frac{\partial^2 w(x,y)}{\partial y^2}\right],$$
$$M_2(x,y) = -D(x,y)\left[\frac{\partial^2 w(x,y)}{\partial y^2} + \sigma\frac{\partial^2 w(x,y)}{\partial x^2}\right], \quad (2)$$
$$M_{12}(x,y) = (1-\sigma)D(x,y)\frac{\partial^2 w(x,y)}{\partial x \partial y},$$

M_1 and M_2 are bending moments, M_{12} is twisting moment, $D(x,y) = \dfrac{2Eh^3(x,y)}{3(1-\sigma^2)}$ is a flexural-rigidity of the plate, σ is a Poison's ratio.

After substituting (2) into (1) we obtain the following differential equation

$$D\Delta\Delta w + 2\frac{\partial D}{\partial x}\frac{\partial \Delta w}{\partial x} + 2\frac{\partial D}{\partial y}\frac{\partial \Delta w}{\partial y}$$
$$+\Delta D\Delta w - (1-\sigma)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - -2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = q, \quad (3)$$

where Δ is a Laplace operator. We will solve equation (3) under following boundary conditions

$$w(x,0) = w(x,1) = 0, \qquad M_2(x,0) = M_2(x,1) = 0. \quad (4)$$

Taking into account of the form $w(x,y)$, from the equation (3) for $D(x,y)$ we get the elliptic type differential equation as follows

$$\frac{\partial^2 D(x,y)}{\partial y^2} + \sigma\frac{\partial^2 D(x,y)}{\partial x^2} = -\frac{q(x)}{2\varepsilon} \quad (5)$$

under following boundary conditions

$$D(x,0) = D(x,1) = 0 \quad (6)$$

After consideration the following functions

$$D^*(x,y) = D(\sqrt{\sigma}\,x, 1), \quad g''(x) = q(x), \quad \Phi(x,y) = D^*(x,y) + \frac{1}{2\varepsilon}g(x), \quad (7)$$

for $\Phi(x,y)$ we get Dirichlet Problem in the strip Π

$$\Delta\Phi(x,y) = 0; \quad \Phi(x,0) = \Phi(x,1) = \frac{1}{2\varepsilon}g(x). \quad (8)$$

Let $g(x) \in L$, using Fourier transform with respect to the variable x from (7) we obtain

$$\frac{d^2\hat{\Phi}(t,y)}{dy^2} - t^2\hat{\Phi}(t,y) = 0, \quad \hat{\Phi}(t,0) = \hat{\Phi}(t,1) = \frac{1}{2\varepsilon}\hat{g}(x) \quad (9)$$

where

$$\hat{\Phi}(t,y) = F_x[\Phi(x,y)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Phi(x,y)e^{-ixt}dx, \quad \hat{g}(t) = F_x[g(x)].$$

The solution of the problem (9) has the following form

$$\hat{\Phi}(t,y)=\frac{1}{2\varepsilon}\frac{ch\frac{t\beta}{2}}{ch\frac{t}{2}}\hat{g}(t) \tag{10}$$

where $\beta=2y-1$.

In virtue of inverse Fourier transform, after using generalize Parsevals' formulae, from (9) and (10) we get that $D^*(x,y)$ has the form as follows

$$D^*(x,y)=-\frac{1}{2\varepsilon}g(x)+\frac{1}{2\varepsilon\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}g(t)M(x-t)dt, \tag{11}$$

where

$$M(\xi)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{ch\frac{t\beta}{2}e^{i\xi t}}{ch\frac{t}{2}}dt. \tag{12}$$

For the calculation of the last integral we use theory of deductions. We will consider integral on the following rectangle

$$\Sigma=\{-R\le t\le R;\quad 0\le\tau\le 2\pi i\}.$$

After consideration integrals

$$I_c^{(1)}=\int\limits_{-\infty}^{+\infty}\frac{ch\frac{t\beta}{2}e^{i\xi t}}{ch\frac{t}{2}}dt,$$

$$I_s^{(1)}=\int\limits_{-\infty}^{+\infty}\frac{sh\frac{t\beta}{2}e^{i\xi t}}{ch\frac{t}{2}}dt,$$

by view of theory of deductions, we obtain following system

$$\begin{gathered}\left(1+e^{-2\pi\xi}ch\pi i\beta\right)I_c^{(1)}+sh\pi i\beta\cdot e^{-2\pi\xi}I_s^{(1)}=4\pi ch\frac{\pi i\beta}{2}\cdot e^{-\pi\xi},\\ sh\pi i\beta\cdot e^{-2\pi\xi}I_c^{(1)}+\left(1+ch\pi i\beta\cdot e^{-2\pi\xi}\right)I_c^{(1)}=4\pi sh\frac{\pi i\beta}{2}\cdot e^{-\pi\xi}.\end{gathered}$$

From the last system we can calculate $I_c^{(1)}$, which has a form

$$I_c^{(1)}=\frac{4\pi\cos\frac{\pi\beta}{2}\,ch\pi\xi}{ch2\pi\xi+\cos\pi\beta}.$$

Then from (12) we get

$$M(\xi)=2\sqrt{2\pi}\frac{\cos\frac{\pi\beta}{2}\,ch\pi\xi}{ch2\pi\xi+\cos\pi\beta}.$$

Finally, after substituting the last expression into (11), using expressions $D(x, y) = D^*\left(\frac{x}{\sqrt{\sigma}, y}\right)$, $\beta = 2y - 1$, we obtain

$$D(x,y) = -\frac{1}{2\varepsilon} g\left(\frac{x}{\sqrt{\sigma}}\right) + \frac{\sqrt{2\pi}\sin\pi y}{\varepsilon}\int\limits_{-\infty}^{+\infty} g(t)\frac{ch\pi\left(\frac{x}{\sqrt{\sigma}} - t\right)}{ch2\pi\left(\frac{x}{\sqrt{\sigma}} - t\right) - \cos 2\pi y}\,dt. \quad (13)$$

In virtue of method of deductions we get

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} M(\xi) = 1.$$

Using Weierstrass theorem for $\rho > 0$ we have

$$\lim_{\rho\to\infty}\rho[g(t) * M(\rho t)] = \lim_{\rho\to\infty}\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} g\left(\frac{x}{\sqrt{\sigma}} - \frac{t}{\rho}\right) M(t)dt = g\left(\frac{x}{\sqrt{\sigma}}\right). \quad (14)$$

After consideration the limits of (13) when $y \to 0, 1$, by view of the form (14), in case of $\rho = \frac{1}{y}$ and $\rho = \frac{1}{1-y}$, we get

$$\lim_{y\to 0} D(x,y) = \lim_{y\to 1} D(x,y) = 0.$$

So, $D(x, y)$ satisfies boundary conditions (6).

So, for the flexural-rigidity we have got formulae (13), but in practice sometimes it is more advantage to use formula (10). Let us consider the following example for the illustration of the last sentence.

Let $q(x)$ is given by the expression

$$q(x) = \pi^2\sqrt{2\pi}\frac{ch^2\pi x - 2}{ch^3\pi x}. \quad (15)$$

Then we will have

$$g(x) = \frac{\sqrt{2\pi}}{ch\pi x}, \quad \hat{g}(x) = \frac{1}{ch\frac{t}{2}}.$$

Finally, from the formula (11) we get

$$D^*(x,y) = -\frac{\sqrt{2\pi}}{2\varepsilon}\frac{1}{ch\pi x} + \frac{1}{2\sqrt{2\pi}\varepsilon} Re\int\limits_{-\infty}^{\infty}\frac{ch\frac{t\beta}{2}e^{ixt}}{ch^2\frac{t}{2}}dt. \quad (16)$$

Let us consider the following integrals

$$I_c^0 = \int\limits_{-\infty}^{\infty}\frac{ch\frac{t\beta}{2}e^{ixt}}{ch^2\frac{t}{2}}dt, \quad I_s^0 = \int\limits_{-\infty}^{\infty}\frac{sh\frac{t\beta}{2}e^{ixt}}{ch^2\frac{t}{2}}dt.$$

It can be shown that I_c^0 and I_s^0 satisfy the following system

$$I_c^0(1-\cos\pi\beta e^{-2\pi x})-iI_s^0\sin\pi\beta e^{-2\pi x}=4\pi\beta\sin\frac{\pi\beta}{2}e^{-\pi x}+8\pi x\cos\frac{\pi\beta}{2}e^{-\pi x}-8\pi i e^{-\pi x}\cos\frac{\pi\beta}{2},$$

$$\begin{aligned}I_c^0(-i\sin\pi\beta e^{-2\pi x}) \quad &+ \quad I_s^0(1-\cos\pi\beta e^{-2\pi x})\\ &= \quad 8\pi e^{-\pi x}\sin\frac{\pi\beta}{2}-i\left[4\pi\beta\cos\frac{\pi\beta}{2}e^{-\pi x}-8\pi x e^{-\pi x}\sin\frac{\pi\beta}{2}\right],\end{aligned}$$

from this system we get

$$Re\int\limits_{-\infty}^{\infty}\frac{ch\frac{\beta t}{2}e^{ixt}}{ch^2\frac{t}{2}}dt=ReI_c^0=\frac{4\pi(\beta\sin\frac{\pi\beta}{2}\cdot ch\pi x+2xcos\frac{\pi\beta}{2}sh\pi x)}{ch2\pi x-\cos\pi\beta}.$$

By virtue of (16), after substituting in the last expression the following value $\beta=2y-1$, we obtain

$$D^*(x,y)=\frac{\sqrt{2\pi}}{\varepsilon}\frac{2x\sin\pi y sh2\pi x+\sin^2\pi y-2ch^2\pi x(y\cos\pi y+\sin^2\frac{\pi y}{2})}{ch\pi x(ch2\pi x+\cos 2\pi y)}.$$

Finally, for the flexural-rigidity we have formula as follows

$$D(x,y)=\frac{\sqrt{2\pi}}{\varepsilon}\frac{\frac{2x}{\sqrt{\sigma}}\sin\pi y sh\frac{2\pi x}{\sqrt{\sigma}}+\sin^2\pi y-2ch^2\frac{\pi x}{\sqrt{\sigma}}(y\cos\pi y+\sin^2\frac{\pi y}{2})}{ch\frac{\pi x}{\sqrt{\sigma}}(ch\frac{2\pi x}{\sqrt{\sigma}}+\cos 2\pi y)}.$$

From the last formulae it is easy to show that $D(x,y)$ is an even function. Furthermore,

$$\lim_{y\to0}\frac{D(x,y)}{y}=\lim_{y\to1}\frac{D(x,y)}{1-y}=\frac{\sqrt{2\pi}}{\varepsilon}\frac{\frac{2\pi x}{\sqrt{\sigma}}sh\frac{2\pi x}{\sqrt{\sigma}}-2ch^2\frac{\pi x}{\sqrt{\sigma}}}{(ch\frac{2\pi x}{\sqrt{\sigma}}+1)ch\frac{\pi x}{\sqrt{\sigma}}}=\frac{2\sqrt{2\pi}}{\varepsilon}\frac{\frac{\pi x}{\sqrt{\sigma}}sh\frac{\pi x}{\sqrt{\sigma}}-ch\frac{\pi x}{\sqrt{\sigma}}}{ch\frac{2\pi x}{\sqrt{\sigma}}+1}$$

$$\lim_{x\to\infty}D(x,y)=\lim_{x\to-\infty}D(x,y)=0,\quad D(0,\frac{1}{2})=\frac{2-\pi}{2\pi}\frac{2\sqrt{2\pi}}{\varepsilon}.$$

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