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ON THE ANALITIC REPRESANTATION OF ONE CLASS OF GEOMETRIC FIGURES, SURFACES AND LINES

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Abstract

Aim of this article is analitical reprezantation of one class of geometrical figurs, surfaces and lines. This class obteined bay identifying of the opposite ends of the prisms or cilinders under spetial conditions. This class of surfaces appear, when we study the problems of spreading of smoke-rings , also this class of lines describe the complicated orbit of some celestial objects. Very interesting this class figures and shells for sduty of some problems of elasticity theory, but in this article we give only analitic represantation of this figure or surfaces.

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Let $\mathbb{P}R_m \equiv A_1A_2\cdots A_mA'_1A'_2\cdots A'_m$ be an orthogonal prism, whose end-walls $A_1\cdots A_m$ and $A'_1\cdots A'_m$ are $\mathbb{P}_m$- regular m- symmetric plane figurs (in particulary polygons) and m is a number of its angels (verteces). In generally A_jA_{j+1} - edge of this figures are not straight line (for example this figure may be epicycloid or hypocycloid). OO' is axis of symmetry of this prism.

Definition. *Generelized Möbius Listing's body* GML^m_n is obtained by identifying of the opposite end-walls of the prism $\mathbb{PR}_m$ in such a way that:

A) for any $n\in Z$ and $i=\overline{1,m}$, each vertex A_i coincides with $A'_{i+n}\equiv A'_{mod_m(i+n)}$, and each edge A_iA_{i+1} coincides with the edge

$$A'_{i+n}A'_{i+n+1}\equiv A'_{mod_m(i+n)}A'_{mod_m(i+n+1)}$$

correspondingly.[1]

B) $n\in Z$ is a number of rotations of the end-walls of the prism with respect to the axis OO' before the identification.

if $n>0$ rotations are counter-clockwise, and if $n<0$ rotations are clockwise.

Remark 1. Each lateral face of the prism $\mathbb{PR}_m$ after transformation pass into the some surfase, wich is **lateral face** of Generelized Möbius Listing's body GML^m_n. Also each edge $A_jA'_j$ of prism for every $j=1,m-1$ after tramsformation pass into the some curve, wich is **edge** of the GML^m_n , but every edge A_jA_j+1 after transformation is vanish.

[1]If we have two numbers $m\in\mathbb{N}, n\in\mathbb{Z}$, then $n=km+i\equiv km+mod_m(n)$, where $k\in\mathbb{Z}$ and $i\equiv mod_m(n)\in\mathbb{N}\cup\{0\}$.

In particular, if m=2, and $\mathbb{PR}_2 \equiv A_1A_2A_1'A_2'$ is a rectangle, A_1A_2 is a segment of the straight line, and GML_1^2 becomes a classical Möbius band (see for example [1-3]); also he have one edge and one surface, (classical one sided surfase), GML_0^2 is a cylinder, cone, frustzum of a cone or a ring. But may be A_1A_2 is a epicycloid were $m = 2$ and in this case GML_0^2 is torus-shaped body wiht radial crossection - epicycloid.

Definition. Some body bounded by **k-colored surface** - if it is possible to paint the exterior (or interior) lateral faces of this surface in k different colours without taking away of the brush. It is prohibited to cross the edges of this surface.

I. In this part of the article we give parametric represantation of the GML_n^m under the following restrictions:

i) middle line OO' transforms in the some plane closed courve;

ii) the end rotation is semi-regular.

Let

$$\begin{cases} x = p(\tau, \psi), \\ z = q(\tau, \psi) \end{cases} \tag{1}$$

parametric represantation of the regular m - symmetric figures $\mathbb{P}_m$, where $(\tau, \psi) \in Q \subset$ $\subset \mathbb{R}^2$, such that $p(0,0) \equiv q(0,0) = 0$, the point $(0,0)$ be a center of symmetry of the $\mathbb{P}_m$.

Let

$$\mathbb{L}_\rho = \begin{cases} x = f_1(\rho, \theta) \\ y = f_2(\rho, \theta) \end{cases} \tag{2}$$

be some one-parametric familly of closed courves, morever:

a) for every fixed $\rho \in [0, \rho^*]$, L_ρ is a closed courve and $f_i(\rho, \theta + 2\pi) = f_i(\rho, \theta)$, $i =$ $= \overline{1, 2}$

b) for any $\rho_1, \rho_2 \in [0, \rho^*]$, $\rho_1 \neq \rho_2$, courves $\mathbb{L}_{\rho_1}$ and $\mathbb{L}_{\rho_2}$ have not common points.

Let

$$g(\theta) : [0, 2\pi] \to [0, 2\pi] \tag{3}$$

be arbitrary functions and for every

$\Phi \in [0, 2\pi]$ exist $\theta \in [0, 2\pi]$ such that $\Phi = g(\theta)$.

Let

$$\Omega = \{(x, z, \theta) \in \mathbb{R}^3; \ \ (x, z) \in \mathbb{P}_m, \ \ 0 \le \theta < 2\pi R\}$$

and

$$\Omega^* = \{(\tau, \psi, \theta) \in \mathbb{R}^3; \ \ (\tau, \psi) \in Q, \ 0 \le \theta < 2\pi R\}.$$

Theorem 1. *The transformation $F : \Omega^* \to GML_n^m$ with*

$$F = \begin{cases} x(\tau,\psi,\theta) &= f_1\left(\left(R + \rho(\tau,\psi)\cos\dfrac{ng(\theta)}{mR} - q(\tau,\psi)\sin\dfrac{ng(\theta)}{mR}\right), \dfrac{\theta}{R}\right), \\ \varphi(\tau,\psi,\theta) &= f_2\left(\left(R + \rho(\tau,\psi)\cos\dfrac{ng(\theta)}{mR} - q(\tau,\psi)\sin\dfrac{ng(\theta)}{mR}\right), \dfrac{\theta}{R}\right), \\ z(\tau,\psi,\theta) &= \rho(\tau,\psi)\sin\dfrac{ng(\theta)}{mR} + q(\tau,\psi)\cos\dfrac{ng(\theta)}{mR}, \end{cases} \tag{4}$$

where $(\tau, \psi, \theta) \in \Omega^*$, is parametric represantation of GML_n^m. R is a arbitrary positive number, but $R > \rho(0, A_i)$ - is a distance between center of symmetry of the polygon $\mathbb{P}_m$

and its vertex A_i - if $\mathbb{P}_m$ - is convex domain. If figure $\mathbb{P}_m$ is non-convex, then R is an arbitrary positive number, which is greater then distance bitween center of symmetry of figure and maximaly distant points of its boundary.

Remark 2. If (1) is a parametric represantation of an arbitrary plane figure, then in formula (4) $m \equiv 1$, for any $n \in Z$.

In this case if prism $\mathbb{PR}_m$ have k different lateral face, then GML_n^1 have k - colored external (or internal) surface. Also if prism $\mathbb{PR}_m$ have l different lateral edges, then GML_n^1 have l different edges;

If figure $\mathbb{P}_m$ have a smooth boundary, then $l = 0$.

Remark 3. If $\mathbb{P}_\infty$ is a disk, then in formula (4) $m \equiv 1$ and n is an arbitrary real number.In this case GML_n^∞ have a one colored external (or internal) surface and have not edge.

Remark 4. If (τ_0, ψ_0) is an arbitrary fixed point of $\partial\mathbb{P}_\infty$ (circle), then transformation

$$l_n(\theta) = (x(\tau_0, \psi_0, \theta), y(\tau_0, \psi_0, \theta), z(\tau_0, \psi_0, \theta)),$$

when $-\infty < \theta < \infty$, is a courve lying on the GML_n^∞ (in particulary torus).

a) If $n \in Z$, the $l_n(\theta) = l_n(\theta + 2\pi)$ is a closed courve, and n is a number of coils around of little parts of the GML_n^∞.

b) If $n = \dfrac{1}{k}, k \in Z$, then $l_n(\theta) = l_n(\theta + 2\pi k)$ is a closed courve, but after k rotations around of big parts of the GML_n^∞ we have only one coil around of little part of the GML_n^∞.

c) If $n = \dfrac{p}{k},\ p, k \in Z$, then $l_n(\theta) = l_n(\theta + 2\pi k)$ is a closed courve, and after k rotations around of big parts of the GML_n^∞ we have p coils around of little part of the GML_n^∞. d) If $n \in R\backslash Q$ is irrational number, then $l_n(\theta)$ is nonclosed courve. This courve makes infinite coils after infinite circuits arournd the GML_n^∞, but this courve is not self-crossing.

Remark 5. If k is the greatest common divisor of m and $mod_m(n)$ then GML_n^m have k - colored surface. Also GML_n^m have k different edges (closed curves). In particular if $k = 1$, then Generelized Möbius Listing's body have a one colored surface.

Remark 6. It is different one or two colored surface and one or two sided surface. But when GML_1^2 a classical Möbius band is one sided and one colored surface, also this surface have only one edge. In this case $\mathbb{PR}_2 \equiv A_1A_2A_1'A_2'$ is a rectangle, A_1A_2 is a segment of the straight line.

Examples: A) If, in particular, middle line OO' transforms in the circle

$$x = f_1(\rho, \theta) = \rho\cos(\theta), \quad y = f_2(\rho, \theta) = \rho\sin(\theta)$$

and the end-wall rotation is evenly along the middle line

$$g(\theta) = \theta$$

then the transformation (4) have following form

$$F = \begin{cases} x(\tau,\psi,\theta) = \left(R + p(\tau,\psi)\cos\dfrac{n\theta}{mR} - q(\tau,\psi)\ \sin\dfrac{n\theta}{mR}\right)\cos\dfrac{\theta}{R}, \\ y(\tau,\psi,\theta) = \left(R + p(\tau,\psi)\cos\dfrac{n\theta}{mR} - q(\tau,\psi)\ \sin\dfrac{n\theta}{mR}\right)\sin\dfrac{\theta}{R}, \\ z(\tau,\psi,\theta) = p(\tau,\psi)\sin\dfrac{n\theta}{mR} + q(\tau,\psi)\cos\dfrac{n\theta}{mR}, \end{cases} \tag{5}$$

where $(\tau,\psi,\theta)\in\Omega^*$ and this is parametric represantation of GML_n^m.

If $m = 2$, and

$$\begin{array}{ll} x = p(\tau,\psi) = \tau\cos\psi_0 & \tau\in(-\tau^*,\tau^*) \\ z = q(\tau,\psi) = \tau\sin\psi_0 & \psi_0 = \text{const.} \end{array}$$

we have following cases:

1) if $\psi_0 = 0$ or $\psi_0 = \pi$, then $m = 2,\ n = 0,\ q(\tau,\psi)\equiv 0,\ p(\tau,\psi)\equiv\tau,\ -\tau^*<\tau<\tau^*$, then GML_0^2 is not tree dimensional body, but is a circular ring , two-sided and two-colored surfece, with two edges;

2) if $\psi_0 = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ then $m = 2,\ n = 0,\ p(\tau,\psi)\equiv 0,\ q(\tau,\psi)\equiv\tau,\ -\tau^*<\tau<\tau^*$, then GML_0^2 is a cylinder-surface , also two-sided and two-colored surfece, with two edges ;

3) if $\psi_0 \neq 0,\ \dfrac{\pi}{2},\pi,\dfrac{3\pi}{2}$, then

i) when $R = |\tau^*||\cos\psi_0|$ then GML_0^2 is a surface of cone.

ii) when $R > |\tau^*||\cos\psi_0|$ then GML_0^2 is a frustrum of a cone.

4) if $m = 2$, $n = 1$, then (5) is a parametric represantation of classic regular Möbius band (see for example [2]), one-sided and one colored surface with one edge;

5) if $m = 2$, n is even number, then $GML_n^2 \equiv M_n$ is Möbius-Listing's type surface (see [4]) which is one-sided surface and one colored surface with one edge; if n is an odd number, then $GML_n^2 = M_n$ is two-sided and two-colored surfece, with two edges.

B) (Limiting case) If $m = \infty$, then $\mathbb{PR}_\infty$ is circular cylinder and its end $\mathbb{P}_\infty$ is a disk

$$\begin{array}{ll} p(\tau,\psi) = \tau\cos\psi, & \tau\in(0,\tau^*), \\ q(\tau,\psi) = \tau\sin\psi, & \psi\in(0,2\pi). \end{array} \tag{6}$$

In this case transformation (4) has the following form:

$$F = \begin{cases} x(\tau,\psi,\theta) = \left(R + \tau\cos\psi\cos\dfrac{n\theta}{R} - \tau\sin\psi\sin\dfrac{n\theta}{R}\right)\cos\dfrac{\theta}{R}, \\ y(\tau,\psi,\theta) = \left(R + \tau\cos\psi\cos\dfrac{n\theta}{R} - i\sin\psi\sin\dfrac{n\theta}{R}\right)\sin\dfrac{\theta}{R}, \\ z(\tau,\psi,\theta) = \tau\cos\psi\sin\dfrac{n\theta}{R} + \tau\sin\psi\cos\dfrac{n\theta}{R}, \end{cases} \tag{7}$$

or

$$F = \begin{cases} x(\tau,\psi,\theta) = \left(R + \tau\cos\left(\psi + \dfrac{n\theta}{R}\right)\right)\cos\dfrac{\theta}{R} \\ y(\tau,\psi,\theta) = \left(R + \tau\cos\left(\psi + \dfrac{n\theta}{R}\right)\right)\sin\dfrac{\theta}{R} \\ z(\tau,\psi,\theta) = \tau\sin\left(\psi + \dfrac{n\theta}{R}\right) \end{cases}$$

where n is any real number.

Remark 7. If $n=0$, formula (4) gives a parametric represantation of the classical torus (see, e.g., [3]).

a) Torus have one colored exterior surface without edges.

b) If $\tau\in(\tau_1,\tau^*)$ then (7) gives a parametric represantation of the torus-shell, with thickness $\tau^*-\tau_1$.

C) If in formula (6) plane figure $\mathbb{P}_4$ - is a rectangle, then we have two different possibility:

1) Edges of rectangle are parallel to the coordinate axis:

$$\begin{cases} x=\tau, & \tau\in(-\tau^*,\tau^*) \\ z=\psi & \psi\in(-\psi^*,\psi^*) \end{cases} \tag{8}$$

in this case body

a)- GML_0^4- is a cilinder, with height $2\psi^*$ and thickness $2\tau^*$. This body have 4-colored exterior or interior surface and 4 edges ;

2)Vertex of square lying on the coordinate axis:

$$\begin{cases} x=\dfrac{\tau}{|\sin\psi|+|\cos\psi|}\cos\psi & \tau\in(0,\tau^*) \\ z=\dfrac{\tau}{|\sin\psi|+|\cos\psi|}\sin\psi & \psi\in(0,2\pi) \end{cases} \tag{9}$$

a*)- GML_0^4- is torus-shaped body, but is not cilinder, with cross-section - square. This body have 4-colored exterior or interior surface and 4 edges ;

Two following examples are common for both cases and when in (8) $\tau^*=\psi^*$:

b)- GML_1^4 or GML_3^4 - is body with one colored exterior or interior surface and one edge;

c)- GML_2^4- is body with two colored exterior or interior surface and two edge;

II. In this part of the articl we give some examples of GML_n^m and its parameric represantation when

i') middle line 00' transforms in one class of tree dimensional courves (closed or nonclosed).

1) If middle line $00'$ transforms in lemniscate

$$\begin{aligned} x&=R\sqrt{2\cos2\theta}\cos\theta && 0\le\theta\le2\pi \\ y&=R\sqrt{2\cos2\theta}\sin\theta && R=\text{const}>0 \\ z&=K\sin\frac{\theta}{2} && K=\text{const}>0 \end{aligned}$$

($K=0$ classical lemniscate of Bernulli, but this courve is self-crossing); then parametric represantation of the lody GML_n^m is

$$\begin{cases} x(\tau,\psi,\theta)=\left(R+p(\tau,\psi)\cos\dfrac{n\theta}{mR}-q(\tau,\psi)\sin\dfrac{n\theta}{mR}\right)\sqrt{2\cos2\theta}\cos\theta \\ y(\tau,\psi,\theta)=\left(R+p(\tau,\psi)\cos\dfrac{n\theta}{mR}-q(\tau,\psi)\sin\dfrac{n\theta}{mR}\right)\sqrt{2\cos2\theta}\sin\theta \\ x(\tau,\psi,\theta)=K\sin\dfrac{\theta}{2}+p(\tau,\psi)\sin\dfrac{n\theta}{mR}+q(\tau,\psi)\cos\dfrac{n\theta}{mR}. \end{cases}$$

This tree-dimensional body is not self-crossing if in addition of $\mathbb{P}_m$ is smaller then K.

2) If middle line $00'$ transforms in line (see remark 4) lying on the classical torus

$$\begin{cases} x = (R + \tau_0 \cos(\psi_0 + n\theta)) \cos\theta, & \tau_0 = \text{const}, \quad \psi_0 = \text{const}, \\ y = (R + \tau_0 \cos(\psi_0 + n\theta)) \sin\theta, & \theta \in (-\infty, \infty) \\ z = \tau_0 \sin(\psi_0 + n\theta). & n \in \mathbb{R}^1 \end{cases}$$

In this case GML_e^∞ represented by following formula

$$\begin{cases} x = (R + \tau_0 \cos(\psi_0 + n\theta) + \tau \cos(\psi_0 + \psi + n\theta + l\theta)) \cos\theta \\ y = (R + \tau_0 \cos(\psi_0 + n\theta) + \tau \cos(\psi_0 + \psi + n\theta + l\theta)) \sin\theta \\ x = \tau_0 \sin(\psi_0 + n\theta) + \tau \cos(\psi_0 + \psi + n\theta + l\theta) \end{cases}$$

where $\tau \in (0, \tau^*),\ \psi \in (0, 2\pi),\ \theta \in (-\infty, \infty), l \in \mathbb{Z}$.

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