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THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF ONE NONLINEAR INTEGRO-DIFFERENTIAL MODEL

Kiguradze Z.

I. Javakhishvili Tbilisi State University

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Abstract

The asymptotic behavior as $t \to \infty$ of solutions for a nonlinear system of integro-differential equations is studied. The system arises as a model describing the penetration of the electromagnetic field in a substance.

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AMS subject classification: 35K55, 45K05.

1. Introduction

A great variety of applied problems are modelled by such nonlinear equations, which side by side with partial derivatives of the unknown function consist of the integrals from it and its derivatives. For instance such systems arise for mathematical modelling of the process of penetration of electromagnetic field in the substance [1]

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2 d\tau\right) rotH\right], \tag{1.1}$$

where function $a = a(s)$ is defined for $s \in [0, \infty)$.

Let's consider the following magnetic field H, having the form $H = (0, 0, U)$, where $U = U(x, y, t)$ is a scalar function of time and of two space variables. Then $rotH = \left(\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial x}, 0\right)$ and the system (1.1) have the form

$$\frac{\partial U}{\partial t} = \nabla\left[a\left(\int_0^t |\nabla U|^2 d\tau\right) \nabla U\right]. \tag{1.2}$$

The study of the equations of type (1.1), (1.2) have been begun in the work [1]. In this work, in particular, are proved the theorems of existence of solution of the first boundary value problem for one-dimensional space case while $a(s) = 1 + s$ and the uniqueness for more general cases.

One-dimensional variant for the case $a(s) = (1 + s)^p$, $0 < p \le 1$ is studied in [2]. In [2] the theorems of existence and uniqueness of solution of the first boundary value problem in the space $L_{2p+2}(0, T; \overset{o}{W}{}^1{}_{2p+2}(0, 1))$ are proved.

Investigations for multidimensional space cases at first are carried out in the work [3].

In the work [4] there is proposed the operational scheme with so called conditionally closed operators.

In the work [4] is proposed some generalization of the equations of type (1.1), (1.2):

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_\Omega |\nabla U|^2 \, dx d\tau\right) \Delta U. \tag{1.3}$$

A lot of scientific works are devoted to the investigation of the equations (1.2), (1.3) and similar system of equations too (see, for example [1-11]).

In the present work is studied first boundary value problem for the system of equations of type (1.3). Attention is paid to the study of asymptotic behavior of the solutions as $t \to \infty$. Note that the brief variant of the section 3 is published in [10].

2. Problem with homogeneous boundary conditions

Let us consider the following system:

$$\frac{\partial U}{\partial t} = a(S)\frac{\partial^2 U}{\partial x^2}, \qquad \frac{\partial V}{\partial t} = b(S)\frac{\partial^2 V}{\partial x^2}, \tag{2.1}$$

where

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx d\tau$$

and $a = a(S)$ and $b = b(S)$ are given functions.

In the domain $Q = (0,1) \times \{t > 0\}$ for the system (2.1) let us consider the following initial-boundary value problem:

$$U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \geq 0, \tag{2.2}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \tag{2.3}$$

where U_0 and V_0 are given functions.

The following statement is true.

Theorem 1. *If $a(S) \geq a_0 = const > 0$, $b(S) \geq b_0 = const > 0$, $c_0 = min(a_0, b_0)$, $U_0(0) = U_0(1) = V_0(0) = V_0(1) = 0$, $U_0, V_0 \in W_2^1(0,1)$, then for the problem (2.1)-(2.3) the following estimate is true*

$$\|U\|_{W_2^1} + \|V\|_{W_2^1} \leq Ce^{-c_0 t}. \tag{2.4}$$

Here and below C denote positive constants dependent only on U_0, V_0 and consequently independent from t.

Proof. Let's multiply the first equation of the system (2.1) by U and integrate on the $(0,1)$. Using boundary conditions (2.2) and formula of integrating by parts we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + a_0\left\|\frac{\partial U}{\partial x}\right\|^2 \le 0. \tag{2.5}$$

Using Poincare's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + a_0\|U\|^2 \le 0. \tag{2.6}$$

Analogously,

$$\frac{1}{2}\frac{d}{dt}\|V\|^2 + b_0\left\|\frac{\partial V}{\partial x}\right\|^2 \le 0, \quad \frac{1}{2}\frac{d}{dt}\|V\|^2 + b_0\|V\|^2 \le 0. \tag{2.7}$$

Let's multiply the first equation of the system (2.1) by $\dfrac{\partial^2 U}{\partial x^2}$. Using formula of integrating by parts we have

$$\left.\frac{\partial U}{\partial t}\frac{\partial U}{\partial x}\right|_0^1 - \int_0^1 \frac{\partial^2 U}{\partial t\partial x}\frac{\partial U}{\partial x}dx = a(S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2.$$

Taking into account (2.2), from the last equality we get

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2 \le 0. \tag{2.8}$$

Analogously,

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial V}{\partial x}\right\|^2 \le 0. \tag{2.9}$$

Assume $c_0 = min(a_0, b_0)$. Let's multiply the inequalities (2.5)-(2.9) by $e^{c_0 t}$, we have

$$e^{c_0 t}\frac{d}{dt}\left(\|U\|^2 + \|V\|^2\right) + c_0 e^{c_0 t}\left(\|U\|^2 + \|V\|^2\right) +$$

$$+e^{c_0 t}\frac{d}{dt}\left(\left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2\right) + c_0 e^{c_0 t}\left(\left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2\right) \le 0.$$

From this we get

$$\frac{d}{dt}\left[e^{c_0 t}\left(\|U\|^2 + \|V\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 + \left\|\frac{\partial V}{\partial x}\right\|^2\right)\right] \le 0.$$

At last from this inequality after integrating on the interval $(0,t)$ we get (2.4) and the proof of the Theorem is over.

3. Problem with nonhomogeneous boundary conditions

Now let us consider the system:

$$\frac{\partial U}{\partial t} = S^p \frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial V}{\partial t} = S^q \frac{\partial^2 V}{\partial x^2}, \tag{3.1}$$

where $p, q \in R$, $p \neq q$ and

$$S(t) = 1 + \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx d\tau. \tag{3.2}$$

In the domain Q for the system (3.1), (3.2) let us consider the following initial-boundary value problem:

$$U(0,t) = V(0,t) = 0, \;\; U(1,t) = \psi_1, \;\; V(1,t) = \psi_2, \quad t \geq 0, \tag{3.3}$$

$$U(x,0) = U_0(x), \;\; V(x,0) = V_0(x), \;\; x \in [0,1], \tag{3.4}$$

where $\psi_1 = Const$, $\psi_2 = Const$; U_0 and V_0 are given functions.

We assume that $(U,V) = (U(x,t), V(x,t))$ is a solution of (3.1)-(3.4) on $[0,1] \times [0,\infty)$ such that $U, V, \frac{\partial U}{\partial x}, \frac{\partial V}{\partial x}, \frac{\partial U}{\partial t}, \frac{\partial V}{\partial t}, \frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 V}{\partial x^2}$ are all in $C^0([0,\infty); L_2(0,1))$, while $\frac{\partial^2 U}{\partial t \partial x}, \frac{\partial^2 V}{\partial t \partial x}$ are in $C^0((0,\infty); L_2(0,1))$ and $\frac{\partial^2 U}{\partial t^2}, \frac{\partial^2 V}{\partial t^2}$ are in $L_{2,loc}((0,\infty); L_2(0,1))$ (see, [1],[2],[4],[12]).

The asymptotic behavior of the solution of the problem (3.1)-(3.4) for the case $p = q \in (0,1]$ is considered in [9]. In the present work the case $p \neq q$ is investigated.

The main result of this section can now be formulated.

Theorem 2. *If* $1 + p + q > 0$, $p > -1$, $q > -1$, $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1$, $V_0(1) = \psi_2$, $\psi_1^2 + \psi_2^2 \neq 0$, U_0, $V_0 \in W_2^2(0,1)$, *then for the solution of the problem (3.1)-(3.4) the following estimates are true as* $t \to \infty$*:*

$$\frac{\partial U(x,t)}{\partial x} = \psi_1 + O(t^{-1-p}), \quad \frac{\partial V(x,t)}{\partial x} = \psi_2 + O(t^{-1-q}), \tag{3.5}$$

$$\frac{\partial U(x,t)}{\partial t} = O(t^{-1}), \quad \frac{\partial V(x,t)}{\partial t} = O(t^{-1}). \tag{3.6}$$

Before we proceed to the proof of the theorem, we state some auxiliary lemmas.

Lemma 1. *If* $1 + p + q > 0$, *then the following estimations are true:*

$$c\varphi^{\frac{1}{1+p+q}}(t) \leq S(t) \leq C\varphi(t)^{\frac{1}{1+p+q}}, \quad t \geq 0,$$

where

$$\varphi(t) = 1 + \int_0^t \int_0^1 S^{p+q} \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx d\tau.$$

Here and below c, C and C_i, denote positive constants dependent only on U_0, V_0 and consequently independent from t.

Proof. From (3.2) it follows that

$$\frac{dS}{dt} = \int_0^1 \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx, \quad S(0) = 1. \tag{3.7}$$

Let us multiply the equation (3.7) on S^{p+q} and introduce following notations:

$$\sigma_1 = S^p \frac{\partial U}{\partial x}, \quad \sigma_2 = S^q \frac{\partial V}{\partial x}.$$

We have

$$\frac{1}{1+p+q} \frac{dS^{1+p+q}}{dt} = \int_0^1 \left(S^{q-p} \sigma_1^2 + S^{p-q} \sigma_2^2 \right) dx.$$

Integrating this equation on $(0, t)$ we get

$$\frac{1}{1+p+q} S^{1+p+q} = \int_0^t \int_0^1 \left(S^{q-p} \sigma_1^2 + S^{p-q} \sigma_2^2 \right) dx d\tau + \frac{1}{1+p+q}.$$

If $0 < \dfrac{1}{1+p+q} \le 1$ then we have

$$\varphi^{\frac{1}{1+p+q}}(t) \le S(t) \le [(1+p+q)\varphi(t)]^{\frac{1}{1+p+q}},$$

and if $\dfrac{1}{1+p+q} \ge 1$, then

$$[(1+p+q)\varphi(t)]^{\frac{1}{1+p+q}} \le S(t) \le \varphi^{\frac{1}{1+p+q}}(t).$$

So, Lemma 1 is proved.

Lemma 2. *The following estimations are true*

$$c\varphi^{\frac{p+q}{1+p+q}}(t) \le \int_0^1 \left(S^{q-p} \sigma_1^2 + S^{p-q} \sigma_2^2 \right) dx \le C\varphi^{\frac{p+q}{1+p+q}}(t).$$

proof. Taking into account Lemma 1 we get

$$\int_0^1 \left(S^{q-p} \sigma_1^2 + S^{p-q} \sigma_2^2 \right) dx = \int_0^1 S^{p+q} \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx \ge c\varphi^{\frac{p+q}{1+p+q}}(t) \int_0^1 \left[\left(\frac{\partial U}{\partial x} \right)^2 + \right.$$

$$\left. + \left(\frac{\partial V}{\partial x} \right)^2 \right] dx \ge c\varphi^{\frac{p+q}{1+p+q}}(t) \left\{ \left[\int_0^1 \frac{\partial U}{\partial x} dx \right]^2 + \left[\int_0^1 \frac{\partial V}{\partial x} dx \right]^2 \right\} = \left(\psi_1^2 + \psi_2^2 \right) c\varphi^{\frac{p+q}{1+p+q}}(t),$$

or

$$\int_0^1 \left(S^{q-p}\sigma_1^2 + S^{p-q}\sigma_2^2\right) dx \geq c\varphi^{\frac{p+q}{1+p+q}}(t). \tag{3.8}$$

Let's multiply the first equation of the system (3.1) by $S^{-p}\partial U/\partial t$ and integrate on the domain $(0,1)\times(0,t)$. Using boundary conditions (3.3), (3.4) and formula of integrating by parts we have

$$\int_0^t \int_0^1 S^{-p}\left(\frac{\partial U}{\partial t}\right)^2 dx d\tau + \frac{1}{2}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx - \frac{1}{2}\int_0^1 \left(\frac{\partial U(x,0)}{\partial x}\right)^2 dx = 0.$$

From this we get

$$\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \leq C. \tag{3.9}$$

Analogously,

$$\int_0^1 \left(\frac{\partial V}{\partial x}\right)^2 dx \leq C. \tag{3.10}$$

From (3.9), (3.10) and Lemma 1 we conclude

$$\int_0^1 \left(S^{q-p}\sigma_1^2 + S^{p-q}\sigma_2^2\right) dx = S^{p+q}\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx \leq C\varphi^{\frac{p+q}{1+p+q}}(t).$$

Now taking into account (3.8) from the last inequality the prove of the Lemma 2 is over.

Lemma 3. *For the function S the following estimate is true*

$$ct \leq S(t) \leq Ct, \quad t \geq 1.$$

Proof. We have

$$\frac{d\varphi(t)}{dt} = \int_0^1 \left(S^{q-p}\sigma_1^2(x,t) + S^{p-q}\sigma_2^2(x,t)\right) dx.$$

From Lemma 2 we get

$$c\varphi^{\frac{p+q}{1+p+q}}(t) \leq \frac{d\varphi(t)}{dt} \leq C\varphi^{\frac{p+q}{1+p+q}}(t),$$

or integrating on $(0,t)$

$$ct^{1+p+q} \leq \varphi(t) \leq Ct^{1+p+q}, \quad t \geq 1.$$

The last estimate and Lemma 2 prove the Lemma 3.

Lemma 4. *For the functions $\partial U/\partial t$ and $\partial V/\partial t$ following inequalities take place:*

$$\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \le Ct^{-2}, \quad \int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 dx \le Ct^{-2}, \quad t \ge 1.$$

Proof. Let us differentiate first equation of the system (3.1) with respect to t

$$\frac{\partial^2 U}{\partial t^2} = S^p \frac{\partial^3 U}{\partial t \partial x^2} + pS^{p-1} \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx \frac{\partial^2 U}{\partial^2 x}. \tag{3.11}$$

Multiplying the equation (3.11) scalarly on $\partial U/\partial t$, applying the formula of integrating by parts, Schwarz's inequality and a priori estimates (3.9), (3.10) we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 S^p \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx \le \frac{1}{2}\int_0^1 S^p \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx +$$

$$+\frac{p^2}{2}\int_0^1 S^{p-2} \left(\frac{\partial U}{\partial x}\right)^2 \left\{\int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx\right\}^2 dx \le$$

$$\le \frac{1}{2}\int_0^1 S^p \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx + Ct^{p-2}.$$

It is clear that

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 S^p \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx \le Ct^{p-2}, \quad t \ge 1. \tag{3.12}$$

Analogously,

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 dx + \int_0^1 S^q \left(\frac{\partial^2 V}{\partial t \partial x}\right)^2 dx \le Ct^{q-2}, \quad t \ge 1. \tag{3.13}$$

So, using Poincare's inequality

$$\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \le \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx,$$

from (3.12) we have

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + ct^p \int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \le Ct^{p-2}.$$

From this last inequality we get the following estimate

$$\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \le Ct^{-2}, \quad t \ge 1.$$

The same estimation is true for the function V. So, Lemma 4 is proved.
Let us now estimate $\partial U^2/\partial x^2$ in $L_1(0,1)$. We have

$$\frac{\partial U}{\partial x} = S^{-p}\sigma_1.$$

From the Lemmas 3 and 4 we conclude

$$\int_0^1 \left|\frac{\partial^2 U}{\partial x^2}\right| dx = \int_0^1 \left|S^{-p}\frac{\partial \sigma_1}{\partial x}\right| dx \le \left[\int_0^1 S^{-2p} dx\right]^{\frac{1}{2}} \left[\int_0^1 \left(\frac{\partial \sigma_1}{\partial x}\right)^2 dx\right]^{\frac{1}{2}} \le$$

$$\le C_1 t^{-p} \left[\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx\right]^{\frac{1}{2}} \le Ct^{-1-p}.$$

Hence, we have

$$\int_0^1 \left|\frac{\partial^2 U(x,t)}{\partial x^2}\right| dx \le Ct^{-1-p}, \ \ t \ge 1.$$

From this estimate, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y} dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2} d\xi dy,$$

it follows that

$$\frac{\partial U(x,t)}{\partial x} - \psi_1 = \int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2} d\xi dy \le \int_0^1 \left|\frac{\partial^2 U(y,t)}{\partial y^2}\right| dy \le Ct^{-1-p}.$$

Thus, the following asymptotic formula takes place

$$\frac{\partial U(x,t)}{\partial x} = \psi_1 + O(t^{-1-p}).$$

The same estimate is valid for $\partial V/\partial x$

$$\frac{\partial V(x,t)}{\partial x} = \psi_2 + O(t^{-1-q}).$$

Let us now prove the asymptotic formulas (3.6). For this let's multiply (3.12) on t^2. Integrating on $(0,t)$, using the formula of integrating by parts, Lemmas 3 and 4 we get

$$\int_0^t \tau^2 \frac{d}{d\tau}\int_0^1 \left(\frac{\partial U}{\partial \tau}\right)^2 dx d\tau + \int_0^t \tau^2 \int_0^1 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx d\tau \le C\int_0^t \tau^p d\tau,$$

$$c\int_0^t \tau^2 \int_0^1 \tau^p \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx d\tau \le -t^2 \int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + 2\int_0^t \tau \int_0^1 \left(\frac{\partial U}{\partial \tau}\right)^2 dx d\tau +$$

$$+Ct^{p+1} \le 2\int_0^t \tau \tau^{-2} d\tau + Ct^{p+1}.$$

It follows that if $p > -1$, then

$$\int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx d\tau \le Ct^{p+1}. \tag{3.14}$$

Multiplying the equation (3.11) scalarly on $t^3 \partial^2 U/\partial t^2$, applying the formula of integrating by parts, Schwarz's inequality and a priori estimates (3.5), (3.14) we get:

$$\int_0^1 t^3 \left(\frac{\partial^2 U}{\partial t^2}\right)^2 dx + \int_0^1 t^3 S^p \frac{\partial^2 U}{\partial t \partial x} \frac{\partial^3 U}{\partial t^2 \partial x} dx +$$

$$+p\int_0^1 t^3 S^{p-1} \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx \frac{\partial U}{\partial x} \frac{\partial^3 U}{\partial t^2 \partial x} dx = 0.$$

$$\int_0^t \int_0^1 \tau^3 \left(\frac{\partial^2 U}{\partial \tau^2}\right)^2 dx d\tau + \frac{1}{2}\int_0^t \int_0^1 \tau^3 S^p \frac{\partial}{\partial \tau}\left[\frac{\partial^2 U}{\partial \tau \partial x}\right]^2 dx d\tau +$$

$$+p\int_0^t \tau^3 S^{p-1} \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx \int_0^1 \frac{\partial U}{\partial x} \frac{\partial}{\partial \tau}\left[\frac{\partial^2 U}{\partial \tau \partial x}\right] dx d\tau = 0.$$

$$\frac{1}{2} t^3 S^p \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx \le \frac{3}{2}\int_0^t \int_0^1 \tau^2 S^p \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx d\tau +$$

$$+\frac{p}{2}\int_0^t \tau^3 S^{p-1} \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx \int_0^1 \left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx d\tau - pt^3 S^{p-1} \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 +\right.$$

$$\left. + \left(\frac{\partial V}{\partial x}\right)^2\right] dx \int_0^1 \frac{\partial U}{\partial x} \frac{\partial^2 U}{\partial t \partial x} dx + 3p\int_0^t \tau^2 S^{p-1} \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 +\right.$$

$$+\left(\frac{\partial V}{\partial x}\right)^2\Bigg]dx\int_0^1\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial\tau\partial x}dx+p(p-1)\int_0^t\tau^3S^{p-2}\left\{\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\right.\right.$$

$$\left.\left.+\left(\frac{\partial V}{\partial x}\right)^2\right]dx\right\}^2\int_0^1\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial\tau\partial x}dxd\tau+p\int_0^t\tau^3S^{p-1}\frac{\partial}{\partial\tau}\left(\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\right.\right.$$

$$\left.\left.+\left(\frac{\partial V}{\partial x}\right)^2\right]dx\right)\int_0^1\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial\tau\partial x}dxd\tau+p\int_0^t\tau^3S^{p-1}\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\right.$$

$$\left.+\left(\frac{\partial V}{\partial x}\right)^2\right]dx\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2dxd\tau\le C_1t^{p+1}+C_2t^{p+1}+$$

$$+\frac{1}{4}t^3S^p\int_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2dx+C_3t^3S^{p-2}+C_4\int_0^t\tau^2S^p\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2dxd\tau+C_4\int_0^t\tau^2S^{p-2}d\tau+$$

$$+C_5\int_0^t\tau^2S^p\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2dxd\tau+C_5\int_0^t\tau^4S^{p-4}d\tau+$$

$$+C_6\int_0^t\tau^3S^{p-1}\int_0^1\left(\left|\frac{\partial^2 U}{\partial\tau\partial x}\right|+\left|\frac{\partial^2 V}{\partial\tau\partial x}\right|\right)dx\int_0^1\left|\frac{\partial^2 U}{\partial\tau\partial x}\right|dxd\tau+C_7\int_0^t\tau^3S^{p-1}\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2dxd\tau.$$

From this take into account Lemma 3 and (3.14), using the Schwarz's inequality we have

$$\frac{c}{4}t^{p+3}\int_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2dx\le C_8t^{p+1}+C_9\int_0^t\tau^3S^{p-1}\left\{\int_0^1\left|\frac{\partial^2 U}{\partial\tau\partial x}\right|dx\right\}^2d\tau+$$

$$+C_9\int_0^t\tau^3S^{p-1}\int_0^1\left|\frac{\partial^2 V}{\partial\tau\partial x}\right|dx\int_0^1\left|\frac{\partial^2 U}{\partial\tau\partial x}\right|dxd\tau\le C_8t^{p+1}+$$

$$+C_{10}\int_0^t\tau^{p+2}\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2dxd\tau+C_{11}\int_0^t\tau^{p+2}\int_0^1\left(\frac{\partial^2 V}{\partial\tau\partial x}\right)^2dxd\tau+$$

$$+C_{11}\int_0^t\tau^{p+2}\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2dxd\tau.$$

Or at last

$$t^{p+3}\int_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2dx\le C_{12}t^{p+1}+C_{11}\int_0^t\tau^{p+2}\int_0^1\left(\frac{\partial^2 V}{\partial\tau\partial x}\right)^2dxd\tau. \qquad (3.15)$$

Multiplying the inequality (3.13) on $t^2 S^{p-q}$, using the formula of integrating by parts, Lemmas 3, 4 and estimate (3.5) we get:

$$\int_0^t \tau^2 S^{p-q} \frac{d}{d\tau} \int_0^1 \left(\frac{\partial V}{\partial \tau}\right)^2 dx d\tau + \int_0^t \tau^2 \int_0^1 S^p \left(\frac{\partial^2 V}{\partial \tau \partial x}\right)^2 dx d\tau \le C \int_0^t \tau^p d\tau,$$

$$c \int_0^t \tau^{p+2} \int_0^1 \left(\frac{\partial^2 V}{\partial \tau \partial x}\right)^2 dx d\tau \le -t^2 S^{p-q} \int_0^1 \left(\frac{\partial V}{\partial t}\right)^2 dx + 2 \int_0^t \tau S^{p-q} \int_0^1 \left(\frac{\partial V}{\partial \tau}\right)^2 dx d\tau +$$

$$+(p-q) \int_0^t \tau^2 S^{p-q-1} \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx \int_0^1 \left(\frac{\partial V}{\partial \tau}\right)^2 dx d\tau + C_1 t^{p+1} \le$$

$$\le \frac{C_2}{|p-q|} t^{p-q} + C_3 t^{p-q} + C_1 t^{p+1}.$$

I.e. if $q > -1$, then (3.15) gives

$$t^{p+3} \int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx \le C t^{p+1}.$$

So, it follows that

$$\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx \le C t^{-2}.$$

From this we obtain

$$\frac{\partial U(x,t)}{\partial t} = O(t^{-1}).$$

Analogously, if $p > -1$, then

$$\frac{\partial V(x,t)}{\partial t} = O(t^{-1}).$$

So, the proof of the main Theorem is over.

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