Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 29, 2003

ON THE INFLUENCE OF A CALCULATION ERROR ON THE SOLUTION OF THE KIRCHHOFF NONLINEAR STATIC EQUATION

Abesadze T., Odisharia V., Peradze J.

I. Vekua Institute of Applied Mathematics,
Department of Applied Mathematics and Computer Sciences of TSU

Reported: 5.06.2002; received: 12.06.2003 ; revised: 20.09.2003

Abstract

The stability of a method of the solution of an integro-differential equation for a string is studied.

Key words and phrases: Kirchhoff nonlinear equation, string, method error, estimate
AMS subject classification: 519.642.2

1. Formulation of the Problem

Let us consider the Kirchhoff equation

$$\Big(1+\delta\int_0^1 (w')^2\,dx\Big)w''=f,\quad 0<x<1, \tag{1.1}$$

describing a static state of a string [1]. Here $w=w(x)$ is the deflection function to be found, $f=f(x)$ is a given load function, $\delta>0$ is the known mechanical parameter.

Let the boundary condition

$$w(0)=\alpha,\quad w(1)=\beta \tag{1.2}$$

be fulfilled.

We write the function $w(x)$ in the form

$$w(x)=W(x)+(\beta-\alpha)x+\alpha, \tag{1.3}$$

where $W(x)$ is a solution of the problem

$$\Big[1+\delta\int_0^1 (W'+\beta-\alpha)^2\,dx\Big]W''=f,\quad 0<x<1, \tag{1.4}$$

$$W(0)=0,\quad W(1)=0. \tag{1.5}$$

2. Method of Solution

To find $W(x)$ we will use the method proposed in [2, 3]. Let us briefly describe its essence. The function $W(x)$ is written in the form

$$W(x) = \lambda\nu(x), \tag{2.1}$$

where λ and $\nu(x)$ are the parameter and the function to be found. Substituting (2.1) into (1.4), we obtain

$$\lambda\Big[1+\delta\int_0^1 (\lambda\nu' + \beta - \alpha)^2\,dx\Big]\nu'' = f. \tag{2.2}$$

Equality (2.2) is replaced by the system of equations

$$\nu'' = f,$$

$$\lambda\Big[1+\delta\int_0^1 (\lambda\nu' + \beta - \alpha)^2\,dx\Big] = 1.$$

As follows from (1.5) and (2.1), the function $\nu(x)$ vanishes on the boundary and hence we obtain the equality $\int_0^1 \nu'\,dx = 0$. Therefore the function $\nu(x)$ and the parameter λ satisfy the relations

$$\nu'' = f, \tag{2.3}$$

$$\nu(0) = 0, \quad \nu(1) = 0 \tag{2.4}$$

and

$$\lambda[1+\delta(\lambda^2 s + c^2)] = 1, \tag{2.5}$$

where

$$s = \int_0^1 (\nu')^2\,dx, \tag{2.6}$$

$$c = \beta - \alpha.$$

Thus problem (1.4),(1.5) has been reduced to finding first the function $\nu(x)$ from (2.3),(2.4) and then the parameter λ from (2.5).

Once the parameter λ and the function $\nu(x)$ have been obtained, the solution of the initial problem (1.1),(1.2) is determined due to (1.3) and (2.1) by the relation

$$w(x) = \lambda\nu(x) + (\beta-\alpha)x + \alpha. \tag{2.7}$$

From the fact that problem (2.3),(2.4) has a unique solution and the cubic equation (2.5) satisfies the uniqueness condition of a real root [4] it follows that the function $w(x)$ constructed by (2.7) is a solution of problem (1.1),(1.2).

Let the function $f(x)$ have a sufficient degree of smoothness. The solution of problem (2.3),(2.4) is written as

$$\nu(x) = (x-1)\int_0^x \zeta f(\zeta)\,d\zeta + x\int_x^1 (\zeta-1)f(\zeta)\,d\zeta, \tag{2.8}$$

while the parameter s, which participates in the derivation of the cubic equation (2.5), is calculated on the strength of (2.6) and (2.8) by

$$s = \int_0^1 \Big[\int_0^x \zeta f(\zeta)\,d\zeta + \int_x^1 (\zeta-1)f(\zeta)\,d\zeta\Big]^2 dx. \tag{2.9}$$

3. Method Error and Its Estimation

The method described above gives an exact solution of equation (1.1) if the boundary condition (1.2) is satisfied and all calculation stages are carried out without any error. However it is natural to assume that integrals (2.8) and (2.9) are found by means of quadrature formulas, i.e. approximately. This circumstance affects the final result. Let us find out how sensitive the method is to an error of the above-mentioned kind.

Note that if round-off errors are not taken into account, in the considered case there are no other reasons for a change of the result .

So, let the realization of (2.8) lead not to an exact solution $\nu_*(x)$, but to its approximation $\widetilde{\nu}(x)$. Analogously, assume that calculations by (2.9) have been carried out approximately. Then in equation (2.5) the exact value of the parameter s_* determined by equality (2.9) is replaced by its approximate value $\widetilde{s}$. As a result, we obtain not the exact solution λ_* of equation (2.5) but its approximation $\widetilde{\lambda}$. Thus formula (2.7) leads not to the exact solution $w_*(x)$ of problem (1.1), (1.2) but to its approximate value $\widetilde{w}(x)$.

From (2.7) we have

$$|\widetilde{w}(x) - w_*(x)| \le |\widetilde{\lambda} - \lambda_*||\widetilde{\nu}(x)| + |\lambda_*||\widetilde{\nu}(x) - \nu_*(x)|. \tag{3.1}$$

Denote by $\lambda(s)$ the functional dependence of the parameter λ on s defined by formula (2.5). Since

$$\widetilde{\lambda} - \lambda_* = \lambda'(\eta)(\widetilde{s} - s_*), \tag{3.2}$$

where η is some point of a segment with boundary points $\widetilde{s}$ and s_*, we have

$$|\widetilde{\lambda} - \lambda_*| \le M\Delta s. \tag{3.3}$$

Here Δs is the upper boundary of the error $|\widetilde{s} - s_*|$, while according to (3.2) and (2.6) the value M is determined by the equality

$$M = \max_s |\lambda'(s)|, \tag{3.4}$$

where $\max(0, \widetilde{s} - \Delta s) \leq s \leq \widetilde{s} + \Delta s$.

Taking (3.3) into account in (3.1), we obtain

$$|\widetilde{w}(x) - w_*(x)| \leq M\Delta s|\widetilde{\nu}(x)| + (|\widetilde{\lambda}| + M\Delta s)\Delta\nu(x) \leq$$

$$\leq M\Delta s(|\widetilde{\nu}(x)| + \Delta\nu(x)) + |\widetilde{\lambda}|\Delta\nu(x), \tag{3.5}$$

where $\Delta\nu(x)$ is the upper boundary of the error $|\widetilde{\nu}(x) - \nu_*(x)|$.

4. Calculation of the Parameter M

Using the Cardano formula for a real root [4] in (2.5), we get the function

$$\lambda(s) = p_1 + p_2,$$

which, after differentiation, can be written as

$$\lambda'(s) = \frac{1}{2}\Big[\frac{1}{s}(p_1^2 + p_2^2) + \delta\frac{q_2^2}{q_1}(p_1^2 - p_2^2)\Big]. \tag{4.1}$$

Here we have used the notation

$$\begin{aligned} p_k &= \Big\{\frac{1}{2\delta s} + (-1)^k\Big[\Big(\frac{1+\delta c^2}{3\delta s}\Big)^3 + \Big(\frac{1}{2\delta s}\Big)^2\Big]^{\frac{1}{2}}\Big\}^{\frac{1}{3}}, \\ q_k &= \Big[3^{k-1}\Big(\frac{1+\delta c^2}{3\delta s}\Big)^3 + 2^{k-1}\Big(\frac{1}{2\delta s}\Big)^2\Big]^{\frac{1}{2}}, \end{aligned} \tag{4.2}$$

$$k = 1, 2.$$

We cannot find M by means of (3.4) and (4.1),(4.2) and have to resort to a differnt way to solve this question.

Since to each s there corresponds a unique λ, there exists an inverse function to $\lambda(s)$, which we denote by $s(\lambda)$. From (2.5) it follows that

$$s(\lambda) = \frac{1}{\delta}\Big(\frac{1}{\lambda^3} - \frac{1+\delta c^2}{\lambda^2}\Big). \tag{4.3}$$

From (2.5) and (2.6) we obtain

$$0 < \lambda \leq \frac{1}{1+\delta c^2} \tag{4.4}$$

and $s \geq 0$, while (4.3) and (4.4) imply

$$s'(\lambda) = \frac{2}{\delta}\Big(\frac{1+\delta c^2}{\lambda^3} - \frac{3}{2\lambda^4}\Big) < 0, \tag{4.5}$$

$$s''(\lambda) = \frac{6}{\delta}\Big(\frac{2}{\lambda^5} - \frac{1+\delta c^2}{\lambda^4}\Big) > 0.$$

Thus $s(\lambda)$ is a decreasing function with nonnegative values and $s'(\lambda)$ is an increasing function with nonnegative values.

Using these properties and the relation

$$\lambda'(s) = \frac{1}{s'(\lambda)}, \tag{4.6}$$

we see that if λ increases, then so does $s'(\lambda)$, and if s decreases, then $s'(\lambda)$ increases. Therefore if s decreases, then so does $\lambda'(s)$. This means that $\lambda'(s)$ is an increasing function.

Now, taking into account that by virtue of (4.5) and (4.6) $\lambda'(s) < 0$ and using (3.4), we arrive at the equality

$$M = -\lambda'(\max(0, \widetilde{s} - \Delta s)). \tag{4.7}$$

Substituting (4.7) into (3.5), we obtain the final estimate

$$|\widetilde{w}(x) - w_*(x)| \leq$$

$$\leq -\lambda'(\max(0, \widetilde{s} - \Delta s))\Delta s(|\widetilde{\nu}(x)| + \Delta\nu(x)) + |\widetilde{\lambda}|\Delta\nu(x). \tag{4.8}$$

About calculation of the right-hand side of (4.8). The function $\widetilde{\nu}(x)$ and the values $\widetilde{s}$ and $\widetilde{\lambda}$ are found by realization of the algorithm. The value of the function $\lambda'(s)$ at the point $\max(0, \widetilde{s} - \Delta s)$ is obtained by means of (4.1) and (4.2). A relatively rare case with $\widetilde{s} - \Delta s < 0$ should be treated individually. We have to calculate $\lambda'(0)$. Then formula (4.2) is not suitable. Our arguments are as follows: (4.3) and (4.5) imply that if $s = 0$, then $\lambda = 1/(1 + \delta c^2)$ and $s'(1/(1 + \delta c^2)) = -(1 + \delta c^2)^4/\delta$. Therefore from (4.6) we have $\lambda'(0) = -\delta/(1 + \delta c^2)^4$.

As to the question of finding the function $\Delta\nu(x)$ and the value Δs, we will discuss it in the next section.

5. Calculation of $\Delta\nu$ and Δs

Let the segment $[0, 1]$ be covered by a grid with pitch $h = 1/n$ and nodes $x_i = ih$, $i = 0, 1, \dots, n$. We will limit our consideration to the case where the solution of problem (1.1), (1.2) by the method described above must be known at the nodes of the introduced grid. Then in equality (2.7) x takes successively the values x_i, $i = 1, 2, \dots, n-1$. Suppose that to calculate integrals in this equality we use some quadrature formula, for instance, a general formula for trapezoids [5], where the nodes of the introduced grid are used. Assume that the same quadrature formula will be used for calculations by (2.8).

Simultaneously with finding $\Delta\nu$ and Δs, we will derive respective formulas for $\widetilde{\nu}(x)$ and $\widetilde{s}$ contained in estimate (4.8).

Note that the restrictions connected with a set of considered values of the variable x and with a choice of a quadrature formula are not of fundamental character.

So, to calculate the integral

$$I = \int_{x_{i_1}}^{x_{i_2}} \varphi(x)\, dx, \tag{5.1}$$

where $0 \le i_1 < i_2 \le n$ and $\varphi(x)$ is some twice continuously differentiable function on the integration segment, we use the general quadrature formula of trapezoids

$$I = h\Big(\frac{\varphi(x_{i_1}) + \varphi(x_{i_2})}{2} + \sum_{i=i_1+1}^{i_2-1} \varphi(x_i)\Big), \tag{5.2}$$

the remainder term of which is estimated by

$$|R| \le \frac{(x_{i_2} - x_{i_1})h^2}{12} \max_{x_{i_1} \le x \le x_{i_2}} |\varphi''(x)|. \tag{5.3}$$

We apply (5.1)–(5.3) first to (2.8) and obtain

$$\widetilde{\nu}(x_i) = (x_i - 1) \sum_{j=1}^{i} x_j f(x_j) + x_i \sum_{j=i+1}^{n-1} (x_j - 1) f(x_j).$$

Since the equality

$$[\psi(\zeta) f(\zeta)]'' = 2f'(\zeta) + \psi(\zeta) f''(\zeta), \tag{5.4}$$

where $\psi(\zeta) = \zeta,\ \zeta - 1$ is fulfilled for the second derivatives of the integrand function in (2.8), we obtain

$$\Delta\nu(x_i) \le \frac{x_i(1 - x_i)h^2}{12} \times$$

$$\times \Big(\max_{0 \le x \le x_i} |2f'(x) + xf''(x)| + \max_{x_i \le x \le 1} |2f'(x) + (x-1)f''(x)| \Big) \le$$

$$\le \frac{x_i(1 - x_i)h^2}{6} \Big(2 \max_{0 \le x \le 1} |f'(x)| + \max_{0 \le x \le 1} |f''(x)| \Big). \tag{5.5}$$

(5.5) contains two estimates for $\Delta\nu(x_i)$, one of which, more complicated than the other, is more exact.

Now about Δs and $\widetilde{s}$. Let us introduce into consideration the function

$$\psi(x) = \int_0^x \zeta f(\zeta)\, d\zeta + \int_x^1 (\zeta - 1) f(\zeta)\, d\zeta.$$

By (2.9)

$$s = \int_0^1 \psi^2(x)\, dx.$$

The integrals occurring in the expression for s are calculated by means of (5.2). The result representing the value of $\widetilde{s}$ is

$$\widetilde{s} = h\Big[\frac{\widetilde{\psi}^2(x_0) + \widetilde{\psi}^2(x_n)}{2} + \sum_{i=1}^{n-1} \widetilde{\psi}^2(x_i)\Big].$$

Here

$$\widetilde{\psi}(x_i) = h\Big[\sum_{j=1}^{i} x_j f(x_j) - \frac{1}{2} f(x_i) + \sum_{j=i+1}^{n-1} (x_j - 1) f(x_j)\Big], \tag{5.6}$$

$i = 0, 1, \dots, n$, is an approximate value of the integral

$$\psi(x_i) = \int\limits_0^{x_i} \zeta f(\zeta)\, d\zeta + \int\limits_{x_i}^{1} (\zeta - 1) f(\zeta)\, d\zeta, \tag{5.7}$$

which is found by means of the quadrature formula (5.2).

To estimate Δs, let us consider this value as a sum of two errors

$$\Delta s = \Delta s_1 + \Delta s_2, \tag{5.8}$$

the first of which is a consequence of the fact that the integral in (2.6) is calculated approximately, while the presence of the second error is explained by the fact that the integrand function in this integral is, as follows from (2.9), a square of the integral expression and thus in its turn is given by an error. More exactly, in (5.8) Δs_1 and Δs_2 are the upper boundaries of the error absolute values obtained respectively because we use the quadrature formula in (2.6) and because in this quadrature formula the values of the function at the nodes are determined by means of an approximate integration formula.

Taking into account (2.3), for the integrand function in (2.6) we have

$$[(\nu')^2]'' = 2\nu'\nu''' + 2(\nu'')^2 = 2(\nu' f' + f^2), \tag{5.9}$$

while from (2.8) we obtain the equality

$$\nu'(x) = \int\limits_0^{x} \zeta f(\zeta)\, d\zeta + \int\limits_{x}^{1} (\zeta - 1) f(\zeta)\, d\zeta,$$

which, after applying the theorem on the mean, leads to an estimate

$$|\nu'(x)| \le \frac{1}{2} \max_{0 \le x \le 1} |f(x)|. \tag{5.10}$$

Using (5.9), (5.10) and (5.3), we write

$$\Delta s_1 \le \frac{h^2}{6}\Big[\frac{1}{2} \max_{0\le x\le 1} |f(x)| \max_{0\le x\le 1} |f'(x)| + \big(\max_{0\le x\le 1} |f(x)|\big)^2\Big]. \tag{5.11}$$

To derive an estimate for Δs_2, we recall how an error of the function of $n+1$ variables $z = \sum\limits_{i=0}^{n} a_i y_i^2$, where $a_i > 0$ are coefficients, is estimated. If instead of exact values of the arguments y_i we use their approximate values $\widetilde{y}_i$, $i = 0, 1, \dots, n$, then instead of the exact value z of the function we obtain its approximate value $\widetilde{z}$, and moreover

$$|\widetilde{z} - z| \le$$

$$\leq \sum_{i=0}^{n} a_i(|\widetilde{y}_i| + |y_i|)|\widetilde{y}_i - y_i| \leq \sum_{i=0}^{n} a_i \sum_{k=1}^{2} (2|\widetilde{y}_i|)^{2-k} |\widetilde{y}_i - y_i|^k. \tag{5.12}$$

If $a_0 = a_n = h/2$, $a_i = h$, $i = 1, 2, \dots, n-1$, $y_i = \psi(x_i)$, $\widetilde{y}_i = \widetilde{\psi}(x_i)$, $i = 0, 1, \dots, n$, then $|\widetilde{z} - z| = \Delta s_2$ and $\widetilde{y}_i - y_i$ is an error that arises because of the replacement of the integral expression (5.7) by sum (5.6). Using (5.12), (5.3) and (5.4) we find

$$\Delta s_2 \leq h \sum_{i=0}^{n} \varepsilon_i \sum_{k=1}^{2} (2|\widetilde{\psi}(x_i)|)^{2-k} \Big[\frac{x_i h^2}{12} \max_{0 \leq x \leq x_i} |2f'(x) + x f''(x)| +$$

$$+ \frac{1 - x_i}{12} h^2 \max_{x_i \leq x \leq 1} |2f'(x) + (x-1) f''(x)| \Big]^k, \tag{5.13}$$

where the parameter ε_i is equal to $1/2$ for $i = 0$ and $i = n$, and is equal to 1 in other cases, while the values $\widetilde{\psi}(x_i)$ are calculated by (5.6) in the process of realization of the algorithm.

Summing (5.11) and (5.13) and taking into account (5.8), we obtain the desired estimate for Δs. Using the latter estimate, we can derive a less exact but more convenient estimate

$$\Delta s \leq \frac{h^2}{6} \Big[\frac{1}{2} \max |f(x)| \max |f'(x)| + (\max |f(x)|)^2 \Big] +$$

$$+ \sum_{k=1}^{2} (\max |f(x)|)^{2-k} \Big[\frac{h^2}{12} (2 \max |f'(x)| + \max |f''(x)|) \Big]^k,$$

where max implies everywhere the relation $0 \leq x \leq 1$.

REFERENCES

1. Kirchhoff G., Vorlesungen über Mechanik. Leipzig. Teubner, 1883.

2. Chipot M., Rodrigues J.-F., On a class of nonlocal elliptic problems. RAIRO Model. Math. Anal. Numer. 26, 1992, 447–467.

3. Chipot M., Lovat D., Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems. Advan. Quench. Dyn. Contin. Discr. Impuls. Syst. Series A. Math. Anal. 8(2001), no. 1, 35–51.

4. Kurosh A.G., A course of higher algebra. Fizmatgiz, Moscow, 1975.

5. Demidovich B.P., Maron I.A., Fundamentals of computational mathematics. Fizmatgiz, Moscow, 1963.