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GENERAL SOLUTIONS OF STRETCH-PRESS AND BENDING EQUATIONS OF BINARY MIXTURES USING I. VEKUA'S METHOD FOR $N = 1$ APPROXIMATION

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The theory of mixtures of elastic materials was originated in 1960. Main mechanical properties of a new model of elastic medium with complicated internal structure were first formulated in the works of C. Truesdell and R. Toupin (see [1]). Later this theory was generalized and developed in many directions. Binary and multicomponent models of different type mixtures were created and studied by means of various mathematical methods. Intensively is being developed also plane theories corresponding to above noted three-dimensional models.

In this paper we consider a version of linear theory for a body composed of two isotropic homogeneous elastic materials suggested by A.E. Green (see [2],[3],[4]). We obtain two-dimensional equations for given plate by means of I. Vekua's method. For $N = 1$ approximation we get the general solutions of stretch-press and bending equations system using the complex variable analytic functions.

1. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a unit vector of Euclidean space. Denote by (x_1, x_2, x_3) the point of Cartesian coordinates system.

Let ω be a finite or infinite domains with smooth boundary in the plane. We define following sets

$$\Omega^h := \omega \times]-h, h[, \quad \Gamma_+ := \omega \times \{h\}, \quad \Gamma_- := \omega \times \{-h\},$$

$$h = const > 0.$$

Assume the domain Ω^h involves mixture of two isotropic homogeneous material. The statical balance equations have the following form

$$\begin{aligned} &\partial_i \left[\sigma'_{ij} - \delta_{ij}\right] + \rho_1 F'_j = 0, \\ &\partial_i \left[\sigma''_{ij} + \delta_{ij}\right] + \rho_2 F''_j = 0, \quad in \ \ \Omega^h, \end{aligned} \tag{1.1}$$

and the generalized Hook's low is written as follows

$$\begin{aligned} \sigma'_{ij} &= (-\alpha_2 + \lambda_1 \varepsilon'_{kk} + \lambda_3 \varepsilon''_{kk})\delta_{ij} + 2\mu_1 \varepsilon'_{ij} + 2\mu_3 \varepsilon''_{ij} - 2\lambda_5 h_{ij}, \\ \sigma''_{ij} &= (\alpha_2 + \lambda_4 \varepsilon'_{kk} + \lambda_2 \varepsilon''_{kk})\delta_{ij} + 2\mu_3 \varepsilon'_{ij} + 2\mu_2 \varepsilon''_{ij} + 2\lambda_5 h_{ij}, \quad in \quad \overline{\Omega}^h, \end{aligned} \tag{1.2}$$

where $\sigma_{ij}^{'}, \sigma_{ij}^{''}$ -are components of strees tensor, δ_{ij} is Kroneker delta, $_j \equiv \partial_j$

$$= \frac{\alpha_2 \rho_2}{\rho} \varepsilon_{kk}^{'} + \frac{\alpha_2 \rho_1}{\rho} \varepsilon_{kk}^{''}, \quad \rho = \rho_1 + \rho_2, \tag{1.3}$$

$\rho_1 > 0, \quad \rho_2 > 0$ are density of mixture, $F_j^{'}, F_j^{''}$ are components of volume forces, $\alpha_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mu_1, \mu_2, \mu_3$ are the modulus of elasticity, besides of $\alpha_2 = \lambda_3 - \lambda_4$, $\varepsilon_{ij}^{'} = \varepsilon_{ji}^{'}, \quad \varepsilon_{ij}^{''} = \varepsilon_{ji}^{''}$ are components of strain tensor

$$\varepsilon_{ij}^{'} = \frac{1}{2}\left(\partial_i u_j^{'} + \partial_j u_i^{'}\right), \quad \varepsilon_{ij}^{''} = \frac{1}{2}\left(\partial_i u_j^{''} + \partial_j u_i^{''}\right), \tag{1.4}$$

$h_{ij} = -h_{ji}$ are components of turn tensor,

$$h_{ij} = \frac{1}{2}\left(\partial_i u_j^{'} - \partial_j u_i^{'} + \partial_j u_i^{''} - \partial_i u_j^{''}\right), \tag{1.5}$$

$\mathbf{u}^{'} = (u_1^{'}, u_2^{'}, u_3^{'}), \quad \mathbf{u}^{''} = (u_1^{''}, u_2^{''}, u_3^{''})$ are displacement vector, $\partial_i := \dfrac{\partial}{\partial x_i}$.

Under repeating indexes we mean summation, the Latin letters are taking the values 1,2,3 and Greek letters are taking the value 1,2.

We introduce the following symbols

$$P_{ij}^{'} := \sigma_{ij}^{'} - \delta_{ij}(- \alpha_2), \quad P_{ij}^{''} := \sigma_{ij}^{''} + \delta_{ij}(- \alpha_2), \tag{1.6}$$

$$P_{ij} := (P_{ij}^{'}, \ \ P_{ij}^{''})^T, \ \ U_j := (u_j^{'}, \ \ u_j^{''})^T, \ \ \in_{ij} := (\varepsilon_{ij}^{'}, \ \ \varepsilon_{ij}^{''})^T, \ \ \hbar_{ij} := (h_{ij}, \ \ h_{ji})^T, \tag{1.7}$$

Using (1.6) and (1.7), relations (1.1) and (1.2) can be rewritten as follows

$$\partial_i P_{ij} + \Phi_j = 0 \qquad in \quad \Omega^h, \tag{1.8}$$

$$P_{ij} = \Lambda \in_{kk} \delta_{ij} + 2M \in_{ij} - 2\lambda_5 \hbar_{ij} \qquad in \quad \overline{\Omega}^h, \tag{1.9}$$

where

$$\Phi_j = (\rho_1 F_j^{'}, \rho_2 F_j^{''})^T,$$

$$\Lambda = \begin{pmatrix} \lambda_1 - \dfrac{\alpha_2 \rho_2}{\rho} & \lambda_3 - \dfrac{\alpha_2 \rho_1}{\rho} \\ \lambda_4 + \dfrac{\alpha_2 \rho_2}{\rho} & \lambda_2 + \dfrac{\alpha_2 \rho_1}{\rho} \end{pmatrix}, \quad M = \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix},$$

$$\in_{ij} = \frac{1}{2}(\partial_i U_j + \partial_j U_i), \tag{1.10}$$

$$\hbar_{ij} = \frac{1}{2} S(\partial_i U_j - \partial_j U_i), \quad S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{1.11}$$

After substituting (1.10) and (1.11) into (1.9), we obtain

$$P_{ij} = \Lambda \partial_k U_k \delta_{ij} + (M - \lambda_5 S)\partial_i U_j + (M + \lambda_5 S)\partial_j U_i. \tag{1.12}$$

By substituting (1.2) into (1.8) we obtain equilibrium equations in components of displacement vector

$$A \Delta U_j + B \partial_j(\partial_k U_k) + \Phi_j = 0, \quad in \quad \Omega^h, \tag{1.13}$$

where $\Delta = \partial_i \partial_i$ is a three-demensional Laplacian,

$$A := M - \lambda_5 S, \quad B := M + \Lambda + \lambda_5 S. \tag{$*$}$$

We reduce three dimensional systems (1.8), (1.9) to the two dimensional problems of the plates with middle surface ω, by means of I. Vekua's method (see [5],[6],[7],[11]).

Let P_{ij}, U_j, Φ_j smooth enough and they can be represented by the uniform convergent series with respect to x_3 on the segment $[-1, 1]$ for all point (x_1, x_2) in domain ω

$$(P_{ij}, U_j, \Phi_j)(x_1, x_2, x_3) = \sum_{m=0}^{\infty} \left(\overset{(m)}{P}_{ij}, \overset{(m)}{U}_j, \overset{(m)}{\Phi}_j \right)(x_1, x_2) P_m\left(\frac{x_3}{h}\right), \tag{1.14}$$

where

$$\left(\overset{(m)}{P}_{ij}, \overset{(m)}{U}_j, \overset{(m)}{\Phi}_j \right) = \frac{2m+1}{2h} \int_{-h}^{h} (P_{ij}, U_j, \Phi_j) P_m\left(\frac{x_3}{h}\right), \quad -h \le x_3 \le h, \tag{1.15}$$

$P_m\left(\frac{x_3}{h}\right)$ is the Legendre polinomial of order m.

By $P^{\pm}_{3k}$ we implies the value of stresses on surfaces Γ^+ and Γ^- accordingly

$$P^{\pm}_{3k} = P_{3k}(x_1, x_2, \pm h). \tag{1.16}$$

By substituting (1.14) into (1.8) and using condition (1.16) on surface of plate we obtain

$$\partial_\alpha \overset{(m)}{P}_{\alpha j} - \frac{2m+1}{h}\left(\overset{(m-1)}{P}_{3j} + \overset{(m-3)}{P}_{3j} + \ldots \right) + \overset{(m)}{\widetilde{F}}_j = 0, \quad in \quad \omega, \quad m = 0, 1, \ldots, \tag{1.17}$$

where

$$\overset{(m)}{\widetilde{F}}_j := \overset{(m)}{\Phi}_j + \frac{2m+1}{2h}(P^+_{3j} - (-1)^m P^-_{3j}),$$

$$\overset{(l)}{P}_{ij} = 0, \quad when \quad l < 0.$$

After substituting (1.12) into (1.15) and taking into account of notation $(*)$, we get

$$\begin{aligned}
&\overset{(m)}{P}_{\alpha\beta} = \Lambda\left(\overset{(m)}{\theta} + \frac{1}{h}\overset{(m')}{U}_3\right)\delta_{\alpha\beta} + A\partial_\alpha \overset{(m)}{U}_\beta + (B - \Lambda)\partial_\beta \overset{(m)}{U}_\alpha, \\
&\overset{(m)}{P}_{\alpha 3} = A\partial_\alpha \overset{(m)}{U}_3 + \frac{1}{h}(B - \Lambda)\overset{(m')}{U}_\alpha, \\
&\overset{(m)}{P}_{3\alpha} = (B - \Lambda)\partial_\alpha \overset{(m)}{U}_3 + \frac{1}{h} A \overset{(m')}{U}_\alpha, \\
&\overset{(m)}{P}_{33} = \Lambda \overset{(m)}{\theta} + \frac{1}{h}(A + B)\overset{(m')}{U}_3, \quad m = 0, 1, \ldots,
\end{aligned} \tag{1.18}$$

where $\overset{(m)}{\theta} = \partial_\gamma \overset{(m)}{U}_\gamma, \quad \overset{(m')}{U}_j = (2m+1)\left(\overset{(m+1)}{U}_j + \overset{(m+3)}{U}_j + \ldots \right).$

Analogously, we reduce the boundary condition given on domain Ω^h to the boundary condition on the boundary $\partial\omega$ of two-dimensional domain ω.

If we substitute (1.18) into (1.17) we obtain

$$
\begin{aligned}
&A\Delta \overset{(m)}{U}_{\alpha} + B\partial_{\alpha} \overset{(m)}{\theta} + \overset{(m)}{M}_{\alpha} + \overset{(m)}{\widetilde{F}}_{\alpha} = 0, \\
&A\Delta \overset{(m)}{U}_{3} + \overset{(m)}{M}_{3} + \overset{(m)}{\widetilde{F}}_{3} = 0, \quad in \ \ \omega, \ \ m = 0, 1, ...,
\end{aligned}
\tag{1.19}
$$

where Δ is a two-dimensional Laplacian $\Delta = \partial_{\gamma}\partial_{\gamma}$. $M_j = (M_j^{'}, M_j^{''})^T$ are linear homogeneous differential expression, which consist the functions $\overset{(m)}{u}\,{}_i^{'}$ and $\overset{(m)}{u}\,{}_i^{''}$, and their first order partial derivative whith respect to x_1, x_2.

Let the displacement vectors $\mathbf{u}^{'}$ and $\mathbf{u}^{''}$ are the same N order polynomial with respect to x_3 (N is a fixed non-negative integer number), i.e.

$$
U_j(x_1, x_2, x_3) = \sum_{m=0}^{N} \overset{(m)}{U}_{j} P_m\left(\frac{x_3}{h}\right), \tag{1.20}
$$

and $\overset{(k)}{U}_{\ j} = 0$, when $k > N$. Furthermore we retain from the system (1.19) the first $6N + 6$ equations. Then we get second order system of $6N + 6$ partial differential equations with respect to two independent variable function.

Let $\mathbf{l} = (l_1, l_2, l_3)$ be a unit vector of the boundary $\partial\omega$.

$$
\mathbf{e}_3 = \mathbf{l} \times \mathbf{s}.
$$

The first and second basic boundary value problem sets following form:

Problem I. $\overset{(m)}{\mathbf{U}} = \overset{(m)}{\mathbf{g}}$ on $\partial\omega$;

Problem II. $\overset{(m)}{\mathbf{P}}_{(l)} = \overset{(m)}{\mathbf{f}}$ on $\partial\omega$, $m = 0, 1, ..., N$,

where $\overset{(m)}{\mathbf{g}} = (\overset{(m)}{\mathbf{g}}{}^{'}, \overset{(m)}{\mathbf{g}}{}^{''})^T$, $\overset{(m)}{\mathbf{f}} = (\overset{(m)}{\mathbf{f}}{}^{'}, \overset{(m)}{\mathbf{f}}{}^{''})^T$, are given functions on $\partial\omega$,

$$
\overset{(m)}{\mathbf{P}}_{(l)} = \overset{(m)}{P}_{ki} l_k \mathbf{e}_i.
$$

Furthermore we get some mixed boundary conditions on $\partial\omega$ (see [6]).

2. Approximation $N = 0$**.**

In this case from (1.20) we get

$$
U_j \approx \overset{(0)}{U}_{j}.
$$

From (1.17) we have equilibrium equations

$$
\partial_{\alpha} \overset{(0)}{P}_{\alpha j} + \overset{(0)}{\widetilde{F}}_{j} = 0, \quad in \ \ \omega, \tag{2.1}
$$

and from $\overset{(0^{'})}{U}_{j} = 0$ and (1.18) we obtain

$$
\begin{aligned}
&\overset{(0)}{P}_{\alpha\beta} = \Lambda \overset{(0)}{\theta} \delta_{\alpha\beta} + A\partial_{\alpha} \overset{(0)}{U}_{\beta} + (B - \Lambda)\partial_{\beta} \overset{(0)}{U}_{\alpha}, \\
&\overset{(0)}{P}_{\alpha 3} = A\partial_{\alpha} \overset{(0)}{U}_{3}, \quad \overset{(0)}{P}_{3\alpha} = (B - \Lambda)\partial_{\alpha} \overset{(0)}{U}_{3}, \quad \overset{(0)}{P}_{33} = \Lambda \overset{(0)}{\theta}.
\end{aligned}
\tag{2.2}
$$

After introducing the complex plane $z = x_1 + ix_2$,

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2), \quad \Delta = 4\partial z\partial_{\bar z},$$

the equilibrium equations can be written as follows

$$\begin{aligned}
&\partial_z \left(\overset{(0)}{P}_{11} - \overset{(0)}{P}_{22} + i(\overset{(0)}{P}_{12} + \overset{(0)}{P}_{21}) \right) + \\
&+\partial_{\bar z} \left(\overset{(0)}{P}_{11} + \overset{(0)}{P}_{22} + i(\overset{(0)}{P}_{12} - \overset{(0)}{P}_{21}) \right) + \overset{(0)}{\widetilde{F}}_{+} = 0, \\
&\partial_z \overset{(0)}{P}_{+} + \partial_{\bar z} \overline{\overset{(0)}{P}_{+}} + \overset{(k)}{\widetilde{F}}_{3} = 0, \quad in \quad \omega,
\end{aligned} \tag{2.3}$$

where $\overset{(0)}{U}_{+} = \overset{(0)}{U}_1 + i\overset{(0)}{U}_2$, $\quad \overset{(0)}{\widetilde{F}}_{+} = \overset{(0)}{\widetilde{F}}_1 + i\overset{(0)}{\widetilde{F}}_2$, from (2.2) we have

$$\begin{aligned}
&\overset{(0)}{P}_{11} - \overset{(0)}{P}_{22} + i(\overset{(0)}{P}_{12} + \overset{(0)}{P}_{21}) = 4M\partial_{\bar z}\overset{(0)}{U}_{+}, \\
&\overset{(0)}{P}_{11} + \overset{(0)}{P}_{22} + i(\overset{(0)}{P}_{12} - \overset{(0)}{P}_{21}) = 2B \overset{(0)}{\theta} - 4\lambda_5 S\partial_z\overset{(0)}{U}_{+}, \\
&\overset{(0)}{P}_{+} = \overset{(0)}{P}_{13} + i\overset{(0)}{P}_{23} = 2A\partial_{\bar z}\overset{(0)}{U}_3, \\
&{}_{+}\overset{(0)}{P} = \overset{(0)}{P}_{31} + i\overset{(0)}{P}_{32} = 2(B - \Lambda)\partial_{\bar z}\overset{(0)}{U}_3, \\
&\overset{(0)}{P}_{33} = \Lambda \overset{(0)}{\theta}.
\end{aligned} \tag{2.4}$$

By substituting (2.4) into (2.3) we obtain

$$A\Delta\overset{(0)}{U}_{+} + 2B\partial_{\bar z} \overset{(0)}{\theta} + \overset{(0)}{\widetilde{F}}_{+} = 0, \tag{2.5}$$

$$A\Delta\overset{(0)}{U}_3 + \overset{(0)}{\widetilde{F}}_3 = 0, \quad in \quad \omega. \tag{2.6}$$

System (2.5) and (2.6) independent each other. First equations are plane strain equation system of binary mixture cylindrical body. The general solution using four complex analytic functions and Kolosov-Muskhelishvili's analogous formula is getting by prof. M. Basheleishvili (see [8],[9],[10]).

$$\begin{aligned}
2\overset{(0)}{U}_{+} = {} & A^*\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \\
& -A^{-1}(2A + B)(A + B)^{-1}\frac{1}{2\pi}\int_\omega \overset{(0)}{\widetilde{F}}_{+}(\xi_1, \xi_2) \ln|\zeta - z| d\xi_1 d\xi_2 + \\
& +A^{-1}B(A + B)^{-1}\frac{1}{4\pi}\int_\omega \overline{\overset{(0)}{\widetilde{F}}_{+}(\xi_1, \xi_2)} \frac{\zeta - z}{\bar\zeta - \bar z} d\xi_1 d\xi_2,
\end{aligned} \tag{2.7}$$

where $\varphi(z) = (\varphi_1(z), \varphi_2(z))^T$, $\psi(z) = (\psi_1(z), \psi_2(z))^T$ are any row matrix of analytic functions in domain ω,

$$A^* = I + 2B^{-1}A,$$

$\zeta = \xi_1 + i\xi_2$, in case of infinity domain implies that

$$\overset{(0)}{\widetilde{F}}_+ = O\left(\frac{1}{|z|^{1+\alpha}}\right), \ \ when \ \ z \to \infty, \ \ \alpha = const > 0.$$

Remark: The elastic constants must to satisfy following condition (see [4])

$$\lambda_5 \leq 0, \ \ \mu_1 > 0, \ \ \lambda_1 - \frac{\alpha_2 \rho_2}{\rho} + \frac{2}{3}\mu_1 > 0, \ \ detM > 0, \quad det\left(\Lambda + \frac{2}{3}M\right) > 0, \tag{2.8}$$

using (2.8) we get $detA > 0$, $det(A + B) > 0$.

From (2.6) we obtain

$$2A\overset{(0)}{U}_3 = f(z) + \overline{f(z)} + \frac{1}{\pi}\int_\omega \overset{(0)}{\widetilde{F}}_3(\xi_1, \xi_2) \ln|\zeta - z| d\xi_1 d\xi_2, \tag{2.9}$$

where $f(z) = (f_1(z), f_2(z))^T$ are row-matrix of arbitrary analytic functions.

Let us $\overset{(0)}{\widetilde{F}}_+ = 0$, $\overset{(0)}{\widetilde{F}}_3 = 0$. By substituting (2.7) and (2.9) into (2.4) we obtain

$$\overset{(0)}{P}_{22} - \overset{(0)}{P}_{11} + i(\overset{(0)}{P}_{12} + \overset{(0)}{P}_{21}) = 2M[\bar{z}\Phi'(z) + \Psi(z)],$$

$$\overset{(0)}{P}_{11} + \overset{(0)}{P}_{22} + i(\overset{(0)}{P}_{12} - \overset{(0)}{P}_{21}) = 2[(A - \lambda_5 S A^*)\Phi(z) + M\overline{\Phi(z)}],$$

$$\overset{(0)}{P}_{13} + i\overset{(0)}{P}_{23} = \overline{f'(z)},$$

$$\overset{(0)}{P}_{31} + i\overset{(0)}{P}_{32} = (B - \Lambda)A^{-1}\overline{f'(z)},$$

where $\Phi(z) = (\varphi_1'(z), \varphi_2'(z))^T$, $\Psi(z) = (\psi_1'(z), \psi_2'(z))^T$.

3. Approximation $N = 1$**.**

In this case by mean of (1.20) we have

$$U_j \approx \overset{(0)}{U}_j + \frac{x_3}{h}\overset{(1)}{U}_j,$$

From (1.17) we get the following equolibrium equations

$$\begin{aligned} &\partial_\alpha \overset{(0)}{P}_{\alpha\beta} + \overset{(0)}{\widetilde{F}}_\beta = 0, \\ &\partial_\alpha \overset{(0)}{P}_{\alpha 3} + \overset{(0)}{\widetilde{F}}_3 = 0, \\ &\partial_\alpha \overset{(1)}{P}_{\alpha\beta} - \frac{3}{h}\overset{(0)}{P}_{3\beta} + \overset{(1)}{\widetilde{F}}_\beta = 0, \\ &\partial_\alpha \overset{(1)}{P}_{\alpha 3} - \frac{3}{h}\overset{(0)}{P}_{33} + \overset{(1)}{\widetilde{F}}_3 = 0. \end{aligned} \tag{3.1}$$

In this case $\overset{(0')}{U}_j = \frac{1}{h}\overset{(1)}{U}_j, \quad \overset{(1')}{U}_j = 0$ and by mean of (1.18) we obtain

$$\begin{aligned}
&\overset{(0)}{P}_{\alpha\beta} = \Lambda(\overset{(0)}{\theta} + \frac{1}{h}\overset{(1)}{U}_3)\delta_{\alpha\beta} + A\partial_\alpha \overset{(0)}{U}_\beta + (B-\Lambda)\partial_\beta \overset{(0)}{U}_\alpha,\\
&\overset{(0)}{P}_{3\alpha} = (B-\Lambda)\partial_\alpha \overset{(0)}{U}_3 + \frac{1}{h}A\overset{(1)}{U}_\alpha,\\
&\overset{(0)}{P}_{\alpha 3} = A\partial_\alpha \overset{(0)}{U}_3 + \frac{1}{h}(B-\Lambda)\overset{(1)}{U}_\alpha,\\
&\overset{(0)}{P}_{33} = \Lambda \overset{(0)}{\theta} + \frac{1}{h}(A+B)\overset{(1)}{U}_3,\\
&\overset{(1)}{P}_{\alpha\beta} = \Lambda \overset{(1)}{\theta} \delta_{\alpha\beta} + A\partial_\alpha \overset{(1)}{U}_\beta + (B-\Lambda)\partial_\beta \overset{(1)}{U}_\alpha,\\
&\overset{(1)}{P}_{\alpha 3} = A\partial_\alpha \overset{(1)}{U}_3,\\
&\overset{(1)}{P}_{3\alpha} = (B-\Lambda)\partial_\alpha \overset{(1)}{U}_3,\\
&\overset{(1)}{P}_{33} = \Lambda \overset{(1)}{\theta},
\end{aligned} \tag{3.2}$$

where $\overset{(0)}{\theta} = \partial_\gamma \overset{(0)}{U}_\gamma, \quad \overset{(1)}{\theta} = \partial_\gamma \overset{(1)}{U}_\gamma,$

$$\overset{(0)}{\widetilde{F}}_j = \overset{(0)}{\Phi}_j + \frac{1}{2h}(P^+_{3j} - P^-_{3j}), \quad \overset{(1)}{\widetilde{F}}_j = \overset{(1)}{\Phi}_j + \frac{3}{2h}(P^+_{3j} + P^-_{3j}), \tag{3.3}$$

By substituting (3.2) into (3.1) we get following system of equation for components of the displacement vector

a) Stretch-press equation system

$$\begin{aligned}
&A\Delta \overset{(0)}{U}_\beta + B\partial_\beta \overset{(0)}{\theta} + \frac{1}{h}\Lambda\partial_\beta \overset{(1)}{U}_3 + \overset{(0)}{\widetilde{F}}_\beta = 0,\\
&A\Delta \overset{(1)}{U}_3 - \frac{3}{h}\Lambda \overset{(0)}{\theta} - \frac{3}{h^2}(A+B)\overset{(1)}{U}_3 + \overset{(1)}{\widetilde{F}}_3 = 0,
\end{aligned} \tag{3.4}$$

b) Bending equation system

$$\begin{aligned}
&A\Delta \overset{(1)}{U}_\beta - \frac{3}{h^2}A\overset{(1)}{U}_\beta + B\partial_\beta \overset{(1)}{\theta} - \frac{3}{h}(B-\Lambda)\partial_\beta \overset{(0)}{U}_3 + \overset{(1)}{\widetilde{F}}_\beta = 0,\\
&A\Delta \overset{(0)}{U}_3 + \frac{1}{h}(B-\Lambda)\overset{(1)}{\theta} + \overset{(0)}{\widetilde{F}}_3 = 0.
\end{aligned} \tag{3.5}$$

Hence, In case of $N = 1$ approximation system of equations split into two independent system (3.4) and (3.5) with unknown $\overset{(0)}{U}_1, \overset{(0)}{U}_2, \overset{(1)}{U}_3$ and $\overset{(1)}{U}_1, \overset{(1)}{U}_2, \overset{(0)}{U}_3$ respectively.

4. The general solution of stretch-press equation system.

Let the volume forces are zero, besides $P^+_{3\beta} = P^-_{3\beta} = 0$, and

$$P^+_{33} = P^-_{33} = p = (p', p'')^T, \;\; p', p'' \;\; are harmonic functions$$

$$\Delta p = 0.$$

Then, from (3.3) we get

$$\overset{(0)}{\widetilde{F}}_j = 0, \quad \overset{(1)}{\widetilde{F}}_\beta = 0, \quad \overset{(1)}{\widetilde{F}}_3 = \frac{3}{h}p.$$

In this case the sytem (3.4) in the complex form will be written as follows

$$\begin{aligned} &A\Delta \overset{(0)}{U}_+ + 2B\partial_{\bar z} \overset{(0)}{\theta} + \frac{2}{h}\Lambda\partial_{\bar z}\overset{(1)}{U}_3 = 0, \\ &A\Delta\overset{(1)}{U}_3 - \frac{3}{h}\Lambda \overset{(0)}{\theta} - \frac{3}{h^2}(A+B)\overset{(1)}{U}_3 + \frac{3}{h}p = 0, \quad in \;\; \omega, \end{aligned} \tag{4.1}$$

where $\overset{(0)}{U}_+ = \overset{(0)}{U}_1 + i\overset{(0)}{U}_2$, $\partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2)$, $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$, $\Delta = 4\partial_z\partial_{\bar z}$.

From the first equation of (4.1) we get

$$\partial_{\bar z}(2A\partial_z \overset{(0)}{U}_+ + B\overset{(0)}{\theta} + \frac{1}{h}\overset{(1)}{U}_3) = 0, \;\; \Rightarrow$$

$$2A\partial_z\overset{(0)}{U}_+ + B\overset{(0)}{\theta} + \frac{1}{h}\Lambda\overset{(1)}{U}_3 = D\varphi'(z), \tag{4.2}$$

where $\varphi(z) = (\varphi_1(z), \varphi_2(z))^T$ are any row-matrix of analytic functions in domain ω, D is any non-degenerate 2×2 matrix.

If we add (4.2) relations to the their conjugation relations and taking into account $\overset{(0)}{\theta} = \partial_z\overset{(0)}{U}_+ + \partial_{\bar z}\overline{\overset{(0)}{U}_+}$, we get

$$(A+B)\overset{(0)}{\theta} + \frac{1}{h}\Lambda\overset{(1)}{U}_3 = \frac{1}{2}D(\varphi'(z) + \overline{\varphi'(z)}). \tag{4.3}$$

From the second equation of (4.1) we get

$$\Lambda\overset{(0)}{\theta} = \frac{h}{3}A\Delta\overset{(1)}{U}_3 - \frac{1}{h}(A+B)\overset{(1)}{U}_3 + p. \tag{4.4}$$

From (4.3) and (4.4) we obtain the following equations ($det(A+B) > 0$ and $detA > 0$) with respect to $\overset{(1)}{U}_3$

$$\begin{aligned} &\Delta\overset{(1)}{U}_3 - \frac{3}{h^2}A^{-1}[A+B-\Lambda(A+B)^{-1}]\overset{(1)}{U}_3 = \\ &\frac{3}{2h}A^{-1}\Lambda(A+B)^{-1}D(\varphi'(z) + \overline{\varphi'(z)}) - \frac{3}{h}A^{-1}(I - \Lambda(A+B)^{-1})p. \end{aligned} \tag{4.5}$$

Introduce by $\widetilde{A}$ the following matrix

$$\widetilde{A} := A^{-1}[A+B-\Lambda(A+B)^{-1}]$$

and imply that they are non-degenerate and have simple proper number. Then from (4.5) we get

$$\overset{(1)}{U}_3 = -\frac{h}{2}\widetilde{A}^{-1}A^{-1}\Lambda(A+B)^{-1}D(\varphi'(z)+\overline{\varphi'(z)}) + L\chi(z,\bar{z}) + hA_0p, \tag{4.6}$$

where $\chi(z,\bar{z}) = (\chi_1(z,\bar{z}), \chi_2(z,\bar{z}))^T$, χ_1 and χ_2 are the general solution of following Helmholtz equation

$$\Delta\chi_1 - \frac{3}{h^2}æ_1\chi_1 = 0, \quad \Delta\chi_2 - \frac{3}{h^2}æ_2\chi_2 = 0, \tag{4.7}$$

where $æ_1$ and $æ_2$ are the proper number of matrix $\widetilde{A}$, and L is a 2×2 matrix which columns values are proper value of the coresponding proper vectors,

$$A_0 := \widetilde{A}^{-1}A^{-1}(I - \Lambda(A+B)^{-1}).$$

By substituting (4.6) into (4.3) for $\overset{(0)}{\theta}$ we get

$$\begin{aligned}\overset{(0)}{\theta} = (A+B)^{-1}\Big\{&\frac{1}{2}[I + \Lambda\widetilde{A}^{-1}A^{-1}\Lambda(A+B)^{-1}]D(\varphi'(z)+\overline{\varphi'(z)}) - \\ &-\frac{1}{h}\Lambda L\chi(z,\bar{z}) - \Lambda A_0 p\Big\}.\end{aligned} \tag{4.8}$$

From the (4.2), (4.6) and (4.8) we will obtain

$$\begin{aligned}2A\partial_z\overset{(0)}{U}_+ = &\left[I - \frac{1}{2}B(A+B)^{-1} + \frac{1}{2}A(A+B)^{-1}\Lambda\widetilde{A}^{-1}A^{-1}\Lambda(A+B)^{-1}\right]D\varphi'(z) - \\ &-\frac{1}{2}\left[B - A(A+B)^{-1}\Lambda\widetilde{A}^{-1}A^{-1}\Lambda\right](A+B)^{-1}D\overline{\varphi'(z)} - A(A+B)^{-1}\Lambda A_0 p - \\ &-\frac{1}{h}A(A+B)^{-1}\Lambda L\chi(z,\bar{z}).\end{aligned} \tag{4.9}$$

Assume the matrix $B - A(A+B)^{-1}\Lambda\widetilde{A}^{-1}A^{-1}\Lambda$ be a non-degenerate then after integration of (4.9) we have

$$\begin{aligned}2\overset{(0)}{U}_+ = &A^*\varphi(z) - z\overline{\varphi'(z)} - \psi(z) - \frac{4h}{3}(A+B)^{-1}\Lambda\widetilde{A}^{-1}L\partial_{\bar{z}}\chi(z,\bar{z}) - \\ &-(A+B)^{-1}\Lambda A_0\int p\,dz,\end{aligned} \tag{4.10}$$

where $\psi(z) = (\psi_1(z), \psi_2(z))^T$ is a row-matrix of any two analytic functions, D be given as follows

$$D = 2(A+B)[B - A(A+B)^{-1}\Lambda\widetilde{A}^{-1}A^{-1}\Lambda]^{-1}A,$$

now A^* matrix has the form

$$A^* = A^{-1}D - I.$$

Taking into account (3.2) we obtain the following formulas

$$\begin{aligned}
&\overset{(0)}{P}_{22}-\overset{(0)}{P}_{11}+i(\overset{(0)}{P}_{12}+\overset{(0)}{P}_{21})=-4M\partial_z\overline{\overset{(0)}{U}_{+}},\\
&\overset{(0)}{P}_{11}+\overset{(0)}{P}_{22}+i(\overset{(0)}{P}_{12}-\overset{(0)}{P}_{21})=2[B\overset{(0)}{\theta}+\frac{1}{h}\Lambda\overset{(1)}{U}_{3}-2\lambda_5 S\partial_z\overset{(0)}{U}_{+}],\\
&\overset{(1)}{P}_{+}=\overset{(1)}{P}_{13}+i\overset{(1)}{P}_{23}=2A\partial_{\bar z}\overset{(1)}{U}_{3},\\
&{}_{+}\overset{(1)}{P}=\overset{(1)}{P}_{31}+i\overset{(1)}{P}_{32}=2(B-\Lambda)\partial_{\bar z}\overset{(1)}{U}_{3}.
\end{aligned}\tag{4.11}$$

By substituting (4.10) and (4.6) into (4.11) we get

$$\overset{(0)}{P}_{22}-\overset{(0)}{P}_{11}+i(\overset{(0)}{P}_{12}+\overset{(0)}{P}_{21})=2M\left\{\bar z\varphi''(z)+\psi'(z)+\right.$$
$$\left.+\frac{4h}{3}(A+B)^{-1}\Lambda\widetilde{A}^{-1}L\partial^2_{zz}\chi(z,\bar z)+(A+B)^{-1}\Lambda A_0\int\partial_z p\,d\bar z\right\},$$

$$\overset{(0)}{P}_{11}+\overset{(0)}{P}_{22}+i(\overset{(0)}{P}_{12}-\overset{(0)}{P}_{21})=2\left\{(A-\lambda_5SA^*)\varphi'(z)+M\overline{\varphi'(z)}+\right.$$
$$\left.+\frac{1}{h}M(A+B)^{-1}\Lambda L\chi(z,\bar z)+M(A+B)^{-1}\Lambda A_0 p\right\},$$

$$\overset{(1)}{P}_{+}=A\left\{-h\widehat{A}\overline{\varphi''(z)}+2L\partial_{\bar z}\chi(z,\bar z)+2hA_0\partial_{\bar z}p\right\},$$
$${}_{+}\overset{(1)}{P}=(B-\Lambda)\left\{-h\widehat{A}\overline{\varphi''(z)}+2L\partial_{\bar z}\chi(z,\bar z)+2hA_0\partial_{\bar z}p\right\},$$

where $\hat A:=\tilde A^{-1}A^{-1}\Lambda(A+B)^{-1}A(I+A^*)$.

Thus the general solution of stretch-press equations systems may be written by four analytic functions and two general solution of Helmholtz equation for $N=1$ approximation. By these functions can be set six independent boundary conditions on boundary of domain ω.

5. The general solution of bending equations.

Assume that $\overset{(1)}{\widetilde F}_j=0$, $\overset{(0)}{\widetilde F}_\beta=0$, $\overset{(0)}{\widetilde F}_3=\frac{1}{h}q=\frac{1}{h}(q',q'')^T$, where q',q'' are constsnts. In this case system (3.5) in the complex form can be written as follows

$$\begin{aligned}
&A\Delta\overset{(1)}{U}_{+}-\frac{3}{h^2}A\overset{(1)}{U}_{+}+2B\partial_{\bar z}\overset{(1)}{\theta}-\frac{6}{h}(B-\Lambda)\partial_{\bar z}\overset{(0)}{U}_{3}=0,\\
&A\Delta\overset{(0)}{U}_{3}+\frac{1}{h}(B-\Lambda)\overset{(1)}{\theta}+\frac{1}{h}q=0,\qquad in\quad\omega,
\end{aligned}\tag{5.1}$$

here $\overset{(1)}{U}_{+}=\overset{(1)}{U}_1+i\overset{(1)}{U}_2$, $\overset{(1)}{\theta}=\partial_z\overset{(1)}{U}_{+}+\partial_{\bar z}\overline{\overset{(1)}{U}_{+}}$.

From the second equation of the system (5.1) we get ($detA>0$)

$$\Delta\overset{(0)}{U}_3+\frac{1}{h}A^{-1}(B-\Lambda)\overset{(1)}{\theta}+\frac{1}{h}A^{-1}q=0.\tag{5.2}$$

By (5.2) we obtain $(det(B-\Lambda)\neq 0)$

$$\overset{(1)}{U}_{+}=-2h(B-\Lambda)^{-1}A\partial_{\bar z}\overset{(0)}{U}_3-\frac{1}{2}(B-\Lambda)^{-1}qz+i\partial_{\bar z}\omega, \tag{5.3}$$

where $\omega=(\omega_1,\omega_2)^T$ is the row-matrix consist with unknown real valued functions ω_1 and ω_2.

Substitute (5.3) into first equations of system (5.1) and change $\overset{(1)}{\theta}$ with the following formula

$$\overset{(1)}{\theta}=-h(B-\Lambda)^{-1}A\Delta\overset{(1)}{U}_3-(B-\Lambda)^{-1}q \tag{5.4}$$

we get

$$\partial_{\bar z}\Big\{-2h(A+B)(B-\Lambda)^{-1}A\Delta\overset{(0)}{U}_3+\frac{6}{h}[A(B-\Lambda)^{-1}A-B+\Lambda]\overset{(0)}{U}_3+$$

$$+Ai(\Delta\omega-\frac{3}{h^2}\omega)-2(A+B)(B-\Lambda)^{-1}q+\frac{3}{2h^2}A(B-\Lambda)^{-1}qz\bar z\Big\}=0,$$

it follows from this formula

$$-2h(A+B)(B-\Lambda)^{-1}A\Delta\overset{(0)}{U}_3+\frac{6}{h}[A(B-\Lambda)^{-1}A-B+\Lambda]\overset{(0)}{U}_3+ \tag{5.5}$$

$$+Ai(\Delta\omega-\frac{3}{h^2}\omega)-2(A+B)(B-\Lambda)^{-1}q+\frac{3}{2h^2}A(B-\Lambda)^{-1}qz\bar z=Gf(z),$$

where $f(z)=(f_1(z),f_2(z))^T$, $f_1(z),f_2(z)$ -are any analytic functions in domain ω, G is 2×2 any matrix.

After adding to the both side of (5.5) their conjugations relations and them consider the difference bitween the obtining relations and conjugations relations of both side of (5.5) we obtain

$$\begin{aligned}&\Delta\overset{(0)}{U}_3-\frac{3}{h^2}A^{-1}(B-\Lambda)(A+B)^{-1}[A(B-\Lambda)^{-1}-(B-\Lambda)A^{-1}]A\overset{(0)}{U}_3=\\&=-\frac{1}{4h}A^{-1}(B-\Lambda)(A+B)^{-1}G(f(z)+\overline{f(z)})-\frac{1}{h}A^{-1}q+\\&+\tfrac{3}{4h^3}A^{-1}(B-\Lambda)(A+B)^{-1}\Lambda(B-\Lambda)^{-1}qz\bar z,\end{aligned} \tag{5.6}$$

$$2Ai(\Delta\omega-\frac{3}{h^2}\omega)=G(f(z)-\overline{f(z)}), \tag{5.7}$$

now introduce by $\widetilde{A}$ followin matrix

$$\widetilde{A}:=A^{-1}(B-\Lambda)(A+B)^{-1}[A(B-\Lambda)^{-1}-(B-\Lambda)A^{-1}]A. \tag{5.8}$$

Imply that are non-degenerate and have simple proper number, then general solution of (5.6) may be written us follows

$$\overset{(0)}{U}_3=\frac{h}{12}[A(B-\Lambda)^{-1}-(B-\Lambda)A^{-1}]AG(f(z)+\overline{f(z)})+L\chi(z,\bar z)+hB_0q+\frac{1}{h}A_0qz\bar z, \tag{5.9}$$

where $\chi=(\chi_1,\chi_2)^T$, χ_1,χ_2 are general solution of (4.7), where æ$_1$ and æ$_2$ are proper number matrix $\widetilde{A}$, which define by (5.8),

$$A_0:=-\frac{1}{4}A^{-1}(B-\Lambda)(A+B)^{-1}\Lambda(B-\Lambda)^{-1},$$

and we have

$$3B_0=\widetilde{A}^{-1}(4A_0+A^{-1}).$$

From (5.7) we get

$$\omega=\frac{ih^2}{6}A^{-1}G(f(z)-\overline{f(z)})+\tau(z,\bar z), \tag{5.10}$$

where $\tau=(\tau_1,\tau_2)^T$, τ_1 and τ_2 are general solution of Helmholtz equation

$$\Delta\tau-\frac{3}{h^2}\tau=0.$$

From (5.4) and (5.9) we obtain

$$\overset{(1)}{\theta}=-(B-\Lambda)^{-1}A\left\{\frac{3}{h}\widetilde{A}L\chi(z,\bar z)+(4A_0+A^{-1})q\right\}.$$

By substituting (5.9) and (5.10) into (5.3) we get

$$\begin{gathered}\overset{(1)}{U}_+=i\partial_{\bar z}\tau(z,\bar z)+\frac{h^2}{6}[A^{-1}-(B-\Lambda)^{-1}A(A(B-\Lambda)^{-1}-(B-\Lambda)A^{-1})A]G\overline{f'(z)}-\\-2h(B-\Lambda)^{-1}AL\partial_{\bar z}\chi(z,\bar z)-(B-\Lambda)^{-1}(2AA_0+\frac{1}{2}I)qz.\end{gathered} \tag{5.11}$$

Taking into account of (3.2) following formulas are hold

$$\begin{aligned}&\overset{(1)}{P}_{22}-\overset{(1)}{P}_{11}+i(\overset{(1)}{P}_{12}+\overset{(1)}{P}_{21})=-4M\partial_z\overline{\overset{(1)}{U}_+},\\&\overset{(1)}{P}_{11}+\overset{(1)}{P}_{22}+i(\overset{(1)}{P}_{12}-\overset{(1)}{P}_{21})=2[B\overset{(1)}{\theta}-2\lambda_5S\partial_z\overset{(1)}{U}_+],\\&\overset{(0)}{P}_+=\overset{(0)}{P}_{13}+i\overset{(0)}{P}_{23}=2A\partial_{\bar z}\overset{(0)}{U}_3+\frac{1}{h}(B-\Lambda)\overset{(1)}{U}_+=\frac{1}{h}(B-\Lambda)i\partial_{\bar z}\omega,\\&{}_+\overset{(0)}{P}=\overset{(0)}{P}_{31}+i\overset{(0)}{P}_{32}=2(B-\Lambda)\partial_{\bar z}\overset{(0)}{U}_3+\frac{1}{h}A\overset{(1)}{U}_+.\end{aligned}$$

If substitute (5.9) and (5.11) into last relation ($q=0$) we obtain

$$\begin{aligned}&\overset{(1)}{P}_{22}-\overset{(1)}{P}_{11}+i(\overset{(1)}{P}_{12}+\overset{(1)}{P}_{21})=-4M\Big\{-i\partial^2_{zz}\tau(z,\bar z)+\frac{h^2}{6}[A^{-1}-\\&-(B-\Lambda)^{-1}A(A(B-\Lambda)^{-1}-(B-\Lambda)A^{-1})A]Gf''(z)-2h(B-\Lambda)^{-1}AL\partial^2_{zz}\chi(z,\bar z)\Big\},\\&\overset{(1)}{P}_{11}+\overset{(1)}{P}_{22}+i(\overset{(1)}{P}_{12}-\overset{(1)}{P}_{21})=2\Big\{-\frac{3i}{2h^2}\lambda_5S\tau(z,\bar z)-\\&-\frac{3}{h}(B-\lambda_5S)(B-\Lambda)^{-1}A\widetilde{A}L\chi(z,\bar z)\Big\},\\&\overset{(0)}{P}_+=\frac{1}{h}(B-\Lambda)i\partial_{\bar z}\tau(z,\bar z)+\frac{h}{6}(B-\Lambda)A^{-1}G\overline{f'(z)},\\&{}_+\overset{(0)}{P}=\frac{1}{h}Ai\partial_{\bar z}\tau(z,\bar z)+\frac{h}{6}G\overline{f'(z)}.\end{aligned}$$

Thus the general solution of bending equations system is written by means of two analytic functions and general solution of four Helmholtz equations. By these functions can be satisfy six independent boundary condition.

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