Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 30, 2004

ON A SYSTEM OF SECOND ORDER DEGENERATE ELLIPTIC EQUATIONS

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Received: 20.10.2004 revised: 17.11.2004

Abstract

The bending of a prismatic cusped shell described by the zero approximation of I.Vekua's version of the theory of elastic prismatic shells is considered. Mathematically it leads to a dirichlet type boundary value problem for a strongly elliptic system of differential equations with order degeneration on the boundery. The existence and uniqueness of generalised solutions of the coresponding boundary value problems in the weighted Sobolev spaces are proved.

Key words and phrases: Elliptic systems with order degeneration, weighted Sobolev space, bending of prismatic cusped shells.

AMS subject classification: primary 35j70, seqondary 74b05.

1. Introduction

Consider the following system

$$\begin{cases} -\dfrac{\partial}{\partial x}(\mu h\dfrac{\partial u}{\partial x}) - \dfrac{\partial}{\partial y}(\mu h\dfrac{\partial u}{\partial y}) = f(x,y), \\ -\dfrac{\partial}{\partial x}\left((\lambda+2\mu)h\dfrac{\partial u_1}{\partial x}\right) - \dfrac{\partial}{\partial y}(\mu h\dfrac{\partial u_1}{\partial y}) - \dfrac{\partial}{\partial x}(\lambda h\dfrac{\partial u_2}{\partial y}) - \dfrac{\partial}{\partial y}(\mu h\dfrac{\partial u_2}{\partial x}) = f_1(x,y), \\ \dfrac{\partial}{\partial x}(\lambda h\dfrac{\partial u_2}{\partial x}) - \dfrac{\partial}{\partial y}\left((\lambda+2\mu)h\dfrac{\partial u_2}{\partial y}\right) - \dfrac{\partial}{\partial x}(\mu h\dfrac{\partial u_1}{\partial y}) - \dfrac{\partial}{\partial y}\left(\lambda h\dfrac{\partial u_1}{\partial x}\right) = f_2(x,y), \end{cases} \quad (1)$$

where u, u_1, u_2 are unknown functions and are called the moments of the displacement vector $\lambda > 0$ and $\mu > 0$ are Lame's constants,f, f_1, f_2 are the Fourier-Legendre moments of a given volume forse,h is the thickness of the plate.

In this paper we study the case when the middle surface ω is a plane bounded domain with a smooth boundary $\partial\omega \in c^1$, where

$$\partial\omega = \Gamma_0 \bigcup \Gamma_1,$$

$$\Gamma_0 = \{(x,0) : a \le x \le b \quad a,b \in \mathbb{R}, a \in \mathbb{R}, a < b\},$$

$$\Gamma_1 = \{(x,y) : (x,y) \in \partial\omega \quad y \le 0\},$$

$$\Gamma_0 \bigcap \Gamma_1 = \{(a,0);(b,0)\}.$$

Let the thickness h be given by the function

$$h = h(x, y) = y^m \quad m > 0.$$

The system (1) is strongly elliptic on $\omega \backslash \Gamma_0$ with order degeneration on Γ_0.

2. Auxiliary material. Let $D(\omega)$ be a set of infinitely differentiable compactly supported functions on ω. We define a scalar product and a norm on $D(\omega)$ according to the formulas:

$$(u, v)_m \equiv \int\limits_{\omega} y^m [\nabla u \nabla v + uv] d\tau, \tag{2}$$

$$\|u\|_m \equiv \left(\int\limits_{\omega} y^m [\nabla u \nabla u + u^2] d\tau \right)^{\frac{1}{2}}, \tag{3}$$

where $u, v \in D(\omega)$.

We complete $D(\omega)$ by the norm (3) to obtain the Hilbert space $\overset{0}{H}_m(\omega)$, where a scalar product and a norm are defined by formulas (2) and (3).

Lemma 1. *For every $u \in \overset{0}{H}_m(\omega)$ there hold the inequality:*

$$\int\limits_{\omega} y^{m-2} u^2 d\tau \leq c \int\limits_{\omega} y^m \left(\frac{\partial u}{\partial y} \right)^2 d\tau \quad \text{for} \quad m \neq 1, \tag{4}$$

$$\int\limits_{\omega} y^{-1} \mid \ln ky \mid^{-2-\varepsilon} u^2 d\tau \leq c \int\limits_{\omega} y \left(\frac{\partial u}{\partial y} \right)^2 d\tau \quad \text{for} \quad m = 1, \tag{5}$$

where $k > 0$ is some positive constant such that $ky < 1$, for all $(x, y) \in \overline{\omega}$, and $c > 0$ is a constant independent of u.

Proof. Since ω is a bounded domain, there exists $d = const > 0$ such that $\omega \subset [a; b] \times [0, d] \equiv E$. Let $u \in D(\omega)$. Clearly, by extension (and preserving the notation) we can assume that $u \in D(E)$. First we consider the case $m \neq 1$:

$$\int\limits_0^d y^{m-2} u^2 dy = \int\limits_0^d \frac{1}{m-1} u^2 dy^{m-1} = \frac{1}{m-1} y^{m-1} u^2 \mid_0^d$$

$$-\frac{1}{m-1} \int\limits_0^d y^{m-1} 2u \frac{\partial u}{\partial y} dy = -\frac{2}{m-1} \int\limits_0^d y^{m-1} u \frac{\partial u}{\partial y} dy$$

$$= -\frac{2}{m-1} \int\limits_0^d \frac{1}{3} y^{\frac{m-2}{2}} u 3 y^{\frac{m}{2}} \frac{\partial u}{\partial y} dy \leq (2ab \leq a^2 + b^2)$$

$$\leq \frac{1}{9}\int_0^d y^{m-2}u^2dy + \frac{9}{(m-1)^2}\int_0^d y^m\left(\frac{\partial u}{\partial y}\right)^2 dy.$$

Hence,

$$\frac{8}{9}\int_0^d y^{m-2}u^2dy \leq \frac{9}{(m-1)^2}\int_0^d y^m\left(\frac{\partial u}{\partial y}\right)^2 dy,$$

i.e.,

$$\int_0^d y^{m-2}u^2dy \leq \frac{81}{8(m-1)^2}\int_0^d y^m\left(\frac{\partial u}{\partial y}\right)^2 dy.$$

If we integrate the last inequality by x over the interval $[a,b]$, we obtain

$$\int_a^b\int_0^d y^{m-2}u^2dydx \leq \frac{81}{8(m-1)^2}\int_a^b\int_0^d y^m\left(\frac{\partial u}{\partial y}\right)^2 dydx.$$

Since $u(x,y)=0$ for $(x,y)\in E\backslash\omega$ we have

$$\int_\omega y^{m-2}u^2d\tau \leq c\int_\omega y^m\left(\frac{\partial u}{\partial y}\right)^2 d\tau. \qquad (6)$$

Now, let $u\in \overset{0}{H}_m(\omega)$ and $\{u_n\}_{n=1}^\infty$ with $u_n\in D(\omega)$, be a sequence such that

$$\lim_{n\to\infty}\|u-u_n\|_m = 0.$$

According to (6) the sequence $\left\{y^{\frac{m-2}{2}}u_n\right\}_{n=1}^\infty \subset D(\omega)$ converges in $L_2(\omega)$. Due to the completness of $L_2(\omega)$ there exists a function $v\in L_2(\omega)$ such that

$$\lim_{n\to\infty}\int_\omega\left(y^{\frac{m-2}{2}}u_n - v\right)^2 d\tau = \lim_{n\to\infty}\int_\omega y^{m-2}\left(u_n - y^{\frac{2-m}{2}}v\right)^2 d\tau = 0.$$

According to (3) it follows that

$$u = y^{\frac{2-m}{2}}v \quad \text{on} \quad \omega,$$

i.e.,

$$\lim_{n\to\infty}\int_\omega y^{m-2}(u_n-u)^2d\tau = 0.$$

If we replace u by u_n in (6) and pass to the limit as $n\to\infty$ we obtain that the inequality (6) is true for every function $u\in \overset{0}{H}_m(\omega)$, $m\neq 1$.

Now we consider the case $m = 1$. Let $u \in D(\omega)$. We have

$$u^2(x,y) = \left(\int_y^d \frac{\partial u}{\partial t}(x,t)dt \right)^2 = \left(\int_y^d t^{-\frac{1}{2}} t^{\frac{1}{2}} \frac{\partial u}{\partial t} dt \right)^2$$

$$\leq \int_y^d t^{-1} dt \int_y^d t \left(\frac{\partial u}{\partial t} \right)^2 dt = (\ln d - \ln y) \int_0^d t \left(\frac{\partial u}{\partial t} \right)^2 dt$$

$$\leq n \mid \ln y \mid \int_0^d t \left(\frac{\partial u}{\partial t} \right)^2 dt.$$

If we multiply the both sides of the last inequality by $y^{-1} \mid \ln ky \mid^{-2-\varepsilon}$ with k as in the lemma and $\varepsilon > 0$, and integrate it first by y and afrerwards by x, we obtain

$$\int_0^d y^{-1} \mid \ln ky \mid^{-2-\varepsilon} u^2(x,y)dy \leq n_1 \int_0^d y^{-1} \mid \ln ky \mid^{-1-\varepsilon} dy \int_0^d t \left(\frac{\partial u}{\partial t} \right)^2 dt$$

$$\leq n_2 \int_0^d y \left(\frac{\partial u}{\partial y} \right)^2 dy.$$

Since

$$\int_0^d y^{-1} \mid \ln ky \mid^{-1-\varepsilon} dy < \infty,$$

$$\int_a^b dx \int_0^d y^{-1} \mid \ln ky \mid^{-2-\varepsilon} u^2(x,y)dy = \int_\omega y^{-1} \mid \ln ky \mid^{-2-\varepsilon} u^2(x,y)d\tau$$

$$\leq c \int_\omega y \left(\frac{\partial u}{\partial y} \right)^2 d\tau.$$

If we repeat the above reasonings we obtain that the inequality (5) is true for every function $u \in \overset{0}{H}_1(\omega)$.

Lemma 1 represent two–dimensional version of Hardy's inequality (see, e.g., [1,7]). ∎

Now we consider the question of a trace of functions from the space $\overset{0}{H}_m(\omega)$.

Lemma 2. *The trace of a function $u \in \overset{0}{H}_m(\omega)$, $0 < m < 1$, on $\partial\omega$ is zero.*

Proof. We introduce the distance function on ω

$$\rho(x,y) = \text{dist}\{(x,y),\ \partial\omega\}.$$

By completing $D(\omega)$ with the norm

$$\|u\|_{\rho^m} = \left(\int\limits_{\omega} \rho^m(x,y)[\nabla u \nabla u + u^2]d\tau\right)^{\frac{1}{2}} \tag{7}$$

we obtain the Hilbert space $\overset{0}{W}{}^1_{2,\frac{m}{2}}(\omega)$ whose properties are well known (see [11,12,14,18]). In particular, the trace of every function $u \in \overset{0}{W}{}^1_{2,\frac{m}{2}}(\omega)$, $-1 < m < 1$, on $\partial\omega$ is zero (see [11], p.393).

It is easy to show that

$$\|u\|_m \geq \|u\|_{\rho^m}$$

for every $u \in D(\omega)$. Therefore,

$$\overset{0}{H}_m(\omega) \subset \overset{0}{W}{}^1_{2,\frac{m}{2}}(\omega),$$

and $u\mid_{\partial\omega}= 0$ for all $u \in \overset{0}{H}_m(\omega)$. ∎

Consider the case $m \geq 1$. The following assertion describes the behaviour of functions from the space $\overset{0}{H}_m(\omega)$ near the boundary Γ_0.

Lemma 3. *Let φ be a continuous function with piecewise continuous first order partial derivatives on ω which are bounded for $y > \varepsilon$ $\forall\varepsilon > 0$. Moreover, let $\varphi\mid_{\Gamma_1}= 0$, $\|\varphi\|_m < \infty$ and*

$$\mid\varphi\mid\leq cy^{\frac{1-m}{2}} \text{ for } m > 1, \qquad \mid\varphi\mid\leq c\mid\ln(ky)\mid^{\frac{1}{2}} \text{ for } m = 1, \tag{8}$$

with k as in Lemma 1 and a positive constant c.

Then φ belongs to the space $\overset{0}{H}_m(\omega)$.

Proof. The proof follows the approach of Vishik [8]. Let $m \geq 1$. We introduce the function

$$\psi_\delta(y) = \begin{cases} 0,\ \ 0 < y \leq \delta, \\ (\ln\mid\ln\delta\mid)^\varepsilon - (\ln\mid\ln y\mid)^\varepsilon,\ \ \delta \leq y \leq \delta_1, \\ 1,\ \ y \geq \delta_1, \end{cases} \tag{9}$$

where δ_1 is a constant such that

$$(\ln\mid\ln\delta\mid)^\varepsilon - (\ln\mid\ln\delta_1\mid)^\varepsilon = 1,\ \ 0 < \varepsilon < \frac{1}{2}. \tag{10}$$

Clearly, δ_1 by δ

$$\delta_1 = exp\left[-exp\left((\ln \mid \ln \delta \mid)^{\varepsilon} - 1\right)^{\frac{1}{\varepsilon}}\right]. \tag{11}$$

From (11) it follows that

$$\lim_{\delta \to 0} \delta_1 = 0. \tag{12}$$

Consider the following function

$$\varphi_\delta(x, y) := \varphi(x, y) \cdot \psi_\delta(y).$$

Evidently $\varphi_\delta \in \overset{0}{W}{}_2^1(\omega)$ ($\overset{0}{W}{}_2^1(\omega)$ is the usual Sobolev space), since φ_δ has square integrable generelized partial derivatives of first order. On the other hand $\overset{0}{W}{}_2^1(\omega) \subset \overset{0}{H}_m(\omega)$ and therefore $\varphi_\delta \in \overset{0}{H}_m(\omega)$.

To complete the proof it is sufficient to show that

$$\lim_{\delta \to 0} \|\varphi - \varphi_\delta\|_m = 0. \tag{13}$$

To this end we calculate $\psi_\delta'(y)$ on $]\delta, \ \delta_1[$:

$$\frac{d\psi_\delta}{dy} = -\varepsilon(\ln \mid \ln y \mid)^{\varepsilon-1} \mid \ln y \mid^{-1} (-y)^{-1} = \varepsilon y^{-1} \mid \ln y \mid^{-1} (\ln \mid \ln y \mid)^{\varepsilon-1}.$$

Hence

$$\frac{d\psi_\delta}{dy} = \begin{cases} 0 \quad \text{for} \quad 0 \le y \le \delta, \\ \varepsilon y^{-1} \mid \ln y \mid^{-1} (\ln \mid \ln y \mid)^{\varepsilon-1} \quad \text{for} \quad \delta < y < \delta_1, \\ 0 \quad \text{for} \quad y \ge \delta_1. \end{cases} \tag{14}$$

Further we derive

$$\int_\omega y^m \left[\left(\frac{\partial}{\partial x}(\varphi - \varphi_\delta)\right)^2 + \left(\frac{\partial}{\partial y}(\varphi - \varphi_\delta)\right)^2\right] d\tau$$

$$= \int_\omega y^m \left[\left(\frac{\partial}{\partial x}((1 - \psi_\delta)\varphi)\right)^2 + \left(\frac{\partial}{\partial y}((1 - \psi_\delta)\varphi)\right)^2\right] d\tau$$

$$= \int_\omega y^m \left[\left((1 - \psi_\delta)\frac{\partial \varphi}{\partial x}\right)^2 + \left((1 - \psi_\delta)\frac{\partial \varphi}{\partial y} + \varphi\frac{\partial}{\partial y}(1 - \psi_\delta)\right)^2\right] d\tau$$

$$= \int_\omega y^m \left[(1 - \psi_\delta)^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left((1 - \psi_\delta)\frac{\partial \varphi}{\partial y} - \varphi\frac{\partial \psi_\delta}{\partial y}\right)^2\right] d\tau$$

$$\leq \int\limits_{\omega} y^m \left[(1-\psi_\delta)^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 + 2(1-\psi_\delta)^2 \left(\frac{\partial \varphi}{\partial y}\right)^2 + 2\varphi^2 \left(\frac{\partial \psi_\delta}{\partial y}\right)^2 \right] d\tau$$

$$\leq 2(I_1^\delta + I_2^\delta),$$

where

$$I_1^\delta := \int\limits_{\omega} y^m (1-\psi_\delta)^2 \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right] d\tau,$$

$$I_2^\delta := \int\limits_{\omega} y^m \varphi^2 \left(\frac{\partial \psi_\delta}{\partial y}\right)^2 d\tau.$$

Let

$$\omega_\delta := \{(x,y): \ (x,y) \in \omega, \ y \leq \delta\},$$
$$\omega^\delta := \{(x,y): \ (x,y) \in \omega, \ y \geq \delta\}.$$

Let us estimate I_1^δ:

$$I_1^\delta = \int\limits_{\omega} y^m (1-\psi_\delta)^2 \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right] d\tau$$

$$= \int\limits_{\omega_{\delta_1}} y^m (1-\psi_\delta)^2 \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right] d\tau.$$

Since $(1-\psi_\delta)^2 \leq 1$ we get

$$I_1^\delta \leq \int\limits_{\omega_{\delta_1}} y^m \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \right] d\tau.$$

From the condition of Lemma 3

$$\|\varphi\|_m < \infty$$

and the equality

$$\lim_{\delta \to 0} mes\omega_{\delta_1} = 0$$

wich follows from (12), we have

$$\lim_{\delta \to 0} I_1^\delta = 0. \tag{15}$$

Now we estimate I_2^δ. Let $m > 1$. If we use the condition of the Lemma 3

$$| \varphi | < k y^{\frac{1-m}{2}}$$

with the help of (14) and (10), we get

$$I_2^\delta \le \int\limits_a^b \int\limits_0^d y^m k^2 y^{1-m} \left(\frac{d\psi_\delta}{dy}\right)^2 dydx$$

$$= k^2 \int\limits_a^b \int\limits_\delta^{\delta_1} y\varepsilon^2 y^{-2} \mid \ln y \mid^{-2} (\ln \mid \ln y \mid)^{2\varepsilon-2} dydx$$

$$= k^2\varepsilon^2(b-a) \int\limits_\delta^{\delta_1} y^{-1} \left|\ln y\right|^{-2} (\ln\left|\ln y\right|)^{2\varepsilon-2} dy.$$

Since

$$\int\limits_0^d y^{-1} \left|\ln y\right|^{-2} dy < \infty,$$

we get

$$\lim_{\delta\to 0} I_2^\delta = 0. \tag{16}$$

Taking into account (8) and (14), for $m = 1$ we derive

$$I_2^\delta \le \int\limits_a^b \int\limits_0^d yk^2 \mid \ln y \mid \left(\frac{d\psi_\delta}{dy}\right)^2 dydx$$

$$= k^2 \int\limits_a^b \int\limits_\delta^{\delta_1} y \mid \ln y \mid \varepsilon^2 y^{-2} \mid \ln y \mid^{-2} (\ln \mid \ln y \mid)^{2\varepsilon-2} dydx$$

$$= k^2\varepsilon^2(b-a) \int\limits_\delta^{\delta_1} y^{-1} \left|\ln y\right|^{-1} (\ln\left|\ln y\right|)^{2\varepsilon-2} dy.$$

Due to the inequality

$$\int\limits_0^{\delta_1} y^{-1} \left|\ln y\right|^{-1} (\ln\left|\ln y\right|)^{2\varepsilon-2} dy < \infty$$

for every $0 < \varepsilon < \frac{1}{2}$. Hence (14) follows.

Thus, according to (15) and (16), we have (13). ∎

It is easy to see that $\forall u \in \overset{0}{H}_m(\omega)$ the trace $u \mid_{\Gamma_1} = 0$.

Now, with the help of Lemma 3, we can construct functions, belonging to $\overset{0}{H}_m(\omega)$ for $m \ge 1$, which have not traces on Γ_0. To this end let us introduce the function $\psi(x, y) \in C^\infty(\overline{\omega})$,

$$\psi(x, y) \equiv 0 \ \text{ for } \ (x, y) \in \{(x, y) : \ (x, y) \in \omega, \ dist[(x, y), \Gamma_1] < \delta\},$$

$$\psi(x,y) \equiv 1 \ \text{ for } \ (x,y) \in \{(x,y): \ (x,y) \in \omega, \ dist[(x,y),\Gamma_1] > 2\delta\},$$

$([16], p.89)$. Then the function

$$\varphi(x,y) := \begin{cases} \psi(x,y) y^{\frac{1-m+\varepsilon}{2}}, \ \ m > 1, \ \ 0 < \varepsilon < m-1, \\ \\ \psi(x,y) \mid \ln y \mid^{\frac{1-\varepsilon}{2}}, \ \ m = 1, \ \ 0 < \varepsilon < 1, \end{cases}$$

belongs to $\overset{0}{H}_m(\omega)$ and has not a trace on Γ_0.

In what follows we derive a Korn's type weighted inequality in a special functional space which will be employed later on. To this end let us define the vector space

$$\overset{\rightarrow}{\overset{0}{H}}_{m,m}(\omega) = \overset{0}{H}_m(\omega) \times \overset{0}{H}_m(\omega)$$

with the norm:

$$\left\| \vec{u} \right\|^2_{m,m} = \|u_1\|^2_m + \|u_1\|^2_m \ \text{ for } \ \vec{u} = (u,u) \in \overset{\rightarrow}{\overset{0}{H}}_{m,m}(\omega).$$

Clearly, $\overset{\rightarrow}{\overset{0}{H}}_{m,m}(\omega)$ is a Hilbert space.

Lemma 4. (Korn's weighted inequality). *Let* $\vec{u} = (u_1, u_2) \in \overset{\rightarrow}{\overset{0}{H}}_{m,m}(\omega)$, $m \neq 1$. *Then the inequality holds true*

$$\int_\omega y^m[\nabla u_1 \nabla u_1 + \nabla u_2 \nabla u_2] d\tau \le c_1 \int_\omega y^m \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 \right] d\tau,$$

where c_1 *is a positive constant independent of* $\vec{u}$.

Proof. First we prove the lemma for a function $\vec{u} = (u_1, u_2) \in [D(\omega)]^2$.

We have

$$\int_\omega y^m \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 d\tau = \int_\omega y^m \left[\left(\frac{\partial u_1}{\partial y}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2 + 2\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} \right] d\tau.$$

Let us estimate the last summand. With the help of Green's formula we have

$$\left| 2\int_\omega y^m \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} d\tau \right| = \left| 2\int_\omega y^m \frac{\partial^2 u_1}{\partial x \partial y} u_2 d\tau \right|$$

$$= \left| 2\int_\omega y^m \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} d\tau + 2m \int_\omega y^{m-1} \frac{\partial u_1}{\partial x} u_2 d\tau \right|$$

$$\le 2\left| \int_\omega y^m \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} d\tau \right| + 2m \left| \int_\omega y^{m-1} \frac{\partial u_1}{\partial x} u_2 d\tau \right|.$$

We proceed as follows

$$\left|2\int\limits_{\omega} y^m \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} d\tau\right| \le \left|2\int\limits_{\omega} y^{\frac{m}{2}} \frac{\partial u_1}{\partial x} y^{\frac{m}{2}} \frac{\partial u_2}{\partial y} d\tau\right|$$

$$\le \int\limits_{\omega} \left[y^m \left(\frac{\partial u_1}{\partial x}\right)^2 + y^m \left(\frac{\partial u_2}{\partial y}\right)^2 \right] d\tau$$

$$= \int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 \right] d\tau. \tag{17}$$

Using the inequality (4) we have

$$\left|2m\int\limits_{\omega} y^{m-1} \frac{\partial u_1}{\partial x} u_2 d\tau\right| \le \left|2m\int\limits_{\omega} y^{\frac{m}{2}} \frac{\partial u_1}{\partial x} y^{\frac{m-2}{2}} u_2 d\tau\right|$$

$$\le m\int\limits_{\omega} y^m \left(\frac{\partial u_1}{\partial x}\right)^2 d\tau + m\int\limits_{\omega} y^{m-2} u_2^2 d\tau$$

$$\le m\int\limits_{\omega} y^m \left(\frac{\partial u_1}{\partial x}\right)^2 d\tau + m\, c\int\limits_{\omega} y^m \left(\frac{\partial u_2}{\partial y}\right)^2 d\tau, \tag{18}$$

where c is the constant involved in (4).

From (17) and (18) we get

$$\left|2\int\limits_{\omega} y^m \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} d\tau\right| \le \int\limits_{\omega} y^m \left[(1+m)\left(\frac{\partial u_1}{\partial x}\right)^2 + (1+mc)\left(\frac{\partial u_2}{\partial y}\right)^2 \right] d\tau$$

$$\le \alpha \int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 \right] d\tau, \tag{19}$$

where $\alpha = \max(1+m,\ \ 1+mc)$.

Further for $0 < \delta < 1$

$$\int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 \right] d\tau$$

$$= \int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + (1-\delta) \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 + \delta \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 \right] d\tau$$

$$\geq \int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \delta \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 \right] d\tau$$

$$= \int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \delta \left(\frac{\partial u_1}{\partial y} \right)^2 + \delta \left(\frac{\partial u_2}{\partial x} \right)^2 + 2\delta \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} \right] d\tau.$$

With the help of the estimate (19) we derive

$$\int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 \right] d\tau$$

$$\geq \int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \delta \left(\frac{\partial u_1}{\partial y} \right)^2 + \delta \left(\frac{\partial u_2}{\partial x} \right)^2 - \delta\alpha \left(\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 \right) \right] d\tau$$

$$= \int\limits_{\omega} y^m \left[(1-\delta\alpha) \left(\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 \right) + \delta \left(\left(\frac{\partial u_1}{\partial y} \right)^2 + \left(\frac{\partial u_2}{\partial x} \right)^2 \right) \right] d\tau.$$

We choose δ as follows $\delta = (1+\alpha)^{-1}$. From the previous relation we have

$$\int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 \right] d\tau$$

$$\geq \int\limits_{\omega} y^m \left[\left(1 - \frac{\alpha}{1+\alpha} \right) \left(\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 \right) + \frac{1}{1+\alpha} \left(\left(\frac{\partial u_1}{\partial y} \right)^2 + \left(\frac{\partial u_2}{\partial x} \right)^2 \right) \right] d\tau$$

$$= \frac{1}{1+\alpha} \int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \left(\frac{\partial u_1}{\partial y} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 \right] d\tau.$$

If we "close" this inequality in the space $\overset{\overset{\rightarrow}{0}}{H}_{m,m}(\omega)$ we obtain that Korn's weighted inequality is true for every function $\vec{u} = (u_1, u_2) \in \overset{\overset{\rightarrow}{0}}{H}_{m,m}(\omega)$. ∎

Our aim is to detect existence end uniqueness conditions of generalized solution of the system (1). denout first equation of system (1) by (1,1) and seqond and third equations of system (1), by(1,2).

Definition. We say that $u \in \overset{0}{H}_m(\omega)$ is a generalised solution of the equation (1.1) if

$$a(u,v) := \int\limits_{\omega} \mu h \nabla u \nabla v = \int\limits_{\omega} f v d\tau$$

for any $v \in \overset{0}{H}_m(\omega)$.

Let $L^2_{\sigma_m}(\omega)$ be a Hilbert space of measurable functions ψ such that the norm

$$\|\psi\|_{L^2_{\sigma_m}} \equiv \left(\int\limits_{\omega} \sigma_m(y) \psi^2(x,y) d\tau \right)^{\frac{1}{2}} < \infty,$$

were

$$\sigma_m(y) = \begin{cases} y^{2-m}, & m \neq 1, \\ y^{1-\epsilon}, & \epsilon > 0 \quad m = 1. \end{cases}$$

Theorem 1. *If* $f \in L^2_{\sigma_m}$, *then equation* (1.1) *has an unique generalised solution* $u \in \overset{o}{H}_m(\omega)$ *and there holds the estimation*

$$\|u\|_{\overset{0}{H}_m} \leq l \|f\|_{L^2_{\sigma_m}},$$

where l *is a positiv constant independent of* u *and* f.

Proof: Let us show that the form $a(u,v)$ is coercive on $\overset{0}{H}_m(\omega)$.

First we estimate $a(v,v)$ for $v \in \overset{0}{H}_m(\omega)$.

For case $m \neq 1$, according to the Lemma1, we have

$$\int\limits_{\omega} y^m v^2 d\tau \leq c_1 \int\limits_{\omega} y^{m-2} v^2 d\tau \leq c_2 \int\limits_{\omega} y^m \left(\frac{\partial v}{\partial y} \right)^2 d\tau,$$

from which fallows that

$$\int\limits_{\omega} y^m v^2 d\tau \leq c_2 \int\limits_{\omega} y^m \left[\left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] d\tau.$$

Hence we get

$$\mu \int\limits_{\omega} y^m v^2 d\tau + \mu \int\limits_{\omega} y^m \left[\left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] d\tau$$
$$\leq \mu(c_2+1) \int\limits_{\omega} y^m \left[\left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] d\tau$$
$$= \mu(c_2+1) \int\limits_{\omega} h \nabla^2 v d\tau,$$

$$\mu\|v\|^2 \le (c_2+1)a(v,v),$$
$$a(v,v) \ge \frac{\mu}{c_2+1}\|v\|^2,$$

that is

$$a(v,v) \ge c_3\|v\|^2, \qquad m \ne 1. \tag{20}$$

Next we have to consider the case $m=1$. According to Lemma 1

$$\int\limits_{\omega} yv^2 d\tau \le c_4 \int\limits_{\omega} y^{-1} \mid \ln ky \mid^{-2-\epsilon} v^2 d\tau \le c_5 \int\limits_{\omega} y \left(\frac{\partial v}{\partial y}\right)^2 d\tau.$$

Reasoning analogously, using the last inequality, we will obtain

$$a(v,v) \ge c_6\|v\|^2, \qquad m=1.$$

Now we can show that the form $a(u,v)$ is bounded in $\overset{0}{H}{}_{-m}(\omega)$. Let $u, v \in \overset{0}{H}{}_{-m}(\omega)$, then

$$\begin{aligned}
|a(u,v)| &= \left|\int\limits_{\omega} \mu h(\nabla u \nabla v) d\tau\right| \le \mu \int\limits_{\omega} h \mid \nabla u \,|\nabla v|\, d\tau \\
&= \mu \int\limits_{\omega} h^{\frac{1}{2}} h^{\frac{1}{2}} \,|\nabla u|\, |\nabla v|\, d\tau \le \mu \left(\int\limits_{\omega} h(\nabla u)^2 d\tau\right)^{\frac{1}{2}} \left(\int\limits_{\omega} h(\nabla v)^2 d\tau\right)^{\frac{1}{2}} \\
&\le \mu \left(\int\limits_{\omega} h(\nabla^2 u + u^2) d\tau\right)^{\frac{1}{2}} \left(\int\limits_{\omega} h(\nabla^2 v + v^2) d\tau\right)^{\frac{1}{2}} \\
&\le \mu\|u\|_m\|v\|_m = \mu\|u\|_m\|v\|_m.
\end{aligned}$$

So $a(u,v)$ is bounded on $\overset{0}{H}{}_{m}(\omega)$.

Let $v \in \overset{0}{H}{}_{m}(\omega)$, $\quad f \in L^2_{\sigma_m}(\omega)$, then

$$\begin{aligned}
\left|\int\limits_{\omega} fv d\tau\right| &= \left|\int\limits_{\omega} \sigma^{\frac{1}{2}} f \sigma^{\frac{1}{2}} v d\tau\right| \le \{\text{By the Hölder inequality}\} \\
&\le \left(\int\limits_{\omega} \sigma_m f^2 d\tau\right)^{\frac{1}{2}} \left(\int\limits_{\omega} \sigma_m^{-1} v^2 d\tau\right)^{\frac{1}{2}} \le t\|f\|_{L^2_{\sigma_m}}\|v\|_m = \beta\|v\|_m,
\end{aligned}$$

where $\quad \beta \equiv t\|f\|_{L^2_{\sigma_m}}$. $\quad$ Now, the Lax-Milgram theorem(see e.g.[6.10]) completes the proof.

Next we consider the system (1.2).

Definition. We say that $\vec{u}=(u_1,u_2)\in \overset{0}{H}_{m,m}$ is a generalized solution of the system $(1,2)$ if

$$\mathbb{B}(\vec{u},\vec{v})\equiv\int_\omega h\left[\lambda\left(\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+\frac{\partial u_2}{\partial y}\frac{\partial v_1}{\partial y}+\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}+\frac{\partial u_1}{\partial x}\frac{\partial v_2}{\partial y}\right)\right.$$
$$\left.+\mu\left(2\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial_x}+\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+\frac{\partial u_2}{\partial x}\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x}+2\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}+\frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x}\right)\right]d\tau.$$
$$=\int_\omega(f_1v_1+f_2v_2)d\tau,$$

for any $\vec{v}=(v_1,v_2)\in \overset{0}{H}_{(m,m)}(\omega)$.

Let $L^2_{(\sigma_m,\sigma_m)}(\omega)$ be a Hilbert spase of measurable functions $\vec{f}$ with the norm

$$\|\vec{f}\|^2_{L^2_{(\sigma_m,\sigma_m)}}=\|f_1\|^2_{L^2_{\sigma_m}}+\|f_2\|^2_{L^2_{\sigma_m}},$$

where $\vec{f}=(f_1,f_2); f_1,f_2\in L^2_{\sigma_m}$.

Theorem 2. *If $\vec{f}=(f_1,f_2)\in L^2_{\sigma_m,\sigma_m}$, then the system (1.2) has a unique generalized solution $\vec{u}=(u_1,u_2)\in \overset{0}{H}_{m,m}(\omega)$ and there holds the estimation*

$$\|\vec{u}\|_{m,m}\le\gamma\|\vec{f}\|_{L^2_{\sigma_m,\sigma_m}}.$$

Proof. Let us show that the form $\mathbb{B}(\vec{u},\vec{v})$ is coercive on $\overset{0}{H}_{m,m}(\omega)$. First we estimate $\mathbb{B}(\vec{v},\vec{v})$ for $\vec{v}=(v_1,v_2)\in \overset{0}{H}_{m,m}(\omega)$

$$\mathbb{B}(\vec{v},\vec{v})\ge\int_\omega h\left[\left(\frac{\partial v_1}{\partial x}\right)^2+\left(\frac{\partial v_2}{\partial y}\right)^2+\left(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x}\right)^2\right]d\tau\ge$$
$$\{\text{according to the Lema 4}\}\ge c_1\int_\omega h[\nabla^2 v_1+\nabla^2 v_2]d\tau\ge \quad .$$
$$\{\text{according L.1}\}\ge\lambda\int_\omega h[\nabla^2 v_1+\nabla^2 v_2+v_1^2+v_2^2]d\tau=\chi\|\vec{v}\|^2_{m,m}$$

Furter we estimate $\mathbb{B}(\vec{u},\vec{v})$ for $\vec{u}=(u_1,u_2);\vec{v}=(v_1,v_2)\in \overset{0}{H}_{m,m}(\omega)$. We heve

$$\left|\mathbb{B}(\vec{u},\vec{v})\right|\le\int_\omega h\lambda\left[\left|\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}\right|+\left|\frac{\partial u_2}{\partial y}\frac{\partial v_1}{\partial x}\right|+\left|\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}\right|+\left|\frac{\partial u_1}{\partial x}\frac{\partial v_2}{\partial y}\right|\right]d\tau$$
$$+\int_\omega h\mu\left[\left|2\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}\right|+\left|\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y}\right|+\left|\frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y}\right|+\left|\frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial x}\right|+2\left|\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}\right|+\left|\frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x}\right|\right]d\tau.$$

By the Hölder inequality we can estimate each term of the last sum separately. Indeed,

$$\int_\omega h \mid \frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} \mid \leq \left[\int_\omega h\left(\frac{\partial u_1}{\partial x}\right)^2 d\tau\right]^{\frac{1}{2}} \left[\int_\omega h\left(\frac{\partial v_1}{\partial x}\right)^2 d\tau\right]^{\frac{1}{2}} \leq \|u_1\|_m \|v_1\|_m \leq \|u\|_{m,m}\|v\|_{m,m}.$$

Analogously will be estimated other terms and finally we will obtain

$$\left|\mathbb{B}(\vec{u},\vec{v})\right| \leq 4\lambda\|u\|_{m,m}\|v\|_{m,m} + 8\mu\|u\|_{m,m}\|v\|_{m,m} = c_2\|u\|_{m,m}\|v\|_{m,m}.$$

Let $\vec{f} = (f_1, f_2) \in L^2_{\sigma_m,\sigma_m}(\omega)$, $\vec{v} \in \overset{0}{H}_{m,m}$, then there holds the following inequality

$$\begin{aligned}
&\left|\int_\omega \vec{f}\,\vec{v}\, d\tau\right| \leq \left|\int_\omega (f_1v_1 + f_2v_2)d\tau\right| \leq \left|\int_\omega f_1v_1 d\tau\right| + \left|\int_\omega f_2v_2 d\tau\right| \\
&\leq \left|\int_\omega (\sigma_m)^{\frac{1}{2}} f_1 (\sigma_m)^{\frac{1}{2}} v_1 d\tau\right| + \left|\int_\omega (\sigma_m)^{\frac{1}{2}} f_2 (\sigma_m)^{-\frac{1}{2}} v_2 d\tau\right| \\
&\leq \left(\int_\omega \sigma_m f_1^2 d\tau\right)^{\frac{1}{2}} \left(\int_\omega \sigma_m^{-1} v_1^2 d\tau\right)^{\frac{1}{2}} + \left(\int_\omega \sigma_m f_2^2 d\tau\right)^{\frac{1}{2}} \left(\int_\omega \sigma_m^{-1} v_2^2 d\tau\right)^{\frac{1}{2}} \\
&\leq \|f_1\|_{L^2_{\sigma_m}} k_1 \|v_1\|_m + \|f_2\|_{L^2_{\sigma_m}} k_2 \|v_2\|_m \\
&\leq \left(k_1\|f_1\|_{L^2_{\sigma_m}} + k_2\|f_2\|_{L^2_{\sigma_m}}\right) \|\,\vec{v}\,\|_{m,m} = k\|\,\vec{v}\,\|_{m,m},
\end{aligned}$$

i.e.,

$$\left|F\,\vec{v}\right| \equiv \left|\int_\omega \vec{f}\,\vec{v}\, d\tau\right| \leq k\|\,\vec{v}\,\|_{m,m}.$$

Thus we heve shown that $\mathbb{B}(\vec{u},\vec{v})$ is coercive, bounded and the functional F is bounded on $L^2_{\sigma_m,\sigma_m}$.

Now, the Lax-Milgram theorem (see[6.10]) complites the proof.

P.S. From the Theorems:1,2; Lemmas:1,2,3 we can conclude:

1) If $m < 1$, then the Dirichlet type boundary value problem is correct for the system(1);

2) If $m \geq 1$, then the Dirichlet type boundery value problem for the system (1) is not correct in general.For it's correctness it's necessery to make free Γ_0 from the boundary condition.

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