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THE SOLUTION SOME PROBLEMS OF TEMPERATURE STRESSES IN TRANSTROPIC PLATE

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Abstract

In the present paper consider a transtropic homogeneous plate with constant thickness $2h$. For finding temperature stresses in the mentionod body use $N = 1$ approximation of I. Vekua theory and solve some problems. Compere results to the results obtained by classical theory.

Key words and phrases: transtropic homogeneous plate, plane of isotropy, temperature stresses, equilibrium equation, thermal properties, thermal stream, modificed functions.
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Let the Cartesian coordinate system $0x_1x_2x_3$ is chosen following from the supposition that plane of isotropy and complex plane $z = x_1 + ix_2$ are the same. Equilibrium equation in components of displacement vector has the form (we consider the case when temperature field T is symmetric across the middle plane) [1]

$$\begin{cases} \mu\Delta \overset{(0)}{u}_+ + \dfrac{EE'}{(1-\nu)E' - 2\nu'^2 E}\partial_{\bar z}\left(\overset{(0)}{\theta} + \dfrac{2\nu'}{h}\overset{(1)}{u}_3\right) - 2\beta\partial_{\bar z}\overset{(0)}{T} = 0, \\ \mu'\Delta \overset{(1)}{u}_3 - \dfrac{3EE'}{(1-\nu)E' - 2\nu'^2 E}\dfrac{1}{h}\left(\nu'\overset{(0)}{\theta} + \dfrac{E'(1-\nu)}{Eh}\overset{(1)}{u}_3\right) + \dfrac{3\beta'}{h}\overset{(0)}{T} = 0, \end{cases} \tag{1}$$

where $\overset{(0)}{u}_+ = \overset{(0)}{u}_1 + i\overset{(0)}{u}_2$, $\quad \overset{(0)}{\theta} = \partial_z \overset{(0)}{u}_+ + \partial_{\bar z}\overline{\overset{(0)}{u}_+}$, $\quad \overset{(0)}{T}(x_1,x_2) = \dfrac{1}{2h}\displaystyle\int_{-h}^{h} T(x_1,x_2,x_3)dx_3.$

Here μ, E, ν - are correspondingly Lame, Young and Poisson constants in the plane of isotropy, while μ', E', ν' - are some constants in the direction perpendicular to the plane of isotropy. β, β' - are positive values dependent on the thermal properties of the body.

Let us consider stationar case and mean that $\overset{(0)}{T}$ satisfies the following equation $\Delta\overset{(0)}{T} = 0$.

The general solutions of the system (1) have the form

$$2\mu \overset{(0)}{u}_{+} = æ^{*}\varphi(z) - \overline{z\varphi'(z)} - \overline{\psi(z)} - \frac{\nu' E h}{3E'}\partial_{\bar z}\chi(z,\bar z) + \frac{(1-\nu)E'\beta - \nu' E\beta'}{2E'}\int \overset{(0)}{T}\,dz$$

$$2\mu' \overset{(1)}{u}_{3} = \chi(z,\bar z) - \frac{4\mu'\nu' h}{E'}(\varphi'(z) + \overline{\varphi'(z)}) + \frac{2\mu' h[(E' - \nu'^2 E)\beta' - \nu' E'(1+\nu)\beta]}{E'^2}\overset{(0)}{T},$$

here $æ^{*} = \dfrac{3-\nu}{1+\nu}$, $\varphi(z)$, $\psi(z)$ are analytic functions of variable z, $\chi(z,\bar z)$ is general solution of the following Helmholtz's equation

$$\Delta\chi - \eta^2\chi = 0, \quad \eta^2 = \frac{3E'^2}{\mu'(E' - \nu'^2 E)h^2}$$

Complex combinations of stresses are represented as follows

$$\overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{22} = 2\left[\varphi'(z) + \overline{\varphi'(z)}\right] + \frac{EE'\nu'}{2\mu' h(E' - \nu'^2 E)}\chi(z,\bar z) -$$

$$-\frac{(1-\nu)E'\beta - \nu' E\beta'}{E'}\overset{(0)}{T},$$

$$\overset{(0)}{\sigma}_{11} - \overset{(0)}{\sigma}_{22} + 2i\overset{(0)}{\sigma}_{12} = -2\left[z\overline{\varphi''(z)} + \overline{\psi'(z)} + \frac{E\nu' h}{3E'}\partial^2_{\bar z\bar z}\chi(z,\bar z) -\right.$$

$$\left.-\frac{(1-\nu)E'\beta - \nu' E\beta'}{2E'}\int \partial_{\bar z}\overset{(0)}{T}\,dz\right],$$

$$\overset{(1)}{\sigma}_{13} + i\overset{(1)}{\sigma}_{23} = \partial_{\bar z}\chi(z,\bar z) - \frac{4\mu'\nu' h}{E'}\overline{\varphi''(z)} +$$

$$+\frac{2\mu' h[(E' - \nu'^2 E)\beta' - \nu'(1+\nu)E'\beta]}{E'^2}\partial_{\bar z}\overset{(0)}{T}.$$

Let's consider a circular plate with radius R, having center in the bigining of Cartesian coordinate system. Let solve the problem with the following conditions:

$$\begin{cases} \overset{(0)}{\sigma}_{rr} - i\overset{(0)}{\sigma}_{r\theta} = \dfrac{1}{2}\left[\overset{(0)}{\sigma}_{11} + \overset{(0)}{\sigma}_{22} - e^{2i\theta}\left(\overset{(0)}{\sigma}_{22} - \overset{(0)}{\sigma}_{11} + 2i\overset{(0)}{\sigma}_{12}\right)\right] = 0, \quad r = R, \\ \overset{(1)}{\sigma}_{r3} = Re\left[\left(\overset{(1)}{\sigma}_{13} + i\overset{(1)}{\sigma}_{23}\right)e^{-i\theta}\right] = 0, \quad r = R, \quad z = re^{i\theta} \end{cases}$$

i.e.

$$\begin{cases} \varphi'(z) + \overline{\varphi'(z)} - \left[\overline{z}\varphi''(z) + \psi'\right] e^{2i\theta} + \dfrac{\nu' E E'}{4\mu' h (E' - \nu'^2 E)} \chi(z,\bar{z}) - \\ -\dfrac{\nu' E h}{3E'} \partial^2_{zz}(z,\bar{z}) = 0, \\ Re\left[(\partial_{\bar{z}}\chi(z,\bar{z}) - \dfrac{4\mu' h \nu'}{E'}\overline{\varphi''(z)})e^{i\theta}\right] = -\dfrac{h\mu'(\beta' - 2\nu')}{2E'}(F'(z)e^{i\theta} + \overline{F'(z)}e^{-i\theta}), \end{cases} \tag{2}$$

where $\frac{1}{2}(F(z) + \overline{F(z)}) = \overset{(0)}{T}$.

The unknown functions can be represented by the series

$$\begin{gathered} \varphi'(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad \psi'(z) = \sum_{n=0}^{+\infty} a'_n z^n, \\ \chi(z,\bar{z}) = \sum_{-\infty}^{+\infty} b_n I_n(\eta r) e^{in\theta}, \quad F'(z) = \sum_{n=0}^{+\infty} A_n z^n \end{gathered} \tag{3}$$

where $I_n(\eta r)$ is Bessel modificed functions. By subtituting (3) into (2) we obtain the system of equations

$$\begin{cases} \sum\limits_{n=0}^{+\infty}(1-n)R^n a_n e^{in\theta} + \sum\limits_{n=0}^{+\infty} R^n \overline{a_n} e^{-in\theta} - \sum\limits_{n=0}^{+\infty} R^n a'_n e^{i(n+2)\theta} + \\ + \dfrac{\nu' E E'}{2\mu' h \eta R (E' - \nu'^2 E)} \sum\limits_{-\infty}^{+\infty} (1-n) I_{n-1} b_n e^{in\theta} = 0, \\ \dfrac{\eta}{2} \sum\limits_{-\infty}^{+\infty} (I_{n+1} + I_{n-1}) b_n e^{in\theta} - \dfrac{4\mu' h \nu'}{E'} \left(\sum\limits_{n=0}^{+\infty} n R^{n-1} \overline{a_n} e^{-in\theta} + \sum\limits_{n=0}^{+\infty} n R^{n-1} a_n e^{in\theta} \right) = \\ = -\dfrac{h\mu'(\beta' - 2\nu'\beta)}{E'} \left(\sum\limits_{n=0}^{+\infty} R^n A_n e^{i(n+1)\theta} + \sum\limits_{n=0}^{+\infty} R^n \overline{A_n} e^{-i(n+1)\theta} \right) \end{cases}$$

We can determine all the coefficients

$$a_0 = 0, \quad b_0 = 0,$$

$$a_n = \frac{\nu' E E' (1+n)(\beta' - 2\nu'\beta) I_{n+1}}{\eta^2 R^2 E'(E' - \nu'^2 E)(I_{n+1} + I_{n-1}) + 4EE'\nu'^2 I_{n+1} n(1+n)} A_{n-1},$$

$$b_n = \frac{2\mu' h \eta R^{n+1}(E' - \nu'^2 E)(\beta' - 2\nu'\beta)}{\eta^2 R^2 E'(E' - \nu'^2 E)(I_{n+1} + I_{n-1}) + 4EE'\nu'^2 I_{n+1} n(1+n)} A_{n-1},$$

$$a^{'}_{n-2}=(1-n)R^2a_n+\frac{\nu^{'}E^{'}(1-n)I_{n-1}}{2\mu^{'}h\eta R^{n-1}(E^{'}-\nu^{'2}E)}A_{n-1}.$$

Let's consider the infinite plate with a circular hole. Assume the homogeneous thermal stream q is given in infinite along to the axis Ox_1. We consider stationar case too and mean that $\overset{(0)}{T}$ is the solution of following problem [2]

$$\begin{cases} \Delta \overset{(0)}{T}=0, \\ \dfrac{\partial \overset{(0)}{T}}{\partial r}|_{r=R}=0, \\ \overset{(0)}{T}_{\infty}=\dfrac{q}{2}(z+\bar{z}) \end{cases} \tag{4}$$

and has the form

$$\overset{(0)}{T}=\frac{q}{2}(z+\bar{z}+\frac{R^2}{z}+\frac{R^2}{\bar{z}}) \tag{5}$$

We mean that boundary of the plate is free from outer loads. We have the following boundary conditions

$$\begin{cases} \overset{(0)}{\sigma}_{rr}-i\overset{(0)}{\sigma}_{r\theta}=0, \quad r=R, \\ \overset{(1)}{\sigma}_{r3}=0, \quad r=R, \quad z=re^{i\theta}. \end{cases} \tag{6}$$

Let us introduce the functions $\varphi(z)$, $\psi(z)$, and $\chi(z,\bar{z})$ by the series

$$\begin{gathered} \varphi^{'}(z)=az+\sum_{n=1}^{\infty}\frac{a_n}{z^n}, \quad \psi^{'}(z)=\sum_{n=0}^{\infty}\frac{a^{'}_n}{z^n}, \\ \chi(z,\bar{z})=\sum_{-\infty}^{\infty}K_n(\eta r)b_ne^{in\theta}, \end{gathered} \tag{7}$$

where $K_n(\eta r)$ are functions of Macdonald.

Using (5), (6), (7) we get

$$\begin{cases} aRe^{-i\theta}+\displaystyle\sum_{n=1}^{+\infty}\frac{a_n}{R^n}e^{-in\theta}+\sum_{n=1}^{+\infty}\frac{\overline{a_n}}{R^n}e^{in\theta}+\sum_{n=1}^{+\infty}\frac{na_n}{R^n}e^{-in\theta}- \\ -\displaystyle\sum_{n=0}^{+\infty}\frac{a^{'}_n}{R^n}e^{-i(n-2)\theta}+\frac{E^{'}}{4h}\frac{\nu^{'}E}{E^{'}-\nu^{'2}E}\sum_{-\infty}^{+\infty}(K_n(\eta R)-K_{n-2}(\eta R))b_ne^{in\theta}= \\ =\dfrac{(1-\nu)E^{'}\beta-\nu^{'}E\beta^{'}}{4E^{'}}q\left(Re^{i\theta}+3Re^{-i\theta}\right), \\ \dfrac{2\mu^{'}\nu^{'}h}{E^{'}}\left(a-\displaystyle\sum_{n=1}^{+\infty}\frac{na_n}{R^{n+1}}e^{-in\theta}+\overline{a}-\sum_{n=1}^{+\infty}\frac{n\overline{a_n}}{R^{n+1}}e^{in\theta}\right)+ \\ +\dfrac{\eta}{4}\displaystyle\sum_{-\infty}^{+\infty}\left(K_{n-1}(\eta R)+K_{n+1}(\eta R)\right)b_ne^{in\theta}=0. \end{cases}$$

We can write down the functions in the form

$$\varphi^{'}(z) = az, \quad \psi^{'}(z) = \frac{a_1^{'}}{z} + \frac{a_2^{'}}{z^2} + \frac{a_3^{'}}{z^3}, \quad \chi(z,\bar{z}) = K_0(\eta r)b_0,$$

where

$$a = \frac{(1-\nu)E^{'}\beta - \nu^{'}E\beta^{'}}{4E^{'}}q, \quad a_1^{'} = -\frac{(1-\nu)E^{'}\beta - \nu^{'}E\beta^{'}}{4E^{'}}qR^2,$$

$$a_2^{'} = -\frac{2R^2E\nu^{'2}(K_0(\eta R) - K_2(\eta R))}{\eta(E^{'} - \nu^{'2}E)K_1(\eta R)}a \quad a_3^{'} = -\frac{(1-\nu)E^{'}\beta - \nu^{'}E\beta^{'}}{2E^{'}}qR^4$$

$$b_0 = -\frac{2\mu^{'}\nu^{'}h[(1-\nu)E^{'}\beta - \nu^{'}E\beta^{'}]}{\eta E^{'2}K_1}q,$$

Further, for the components of stresses and displacements we obtain

$$\overset{(0)}{\sigma}_{\theta\theta} = \frac{a_2^{'}}{r^2} + \frac{\nu^{'}EE^{'}(K_0(\eta r) + K_2(\eta r))}{4\mu^{'}h(E^{'} - \nu^{'2}E}b_0 -$$

$$-\left[\frac{R^2}{r}a - \frac{1}{r}a_1^{'} - \frac{1}{r^3}a_3^{'}\right]cos\theta$$

$$\overset{(0)}{\sigma}_{rr} = -\frac{a_2^{'}}{r^2} + \frac{\nu^{'}EE^{'}(K_0(\eta r) - K_2(\eta r))}{4\mu^{'}h(E^{'} - \nu^{'2}E}b_0 -$$

$$-\left[\frac{3R^2}{r}a + \frac{1}{r}a_1^{'} + \frac{1}{r^3}a_3^{'}\right]cos\theta$$

$$\overset{(0)}{\sigma}_{r\theta} = -\left[\frac{R^2}{r}a - \frac{1}{r}a_1^{'} + \frac{1}{r}a_3^{'}\right]sin\theta$$

$$\overset{(1)}{\sigma}_{r3} = \frac{4\mu^{'}\nu^{'}h}{E}a + \frac{\eta K_1(\eta r)}{4}b_0 - \frac{\mu^{'}h[(E^{'} - \nu^{'2}E)\beta^{'} - E^{'}\nu^{'}(1+\nu)\beta]}{2E^{'2}}\left(1 - \frac{R^2}{r^2}\right)qcos\theta$$

$$2\mu^{'}\overset{(1)}{u}_3 = K_0(\eta r)b_0 - \left[\frac{8\mu^{'}h\nu}{E^{'}}ar - \frac{\mu^{'}h[(E^{'} - \nu^{'}E)\beta^{'} - \nu^{'}E^{'}(1+\nu)\beta]}{E^{'2}}q(r + \frac{R^2}{r})\right]\cos\theta$$

It's interesting that the stresses obtained by means of the plane theory are dependent on only radii and material, whereas by I. Vekua's theory stresses are dependent on $\frac{R}{h}$.

Let's consider a circular rings. R_1 and R_2 are radii of inner and outer circles ($R_1 < R_2$). We mean that the boundary of the plate is free from outer loads [3].

The boundary conditions have the form

$$\overset{(0)}{\sigma}_{rr} - \overset{(0)}{\sigma}_{r\theta} = \begin{cases} 0, & r = R_1, \\ 0, & r = R_2, \end{cases} \qquad \overset{(1)}{\sigma}_{r3} = \begin{cases} 0, & r = R_1, \\ 0, & r = R_2, \end{cases}$$

hence

$$\begin{cases} \varphi^{'}(z)+\overline{\varphi}^{'}(z)-\left[\bar{z}\varphi^{''}(z)+\psi^{'}\right]e^{2i\theta}+\dfrac{\nu^{'}EE^{'}}{4\mu^{'}(E^{'}-\nu^{'2}E)h}\chi(z,\bar{z})-\dfrac{\nu^{'}Eh}{3E^{'}}\partial_z^2\chi(z,\bar{z})= \\ =\begin{cases} 0, & r=R_1, \\ 0, & r=R_2, \end{cases} \\ Re\left[(\partial_{\bar{z}}\chi(z,\bar{z})-\dfrac{4\mu^{'}\nu^{'}h}{E^{'}}\varphi^{''}(z))e^{i\theta}\right]+\dfrac{h\mu^{'}(\beta^{'}-2\nu^{'}\beta)}{2E^{'}}\left(F^{'}e^{i\theta}+\overline{F}^{'}e^{-i\theta}\right)= \\ =\begin{cases} 0, & r=R_1, \\ 0, & r=R_2, \end{cases} \end{cases} \tag{8}$$

For temperature we have following conditions

$$\begin{cases} \overset{(0)}{T}|_{R_1}=T_1, \\ \overset{(0)}{T}|_{R_2}=T_2, \end{cases} \tag{9}$$

the solution of (9) has the form

$$\overset{(0)}{T}=\frac{T_2-T_1}{\ln R_2-\ln R_1}\ln r+\frac{T_1\ln R_2-T_2\ln R_1}{\ln R_2-\ln R_1}$$

If we introduce the functions $\varphi(z)$, $\psi(z)$, and $\chi(z,\bar{z})$ by the series

$$\varphi^{'}(z)=A\ln z+\sum_{n=0}^{\infty}a_nz^n,\quad \psi^{'}(z)=\sum_{n=0}^{\infty}a_n^{'}z^n,$$

$$\chi(z,\bar{z})=\sum_{-\infty}^{\infty}(b_nI_n(\eta r)+c_nK_n(\eta r))e^{in\theta},$$

and solve the problem (8) we obtain

$$A=-\frac{(1-\nu)E^{'}\beta-\nu^{'}E\beta^{'}}{4E^{'}}\frac{T_2-T_1}{\ln R_2-\ln R_1}$$

$$a_0=\frac{(1-\nu)E^{'}\beta-\nu^{'}E\beta^{'}}{4E^{'}(R_2^2-R_1^2)}\frac{T_2-T_1}{\ln R_2-\ln R_1}(R_2^2\ln R_2-R_1^2\ln R_1+\frac{1}{2}(R_1^2-R_2^2))$$

$$a_{-2}^{'}=\frac{(1-\nu)E^{'}\beta-\nu^{'}E\beta^{'}}{2E^{'}}\frac{T_2-T_1}{\ln R_2-\ln R_1}\frac{R_1^2R_2^2\ln\frac{R_2}{R_1}}{R_2^2-R_1^2}-$$

$$-2\mu^{'}\nu^{'}Eh^2\frac{(E^{'}-\nu^{'2}E)\beta^{'}-(1+\nu)\nu^{'}E^{'}\beta}{3E^{'2}}\frac{T_2-T_1}{\ln R_2-\ln R_1}$$

$$b_0=-\frac{4\mu^{'}h[(E^{'}-\nu^{'2}E)\beta^{'}-(1+\nu)\nu^{'}E^{'}\beta]}{\eta E^{'2}R_1R_2}\times$$

$$\times\frac{R_1K_1(\eta R_1)-R_2K_1(\eta R_2)}{K_1(\eta R_1)I_1(\eta R_2)-K_1(\eta R_2)I_1(\eta R_1)}\frac{T_2-T_1}{\ln R_2-\ln R_1}$$

$$c_0 = -\frac{4\mu' h[(E' - \nu'^2 E)\beta' - (1+\nu)\nu' E' \beta]}{\eta E'^2 R_1 R_2} \times$$
$$\times \frac{R_1 I_1(\eta R_1) - R_2 I_1(\eta R_2)}{K_1(\eta R_1) I_1(\eta R_2) - K_1(\eta R_2) I_1(\eta R_1)} \frac{T_2 - T_1}{\ln R_2 - \ln R_1}$$

For components of stresses we have

$$\overset{(0)}{\sigma}_{rr} = 2A \ln r - A + 2a_0 - \frac{a'_{-2}}{r^2} +$$
$$+ \frac{\nu' E E'}{4\mu' h(E' - \nu'^2 E)} \left[(I_0(\eta r) - I_2(\eta r)) b_0 + (K_0(\eta r) - I_2(\eta r)) c_0 \right],$$

$$\overset{(0)}{\sigma}_{\theta\theta} = 2A \ln r + A + 2a_0 + \frac{a'_{-2}}{r^2} +$$
$$+ \frac{\nu' E E'}{4\mu' h(E' - \nu'^2 E)} \left[(I_0(\eta r) + I_2(\eta r)) b_0 + (K_0(\eta r) + I_2(\eta r)) c_0 \right],$$

$$\overset{(1)}{\sigma}_{r3} = \frac{\eta}{2} \left(I_1(\eta r) b_0 - K_1(\eta r) c_0 \right) + \frac{\mu' h(\beta' - 2\nu' \beta)}{E'} \frac{T_2 - T_1}{\ln R_2 - \ln R_1} \frac{1}{r},$$

$$\overset{(0)}{\sigma}_{r\theta} = 0.$$

We can make the same comment on the obtained result as on the first one.

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