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EFFECTIVE SOLUTION OF THE MIXED BOUNDARY VALUE PROBLEM FOR AN INFINITE ISOTROPIC PLANE WITH AN ELLIPTICAL HOLE

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Applying the represtation of the stress vector the so-called mutually adjount vector-functions [4] we obtain an explicit solution to the basic boundary value problem of the elastic mixture theory for an infinite isopropic plane with elliptic hole.

As is known, a homogenoues system of equations of the elastic mixture theory is written as [4]

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} + \varepsilon^T \frac{\partial^2 \bar{U}}{\partial \bar{z}^2} = 0 \quad U = \{u_1 + iu_2, \ \ u_3 + iu_4\}^T, \tag{1}$$

where $u_p, \ \ p = \overline{1,4}$ are components of the partial displacement vector.

$$\varepsilon^T = -\frac{1}{2} e m^{-1}, \ \ e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \ \ m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \ \ m_k = e_k + 0,5 e_{3+k}, \tag{2}$$

$k = 1, 2, 3$; the $e_q, \ \ q = \overline{1,6}$ are expressed in terms of the elastic constants [4].

Using analogues of the general Kolosov-Muskheliscvili represtations from [4] we can write

$$U = m\varphi(z) + \frac{1}{2} e z \overline{\varphi'}(z) + \bar{\psi}(z), \quad TU = \frac{\partial}{\partial s(x)} \left(-2\varphi(z) + 2\mu U(x)\right), \tag{3}$$

where $TU = \{(TU)_2 - i(TU)_1, \ \ (TU)_4 - i(TU)_3\}^T, \ \ (TU)_p, \ \ p = \overline{1,4}$ are components of the stress vector, $\varphi = \{\varphi_1, \varphi_2\}^T$ and $\psi = \{\psi_1, \psi_2\}^T$ are arbitrary analytics vector-functions, $\mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}$, μ_k $(k = 1, 2, 3)$ are elastic constants, $\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}$, $n = (n_1, n_2)$ is an arbitrary unit vector.

From (3) we obtain

$$mTU = \frac{\partial}{\partial s(x)} [(A^T - E)U - iV], \ A^T = \begin{bmatrix} A_1 & A_3 \\ A_2 & A_3 \end{bmatrix} = 2m\mu; \tag{4}$$

where $V = i\left[-m\varphi(z) + \frac{1}{2} e z \overline{\varphi'(z)} + \bar{\psi}(z)\right]$ is a vector adjount to U [4].

Let an infinite isotropic plate be weakend by an elliptic hole with the semi-axes a and b $(a > b)$. This unbounded domain we denote by D^-. The symmetry axes of the ellipse are taken at the coordinate axes, and the major axis consiides with the real axis ox_1. By L we denote the elliptic curve under consideration. Denote by L_1 that part of the ellipse whith is located in the upper $(x_2 > 0)$ halt-plane and by L_2 the part located in the lower $(x_2 < 0)$ halt-plane.

Consider the following boundary value problem. Define a stress state in the unbounded domain D^-

$$\begin{aligned} &m(TU)^-_{L_1} = \frac{F(\psi_0)}{\sqrt{a^2\sin^2\psi_0 + b^2\cos^2\psi_0}}, \quad 0 \le \psi_0 \le \pi, \\ &(U)^-_{L_2} = f(\psi_0), \quad \pi \le \psi_0 \le 2\pi, \end{aligned} \tag{5}$$

where $(a\cos\psi_0, b\sin\psi_0) \in L$, $f \in C^{0,\alpha}[\pi, 2\pi]$, $\alpha > 0$, $f'(\psi_0)$ and $F(x_0)$ belong to the H^* class on $(\pi, 2\pi)$ and $(0, \pi)$, respectively [5].

The displacement vector for an infinite plane with an elliptic hole will be sought in the form [1]

$$U(z) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{(1-\tau_1\bar\tau_1)g(\varphi_0)}{1-(\tau_1 e^{i\varphi_0} + \bar\tau_1 e^{-i\varphi_0}) + \tau_1\bar\tau_1} - \frac{A_0\varepsilon^T\bar\tau_1 e^{-i\varphi_0}\bar g(\varphi_0)}{(1-\bar\tau_1 e^{-i\varphi_0})^2}\right]d\varphi_0, \tag{6}$$

where $g = (g_1, g_2)$ is an unknown complex vector, ε^T is define by (2), $A_0 = (1 - -\eta_1\bar\eta_1)(\bar\eta_1^{-1} - \eta_2)(\bar\eta_1 - \bar\eta_2)^{-1}$, $\eta_{1,2} = (z \pm \sqrt{z^2 - a^2 + b^2}(a+b)^{-1}$, $\tau_1 = \eta_1^{-1}$, $z = x_1 + ix_2$.

If we calculate the boundary value of the displacement vector by formula (6) at the point $(a\cos\psi_0, b\sin\psi_0)$, $\pi \le \psi_0 \le 2\pi$, of the contour L_2 and take into account the fact that $\tau_1 = e^{-i\psi_0}$, $\bar\tau_1 = e^{i\psi_0}$, then we obtain $g(\psi_0) = (U)^-_{L_2} = f(\psi_0)$, $\pi \le \psi_0 \le 2\pi$.

Consequently, we have defined g on L_2. It remains to define it on L_1 that g would belong to the Hölder class on the entire contour of the ellipse.

Using now (4) and taking into account that $V^- = \frac{1}{2\pi}\int_0^{2\pi} ctg\frac{\varphi_0 - \psi_0}{2}g(\varphi_0)d\varphi_0$, for the boundary value of the stress vector we obtain

$$\begin{aligned} &(A^T - E)g(\varphi_0) + \frac{1}{2\pi i}\int_0^{2\pi} ctg\frac{\varphi_0 - \psi_0}{2}g(\varphi_0)d\varphi_0 = \\ &= m\int_0^{\psi_0}(TU)^-\sqrt{a\sin^2\varphi_0 + b^2\cos^2\varphi_0}d\varphi_0 + c, \end{aligned} \tag{7}$$

where $c = (c_1, c_2)$ is an arbitrary constant vector.

Since $g(\varphi_0)$ for $\pi \le \varphi_0 \le 2\pi$ is known, after the transformartion $t = e^{i\psi_0}$, $\tau = e^{i\varphi_0}$ in view of the obvious identity $\frac{1}{2}ctg\frac{\varphi_0 - \psi_0}{2} = \frac{d\tau}{\tau - t} - \frac{d\tau}{\tau}$, the relation (7) for $0 \le \varphi_0 \le \pi$ will take the form

$$(A^T - E)g(t) + \frac{1}{\pi i}\int_{L^+}\frac{g(\tau)}{\tau - t}d\tau = X(t) + B, \quad t \in L^+, \tag{8}$$

where L^+ is the unit semi-circumference located in the upper half-plane

$$\chi(t) = \{\chi_1, \chi_2\}^T = m \int\limits_0^{\psi_0} F(\varphi_0) d\varphi_0 - \frac{1}{\pi i} \int\limits_{L^-} \frac{f(\tau)}{\tau - t} d\tau, \quad 0 \leq \psi_0 \leq \pi, \tag{9}$$

$$B = \{B_1, B_2\}^T = c + \frac{1}{2\pi i} \int\limits_{L^+} \frac{g(\tau) d\tau}{\tau} + \frac{1}{2\pi i} \int\limits_{L^-} \frac{f(\tau) d\tau}{\tau}$$

is an arbitrary constant vector, and L^- is the unit semi-circumference located in the lower half-plane.

Since f is Hölder continuous, therefore at the ends of the line L_+ vector χ has a logaritmic singularity and belongs to the class H_ε^*[5].

Thus with respect to the vector $g(t)$ for $t \in L^+$ we have obtained a charasteristic system of singular integral equations of the type (8). This system is a normal type since $\det A^T > 0$ and $\det(A^T - 2E) > 0$ (see [3]).

By the method given in [2] we can reduced system (8) to the system of the equations with respect to the scalar unknown functions $h_j = g_1 + y_j g_2$, $j = 1, 2$ on the contour L^+

$$\begin{gathered} p_j h_j + \frac{1}{\pi^2} \int\limits_{L^+} \frac{h_j(\tau)}{\tau - t} d\tau = R_j + Q_j; \\ R_j = \chi_1 + y_j \chi_2, \quad Q_j = B_1 + y_j B_2, \quad j = 1, 2; \end{gathered} \tag{10}$$

where $p_j = 0,5 \left[A_1 + A_4 - 2 - (-1)^j \sqrt{(A_1 + A_4) + 4A_2A_3} \right]$, $p_j + 1 > 0$, $p_j - 1 < 0$, $j = 1, 2$, y_1 and y_2 are roots of the equation $A_2 y^2 + (A_1 - A_4) y - A_3 = 0$.

According to the general theory (see [5]) define the character of the end points of the line of integration. Introduce the number $\gamma_j = \frac{1}{2\pi} \ln \frac{p_j - 1}{p_j + 1} = \frac{1}{2} + i\beta$; $\beta_j = \frac{1}{2\pi} \ln \left| \frac{p_j + 1}{p_j - 1} \right|$, $j = 1, 2$.

Hence the ends of the line L^+ are nonsingular. Solutions of the equations (10) will be sought in the class of functions wich are bounded at the ends. The canonical solutions of the corresponding homogeneous problem of linear conjugation will have the form $X_0(z) = (z+1)^{\gamma_j} (z-1)^{1-\gamma_j}$, $z = x_1 + i x_2$, where the branch is considered which is holomorphic on the plane cut along L^+, and satisfies $z^{-1} \overset{(j)}{X_0}(z) \to 1$ as $z \to \infty$.

Hence we can see that the index of that class is equal -1, and the solutions of the equations (10) have the form (see [5])

$$h_j = \frac{1}{p_j^2} \left[p_j (R_j + Q_j) - \frac{\overset{(1)}{X_0^+}(t)}{\pi i} \int\limits_{L^+} \frac{R_j(\tau) + Q_j}{\overset{(j)}{X_0^+}(\tau)(\tau - t)} d\tau \right], \quad j = 1, 2, \tag{11}$$

proveded the conditions of solvability $\int\limits_{L^+} (R_j(\tau)+Q_j)\left(\overset{(j)}{X_0^+}(\tau)\right)^{-1} d\tau = 0$, $j=1,2$, are satisfied, where $\overset{(j)}{X_0^+}$ is the boundary value of the function $\overset{(j)}{X_0}(z)$ on L^+ from the left.

Taking into account $\int\limits_{L^+} \left(\overset{(j)}{X_0^+}(\tau)\right)^{-1} d\tau = -\pi i(1+p_j)$, $j=1,2$, we can conclude that the solvability conditions for the equations (10) in the class of functions, which are bounded at the ands, are fulfilled.

Having found the functions h_j, $j=1,2$, we can define the vector g on L^+ as $g=(g_1,g_2)^T$, where $g_1=(h_1y_2-h_2y_1)(y_2-y_1)^{-1}$, $g_2=(h_1-h_2)(y_1-y_2)^{-1}$.

Using the results obtained in [5], the vector g is proved to the belong to the Hölder class on the entire contour of the L.

Thus the mixed $BVPs$ is solved. Obviously, the general case where the ellipse is divided into several parts, can be solved in a similar way.

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