Seminar of I. Vekua Institute
of Applied Mathematics
REPORTS, Vol. 30, 2004

ON APPROXIMATE CALCULATION OF SOME IMPROPER INTEGRALS

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Received: 22.09.2004; revised: 29.12.2004

Abstract

Approximate calculation of improper integrals with unbounded integrands is considered. Values of the integrals are found as solutions of a certain Cauchy problem for an differential equation of the first order. A formula of the numerical integration of improper integrals with the remainder term of the forth order with respect to the step of integration is constructed by means of the formulas of the numerical integration and Runge-Kutta.

Key words and phrases: improper integral, approximate calcilation.
AMS subject classification: 65D30, 65D32.

Let a function $y = f(x)$ and all its derivatives including the fifth onder be continuous in the interval $[a, b[,\ b > a$, and unbounded in a neighbourhood of b. Then the improper integrals

$$\Im = \int_a^b f(x)dx \tag{1}$$

and

$$\int_{b-\eta}^{b} f(x)dx, \quad 0 < \eta < b - a$$

are convergent or divergent at the same time and if (1) is convergent, then

$$\lim_{\eta\to 0} \int_{b-\eta}^{b} f(x)dx = 0. \tag{2}$$

Applying the rectangle formula, from (2) we get

$$\lim_{\eta\to 0} \eta f(b-\eta) = 0. \tag{3}$$

Similary, when $y = f(x)$ is defined on the interval $]a, b]$ and unbounded in a neighbourhood of a, then for the convergent improper integral

$$\int_a^b f(x)dx$$

we have

$$\lim_{\eta \to 0} \eta f(a+\eta) = 0. \tag{4}$$

Equalities (3) and (4), i.e., (2), are necessary conditions of the convergence of the above improper integrals. If any of them is not valid, then the corresponding improper integral is divergent.

when $c \in]a;b[$ is a point of discontinuity of $y = f(x)$, then in the case of convergence of (1) the equalities $\lim_{\eta \to 0} \eta f(c-\eta) = 0$, $\lim_{\eta \to 0} \eta f(c+\eta) = 0$ necessarily take place.

Let $y = f(x)$ be unbounded in a neighbourhood of b and the integral (1) be convergent, then

$$\Im = \int_{a}^{b-t} f(x)dx + \int_{b-t}^{b} f(x)dx, \quad 0 < t \leq b-a.$$

Let

$$\phi(t) := \int_{b-t}^{b} f(x)dx. \tag{5}$$

By virtue of equality (2), we have in the sense of limit representation

$$\Phi(0) := \phi(0+) = 0. \tag{6}$$

Evidently,

$$\phi(b-a) = \Im. \tag{7}$$

After differentiation of (5) we obtain

$$\phi'(t) = f(b-t), \quad 0 < t \leq b-a. \tag{8}$$

The equality (8) is a differential aquation with respect to $\phi(t)$ under the initial condition (6). So we have Cauchy problem

$$\begin{cases} \phi'(t) = f(b-t), \quad 0 < t \leq b-a, \\ \phi(0) = 0. \end{cases} \tag{9}$$

The value of the solution of (9) at the end point $t = b - a$ according to (7), gives the value of the improper integral (1).

The problem (9) is incorrect in general, since the right-hand side $f(b - t)$ of the differential equation has a singuliarity at the point $t = 0$. Let us try to avoid this difficulty, using an artificial way.

Dividie the segment $[0; b - a]$ in n ($n \geq 4$) equal parts with the step $h = \dfrac{b-a}{n}$. Denote points of division by $t_i = ih$ and values $\phi(t_i)$ by y_i, $i = 0, 1, ..., n$. Using Runge-Kutta method on the segment $[h; b - a]$ (see[1]), we get

$$y_i = y_{i-1} + \frac{h}{6}\left[f(b-t_{i-1}) + 4f\left(b - \left(t_{i-1} + \frac{h}{2}\right)\right) + f(b-(t_{I-1}+h))\right], \tag{10}$$

$i = 2, ..., n$

For the remainder term of the formula (10) we have the following estimate

$$|R_n[f]| \leq \frac{b-a}{2880} h^4 M, \;\; M := \max_{t \in [h, b-a]} |f^{(IV)}(b-t)|; \tag{11}$$

The formula (10) can be rewritten as follows

$$\begin{cases} y_2 = y_1 + \dfrac{h}{6}(f_1 + 4f_{3/2} + f_2), \\ y_3 = y_2 + \dfrac{h}{6}(f_2 + 4f_{5/2} + f_3), \\ y_4 = y_3 + \dfrac{h}{6}(f_3 + 4f_{7/2} + f_4), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n = y_{n-1} + \dfrac{h}{6}(f_{n-1} + 4f_{\frac{2n-1}{2}} + f_n), \end{cases} \tag{12}$$

where $f_i = f(b - t_i)$.

It is abvious, that from (10) it is not possible to calculate y_1. To this end we use the following formula of numerical differentiation [2].

$$y_2' = \frac{1}{12h}(y_0 - 8y_1 + 8y_3 - y_4) + \frac{h^4}{30} f^{(V)}(\xi), \tag{13}$$

where $h < \xi < 2h$.

Let us expess y_3 and y_4 by y_1 by means of the first three equalities of (12)

$$y_3 = y_1 + \frac{h}{6}(f_1 + 4f_{3/2} + 2f_2 + 4f_{5/2} + f_3),$$

$$y_4 = y_1 + \frac{h}{6}(f_1 + 4f_{3/2} + 2f_2 + 4f_{5/2} + 2f_3 + 4f_{7/2} + f_4).$$

The obtained expessions put in (13). Then taking into account

$$y_2' = \phi'(t_2) = f(b - t_2) = f_2, \quad y_0 = \phi(0) = 0,$$

after some simplifications we get

$$y_1 = \frac{h}{6}(7f_1 + 28f_{3/2} - 58f_2 + 28f_{5/2} + 6f_3 - 4f_{7/2} - f_4). \tag{14}$$

Summing up (12) and (14), we have

$$y_n = \frac{h}{6}[2(f_1 + f_2 + ... + f_{n-1}) + 4(f_{3/2} + f_{5/2} + ... + f_{\frac{2n-1}{2}}) + 6(f_1 + f_3) +$$

$$+28(f_{3/2} + f_{5/2}) - 58f_2 - 4f_{7/2} - f_4 + f_n]. \tag{15}$$

For the reminder term of (15) we have the following estimate

$$|\tilde{R}_n[f]| \leq \frac{b-a+96}{2880} h^4 M,$$

where

$$M := max(M_1; M_2),\ M_1 := \max_{t\in[h;b-a]} |f^{(IV)}(b-t)|,\ M_2 := \max_{t\in[h;4h]} |f^{(V)}(b-t)|.$$

Evidently, when $n \to \infty$ and $h \to 0$ then $f_1 = f(b-h)$ becomes undbounded, but the formula (15) can be applied successfully since, because of (3), $hf_1 = hf(b-h) \to 0$.

The formula (15) is also possible to use for calculation of the improper integral

$$\int_a^{+\infty} f(x)dx.$$

where a function $y = f(x)$ is continuous in the interval $[a; +\infty[$. To this end by change of variables $x = \dfrac{a}{1-t}$ we get

$$\int_a^{+\infty} f(x)dx = a\int_0^1 f\left(\frac{a}{1-t}\right)\frac{dt}{(1-t)^2}. \tag{16}$$

Now, the formula (15) can be applied to the integral (16). After change of variables the integrand of (16) may be continuous on the segment [0,1] and, therefore, the integral becomes proper one. Obviously, the formula (15) can be applied also in this case.

If $y = f(x)$ is unbounded in the neighbourhood of a, then we can represent the integral (1) in the following form

$$\Im = \int_a^{a+t} f(x)dx + \int_{a+t}^{b} f(x)dx, \quad 0 < t \le b-a.$$

Introducing the notation

$$\phi(t) := \int_a^{a+t} f(x)dx,$$

evidently,

$$\phi(0) = 0,\ \ \phi(b-a) = \Im,$$

and we get the following Cauchy problem

$$\begin{cases} \phi'(t) = f(a+t), \quad 0 < t \le b-a, \\ \phi(0) = 0. \end{cases}$$

where $f_i = f(a + t_i)$.

If the function $y = f(x)$ is unbounded in the neighbourhood of the point $c \in]a; b[$, then we represent the integral (1) in the form

$$\Im = \int_a^{c} f(x)dx + \int_c^{b} f(x)dx.$$

Introducing the functions

$$\phi_1(t) := \int\limits_{c-t}^{c} f(x)dx, \ \ \phi_2(t) := \int\limits_{c}^{c+t} f(x)dx,$$

we get, similarly,

$$\begin{cases} \phi_1'(t) = f(c-t), \ 0 < t \le c-a, \\ \phi_1(0) = 0; \end{cases} \quad \begin{cases} \phi_2'(t) = f(c+t), \ 0 < t \le b-c, \\ \phi_2(0) = 0. \end{cases}$$

Obviousely,

$$\Im = \phi_1(c-a) + \phi_2(b-c).$$

Example. Calculate the integral

$$\Im = \int\limits_0^1 \frac{dx}{(1-x)^\alpha}.$$

Acording to (3), as $f(x) = \dfrac{1}{(1-x)^\alpha}$, $b = 1$,

$$\lim_{\eta\to 0} \eta f(b-\eta) = \lim_{\eta\to 0} \frac{\eta}{[1-(1-\eta)]^\alpha} = \lim_{\eta\to 0} \eta^{1-\alpha} = 0,$$

if and only if $1-\alpha > 0$, i.e., $\alpha < 1$.

If $\alpha = \dfrac{1}{2}$, the value of the integral is 2. If $n = 4$, then calculation by the formula (15) gives $\Im \approx 1.6404$ with the absolute error $\Delta\Im = 0,36$ and the relative error $\delta\Im = 18\%$. If $n = 8$, then $\Im \approx 1.9801$; $\Delta\Im = 0,02$, $\delta\Im = 1\%$.

If $\alpha = \dfrac{2}{3}$, $n = 10$, then $\Im \approx 2,9812$.

If $\alpha = \dfrac{3}{4}$, $n = 12$, then $\Im \approx 3,9903$.

If $\alpha = \dfrac{4}{5}$, $n = 12$, then $\Im \approx 4,9806$.

R E F E R E N C E S

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