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ON THE INVESTIGATION OF A DIMENSIONAL REDUCTION METHOD FOR ELLIPTIC PROBLEMS

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Abstract

The present paper is devoted to the analysis of a dimensional reduction method, which is a generalization of I. Vekua's method for general elliptic problems. For $(n+1)$-dimensional boundary value problem we construct the sequence of problems in n-dimensional spaces and prove the well-posedness of the obtained problems. Moreover, we prove convergence of the sequence of vector-functions of $(n+1)$ variables restored from the solutions of reduced n-dimensional boundary value problems to the exact solution of original problem and estimate the rate of convergence.

Key words and phrases: elliptic boundary value problems, Legendre polynomials, error estimates.

AMS subject classification: 65N30,42C10

In the paper [1] was suggested one of the methods for constructing the two-dimensional models of prismatic shells in the theory of elasticity. In this work I. Vekua expands the displacement vector-functions, stress and strain tensors into orthogonal Fourier-Legendre series with respect to the variable of plate thickness and then considering only the first $N+1$ terms of the expansions, he obtaines the two-dimensional model of order N. However, in [1] initial boundary value problems are considered only in the spaces of regular functions and the relation of the constructed models to the original three-dimensional problems is not investigated. For static boundary value problem existence and uniqueness of solution to the reduced problem, obtained by I. Vekua's method, in Sobolev spaces first were investigated in [2]. The rate of approximation of the exact solution to the three-dimensional problem by the vector-functions of three variables restored from the solutions of reduced problems in C^k spaces was estimated in [3]. Later, applying I. Vekua's method hierarchic two-dimensional models of shells and plates in the theory of elasticity were constructed in [4-6].

Note, that first, the systematic investigation of a dimensional reduction method based on I. Vekua's idea, was carried out by M. Avalishvili under supervision of D. Gordeziani in 1998, 1999 and the obtained results were reported on the 58-th, 59-th Students Conferences of Tbilisi State University and on Enlarged Sessions of the Seminar of I. Vekua Institute of Appled Mathematics [7, 8]. Later, the results on the relation of hierarchic models for static and dynamical problems of the theory of elasticity in the case of various spatial domains were presented with complete proofs in the papers prepared by M. Avalishvili under supervision of D. Gordeziani, which win the first prizes at Soros Conferences for Students and Post-Graduate Students in 2000-2002.

More precisely, in the paper presented on the first Soros Conference in 2000 was constructed and investigated in Sobolev spaces a two-dimensional model of static boundary value problem in the case of elastic prismatic shell, an one-dimensional model in the case of elastic rod with variable rectangular cross-section and a new hierarchic model for elastic multistructures. Moreover, in this work was proved convergence of the sequences of vector-functions restored from the solutions of the corresponding lower-dimensional problems to the exact solutions of original problems and first were obtained *a-priori* modelling error estimates in Sobolev spaces. It should be pointed out, that the approach and basic formulas given in this paper can be directly applied without any modifications for various elastic structures, and, in particular, for shells and rods with singularities on the boundary, but in order to simplify notations and reasoning in the paper were considered the cases of elastic bodies with positive thickness and width. In the papers presented on the second (in 2001) and on the third (in 2002) Soros Conferences along with the static problems were considered and completely studied the dynamical lower-dimensional models for initial boundary value problems in the cases of prismatic shells and rods, and their relation to the original three-dimensional problems was investigated. A certain part of the above mentioned results may be found in the papers [9-12].

Let D be an open set of the space $\mathbb{R}^n$, $n \in \mathbb{N}$. For each positive integer s and real $r \geq 1$, $W^{s,r}(D)$ denote the Sobolev spaces of real-valued functions based on $L^r(D)$ [13] and for $r = 2$ these spaces we denote by $H^s(D)$ and the corresponding spaces of vector-valued functions are $\mathbf{H}^s(D) = [H^s(D)]^m$, $m \in \mathbb{N}$. $\mathbf{H}^s_0(D)$ is the closure of the set $[\mathcal{D}(D)]^m$ of infinitely differentiable vector-functions with compact support in D in the space $\mathbf{H}^s(D)$. The generalized partial derivative $\partial/\partial x_\alpha$ with respect to the α-th variable we denote by ∂_α, $1 \leq \alpha \leq n$. For Lipschitz domain D [14], we also require Sobolev space $H^{1/2}(\Gamma_1^D)$ defined on a part Γ_1^D of the boundary ∂D, which is an element of Lipschitz dissection of ∂D and let $\mathbf{H}^{1/2}(\Gamma_1^D) = [H^{1/2}(\Gamma_1^D)]^m$. The dual spaces of $\mathbf{H}^1(D)$ and $\mathbf{H}^{1/2}(\Gamma_1^D)$ we denote by $\widetilde{\mathbf{H}}^{-1}(D)$ and $\mathbf{H}^{-1/2}(\Gamma_1^D)$, respectively.

Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $x = (x_1, ..., x_{n+1})$, $n \in \mathbb{N}$, is a bounded Lipschitz domain, and consider boundary value problem for a linear, second-order system of partial differential equations

$$-\sum_{p=1}^{n+1}\sum_{q=1}^{n+1} \partial_p \left(\boldsymbol{M}_{pq} \partial_q \boldsymbol{u} \right) + \sum_{q=1}^{n+1} \boldsymbol{M}_q \partial_q \boldsymbol{u} + \boldsymbol{M}_0 \boldsymbol{u} = \boldsymbol{f}, \qquad \text{in } \Omega, \tag{1}$$

$$\begin{aligned} &\sum_{p=1}^{n+1}\sum_{q=1}^{n+1} \boldsymbol{M}_{pq} \partial_q \boldsymbol{u} \nu_q + \boldsymbol{M}_{\Gamma_1} \boldsymbol{u} = \boldsymbol{g}, && \text{on } \Gamma_1, \\ &\boldsymbol{u} = \boldsymbol{0}, && \text{on } \Gamma_0, \end{aligned} \tag{2}$$

where $\boldsymbol{M}_{pq}, \boldsymbol{M}_q, \boldsymbol{M}_0$ and $\boldsymbol{M}_{\Gamma_1}$ are $m \times m$, $m \in \mathbb{N}$, matrices with elements from the spaces $L^\infty(\Omega)$ and $L^\infty(\Gamma_1)$, respectively, $\boldsymbol{f}$, $\boldsymbol{g}$ are prescribed m-component vector-functions, $\boldsymbol{\nu} = (\nu_1, ..., \nu_{n+1})^T$ is the outward unit normal to Ω, Γ_0, Γ_1 are elements of Lipschitz dissection of $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_{01} \cup \Gamma_1$ [14], Γ_{01} is a Lipschitz curve, and

$\boldsymbol{u} = (u_1, ..., u_m)^T$ is the unknown vector-function, which satisfies the equation (1) and the boundary conditions (2) in the sense of suitable spaces. The problem (1), (2) admits the following variational formulation: find $\boldsymbol{u} \in V(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega); \boldsymbol{v} = \mathbf{0} \quad \text{on} \ \Gamma_0\}$, such that

$$\sum_{q=1}^{n+1} \int_{\Omega} \left(\sum_{p=1}^{n+1} (\boldsymbol{M}_{pq}\partial_q \boldsymbol{u}, \partial_p \boldsymbol{v}) + (\boldsymbol{M}_q \partial_q \boldsymbol{u}, \boldsymbol{v}) + (\boldsymbol{M}_0 \boldsymbol{u}, \boldsymbol{v}) \right) dx + \\ + \int_{\Gamma_1} (\boldsymbol{M}_{\Gamma_1} \boldsymbol{u}, \boldsymbol{v})\, d\Gamma_1 = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega} + \langle \boldsymbol{g}, tr_{\Gamma_1}(\boldsymbol{v}) \rangle_{\Gamma_1}, \quad \forall \boldsymbol{v} \in V(\Omega), \tag{3}$$

where $(.,.)$ is the scalar product in the space $\mathbb{R}^m$, $\boldsymbol{f} \in \widetilde{\mathbf{H}}^{-1}(\Omega)$, $\boldsymbol{g} \in \mathbf{H}^{-1/2}(\Gamma_1)$, $\langle .,. \rangle_{\Omega}$ and $\langle .,. \rangle_{\Gamma_1}$ denote the duality relations between the spaces $\widetilde{\mathbf{H}}^{-1}(\Omega)$, $\mathbf{H}^1(\Omega)$ and $\mathbf{H}^{-1/2}(\Gamma_1)$, $\mathbf{H}^{1/2}(\Gamma_1)$, respectively. The bilinear and linear forms in the left-hand and right-hand parts of the equation (3) we denote by $B(\boldsymbol{u}, \boldsymbol{v})$ and $L(\boldsymbol{v})$, respectively. The formulated problem (3) has a unique solution $\boldsymbol{u} \in V(\Omega)$ if the matrices $\boldsymbol{M}_{pq}$, $\boldsymbol{M}_q$, $\boldsymbol{M}_0$ and $\boldsymbol{M}_{\Gamma_1}$ are such that for all $\boldsymbol{v} \in V(\Omega)$ the following condition is fulfilled

$$\sum_{q=1}^{n+1} \int_{\Omega} \left(\sum_{p=1}^{n+1} (\boldsymbol{M}_{pq}\partial_q \boldsymbol{v}, \partial_p \boldsymbol{v}) + (\boldsymbol{M}_q \partial_q \boldsymbol{v}, \boldsymbol{v}) + (\boldsymbol{M}_0 \boldsymbol{v}, \boldsymbol{v}) \right) dx + \\ + \int_{\Gamma_1} (\boldsymbol{M}_{\Gamma_1} \boldsymbol{v}, \boldsymbol{v})\, d\Gamma_1 \geq c_B \left\| \boldsymbol{v} \right\|^2_{\mathbf{H}^1(\Omega)}. \tag{4}$$

Note, that static boundary value problems of the theory of linear elasticity [15-17] are particular cases of the problem (3). Indeed, let $\overline{\Omega^*} = \boldsymbol{\theta}(\overline{\Omega})$ be an initial configuration of elastic body, where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain and $\boldsymbol{\theta}$ is a C^2-diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega^*}$, so that the vectors $\boldsymbol{T}_p(x) = \partial_p \boldsymbol{\theta}(x)$, $p = \overline{1,3}$, are linearly independent at all points $x \in \overline{\Omega}$ ($\overline{\Omega}$, $\overline{\Omega^*}$ denote the closures of the domains Ω and Ω^*, respectively). Since $\boldsymbol{\theta}$ is an injective mapping, each point $x^* \in \overline{\Omega^*}$ can be unambiguously written as $x^* = \boldsymbol{\theta}(x)$, $x \in \overline{\Omega}$, and the coordinates x_i $(i = \overline{1,3})$ of x are the curvilinear coordinates of x^*. The vectors $\{\boldsymbol{T}_p(x)\}_{p=1}^3$ and $\{\boldsymbol{T}^p(x)\}_{p=1}^3$ form the covariant and contravariant basises at the point $x^* = \boldsymbol{\theta}(x)$, respectively, where the scalar product of $\boldsymbol{T}^p(x)$ and $\boldsymbol{T}_q(x)$ is equal to $\delta_{pq} = \boldsymbol{T}^p(x) \cdot \boldsymbol{T}_q(x)$, δ_{pq} is the Kronecker delta $(p, q = \overline{1,3})$. Assume, that the body Ω^* consists of arbitrary (i.e., of nonhomogeneous and anisotropic) linearly elastic material, Ω^* is clamped along a part $\boldsymbol{\theta}(\Gamma_0)$, $\Gamma_0 \subset \partial\Omega$ of the boundary $\Gamma^* = \partial\Omega^*$ and surface forces are acting on the rest part of the boundary. The variational formulation of the corresponding static three-dimensional problem of linearized elasticity in terms of curvilinear coordinates is of the form (3), with $n = 2$, $m = 3$, $\boldsymbol{M}_{\Gamma_1} \equiv 0$,

$$\boldsymbol{f} = \widehat{\boldsymbol{f}}\sqrt{T}, \quad \boldsymbol{g} = \boldsymbol{\sigma}\sqrt{T}, \quad B(\widehat{\boldsymbol{v}}, \boldsymbol{v}) = \sum_{p,q,\widehat{p},\widehat{q}=1}^{3} \int_{\Omega} a^{pq\widehat{p}\widehat{q}}(x) e_{\widehat{p}||\widehat{q}}(\widehat{\boldsymbol{v}}) e_{p||q}(\boldsymbol{v}) \sqrt{T} dx,$$

where $a^{pq\widehat{p}\widehat{q}}$ are the contravariant components of the three-dimensional elasticity tensor, $e_{p||q}(\boldsymbol{v}) = 1/2(\partial_p v_q + \partial_q v_p) - \sum_{\widehat{p}=1}^{3} \boldsymbol{T}^{\widehat{p}} \cdot \partial_p \boldsymbol{T}_q(x) v_{\widehat{p}}$ denotes the linearized strains in curvilinear coordinates, T is determinant of the matrix (T_{pq}) with elements $T_{pq} = \boldsymbol{T}_p \cdot \boldsymbol{T}_q$, $\widehat{\boldsymbol{f}} = (\widehat{f}^i)$, $\boldsymbol{\sigma} = (\sigma^i)$, $\widehat{f}^i$, σ^i are the contravariant components of the applied body force and surface force densities, respectively, $\boldsymbol{u} = (u_i)$, u_i are the covariant components of the displacement vector-field $\sum_{i=1}^{3} u_i \boldsymbol{T}^i$ of the points of the body $\overline{\Omega^*}$ $(i, p, q = \overline{1,3})$. From practical point of view it is important to consider the case when the tensor $(a^{pq\hat{p}\hat{q}})$ is symmetric and positive definite, i.e.

$$a^{pq\hat{p}\hat{q}}(x) = a^{qp\hat{p}\hat{q}}(x) = a^{pq\hat{q}\hat{p}}(x), \qquad \sum_{p,q,\hat{p},\hat{q}=1}^{3} a^{pq\hat{p}\hat{q}}(x) \varepsilon_{pq} \varepsilon_{\hat{p}\hat{q}} \geq \widehat{c} \sum_{p,q=1}^{3} (\varepsilon_{pq})^2, \qquad x \in \Omega,$$

for all $\varepsilon_{pq} \in \mathbb{R}$, $\varepsilon_{pq} = \varepsilon_{qp}$, $p, q, \hat{p}, \hat{q} = \overline{1,3}$. From the latter conditions applying a lemma of J.-L. Lions [18] we obtain that the condition (4) is fulfilled and since, in this case, the bilinear form $B(.,.)$ is symmetric, then $\boldsymbol{u}$ is also a unique solution to the following minimization problem: find $\boldsymbol{u} \in V(\Omega)$, such that

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in V(\Omega)} J(\boldsymbol{v}), \qquad J(\boldsymbol{v}) = \frac{1}{2} B(\boldsymbol{v}, \boldsymbol{v}) - L(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V(\Omega).$$

In the present paper we construct an algorithm for approximation of $(n+1)$-dimensional elliptic problem (3) by n-dimensional problems in the case of the following type Lipschitz domain

$$\Omega = \{(x_1, ..., x_n, x_{n+1}) \in \mathbb{R}^{n+1};\ h^-(x_1, ..., x_n) < x_{n+1} < h^+(x_1, ..., x_n), (x_1, ..., x_n) \in \omega\},$$

where $\omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with boundary $\partial\omega$, $h^\pm \in Lip(\omega) \cap C^0(\bar{\omega})$ are Lipschitz continuous in ω, $h^+ = h^-$ on a subset $\omega_0 \subset \overline{\omega}$ with zero measure in $\mathbb{R}^n$ and $h^+ > h^-$ on the rest part $\omega_1 \cup \tilde{\gamma}$ of $\overline{\omega}$, $\omega_1 = \omega \backslash \omega_0$, $\tilde{\gamma} = \partial\omega \backslash \omega_0$. Note, that ω_0 is a closed set and $\omega_0 \subset \partial\omega$, $\omega_1 = \omega$, since $h^\pm$ are continuous functions and Ω is a Lipschitz domain. Denote by Γ^+ and Γ^- the parts of the boundary $\partial\Omega$ defined by the equations $x_{n+1} = h^+(x_1, ..., x_n)$ and $x_{n+1} = h^-(x_1, ..., x_n)$, $(x_1, ..., x_n) \in \overline{\omega}$, respectively, and let $\widetilde{\Gamma} = \{(x_1, ..., x_{n+1}) \in \partial\Omega \backslash (\Gamma^+ \cup \Gamma^-); (x_1, ..., x_n) \in \widetilde{\gamma}\}$ be a part of the lateral boundary where h^+ is greater then h^-.

Let us consider the $(n+1)$-dimensional problem (3) when Γ_0 is a subset of $\widetilde{\Gamma}$, $\Gamma_0 = \{(x_1, ..., x_{n+1}) \in \widetilde{\Gamma};\ (x_1, ..., x_n) \in \gamma_0 \subset \tilde{\gamma}\}$, γ_0 is a Lipschitz curve with positive length, if $\gamma_0 \neq \emptyset$. In order to reduce the problem (3) to n-dimensional one we construct the sequence of subspaces $V_{\mathbf{N}}(\Omega) \subset V(\Omega)$, $\mathbf{N} = (N_1, ..., N_m)$, of vector-functions the i-th component is which is a polynomial of degree $N_i \geq 0$ with respect to the variable x_{n+1}, i.e. the subspace $V_{\mathbf{N}}(\Omega)$ is defined by

$$V_{\mathbf{N}}(\Omega) = \{\boldsymbol{v}_{\mathbf{N}} = (v_{\mathbf{N}i}) \in \mathbf{H}^1(\Omega);\ v_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h} \left(r_i + \frac{1}{2}\right) \overset{r_i}{v}_{\mathbf{N}i}\, P_{r_i}\left(\frac{x_3 - \bar{h}}{h}\right),$$
$$\boldsymbol{v}_{\mathbf{N}} = \mathbf{0} \quad \text{on } \Gamma_0, 0 \leq r_i \leq N_i, i = \overline{1,m}\},$$

where $h = \dfrac{h^+ - h^-}{2}$, $\bar{h} = \dfrac{h^+ + h^-}{2}$ and P_r is the Legendre polynomial of degree $r \in \in \mathbb{N} \cup \{0\}$. For notational brevity in the sequel $(x_{n+1} - \bar{h})/h$ we designate by z and assume, that the indices i, j take their values in the set $\{1, ..., m\}$.

From the problem (3), on the subspace $V_{\mathbf{N}}(\Omega)$ we obtain the following problem: find the unknown vector-function $\boldsymbol{w}_{\mathbf{N}} = (w_{\mathbf{N}i}) \in V_{\mathbf{N}}(\Omega)$, $w_{\mathbf{N}i} = \sum\limits_{r_i=0}^{N_i} \dfrac{1}{h}\left(r_i + \dfrac{1}{2}\right) \overset{r_i}{w}_{\mathbf{N}i} \, P_{r_i}(z)$, which satisfies the equation

$$\sum_{q=1}^{n+1} \int\limits_{\Omega} \left(\sum_{p=1}^{n+1} (\boldsymbol{M}_{pq}\partial_q \boldsymbol{w}_{\mathbf{N}}, \partial_p \boldsymbol{v}_{\mathbf{N}}) + (\boldsymbol{M}_q \partial_q \boldsymbol{w}_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}}) + (\boldsymbol{M}_0 \boldsymbol{w}_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}}) \right) dx +$$
$$+ \int\limits_{\Gamma_1} (\boldsymbol{M}_{\Gamma_1} \boldsymbol{w}_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}}) \, d\Gamma_1 = \langle \boldsymbol{f}, \boldsymbol{v}_{\mathbf{N}} \rangle_{\Omega} + \langle \boldsymbol{g}, tr_{\Gamma_1}(\boldsymbol{v}_{\mathbf{N}}) \rangle_{\Gamma_1}, \qquad \forall \boldsymbol{v}_{\mathbf{N}} \in V_{\mathbf{N}}(\Omega). \quad (5)$$

Since $h^{\pm} \in Lip(\omega)$, by Rademacher's theorem [19] the functions $h^{\pm}$ are differentiable almost everywhere in ω and $\partial_\alpha h^{\pm} \in L^{\infty}(\omega^*)$, $\overline{\omega^*} \subset \omega$, $\alpha = \overline{1, n}$. Therefore, taking into account that h is positive in ω, from the definition of the space $V_{\mathbf{N}}(\Omega)$ it follows, that $\overset{r_i}{v}_{\mathbf{N}i}$ is a function of class H^1 in the interior of the set ω, i.e. $\overset{r_i}{v}_{\mathbf{N}i} \in H^1_{loc}(\omega)$, $0 \leq r_i \leq N_i$, $i = \overline{1, m}$. Moreover, $\|\boldsymbol{v}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)} < \infty$ implies that in the space $[H^1_{loc}(\omega)]^{N_{1,m}}$, $N_{1,m} = = N_1 + ... + N_m + m$, of vector-functions $\vec{v}_{\mathbf{N}}$ with components $\overset{r_i}{v}_{\mathbf{N}i}$ (i.e. $\vec{v}_{\mathbf{N}} = (\overset{0}{v}_{\mathbf{N}1}, ..., \overset{N_1}{v}_{\mathbf{N}1}, ..., \overset{0}{v}_{\mathbf{N}m}, ..., \overset{N_m}{v}_{\mathbf{N}m})^T$) we can define the weighted norm $\|\vec{v}_{\mathbf{N}}\|_* = \|\boldsymbol{v}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)}$, where the function $\boldsymbol{v}_{\mathbf{N}}$ of $(n+1)$ variables corresponds to $\vec{v}_{\mathbf{N}}$.

Hence, the problem (5) is equivalent to the following problem: find $\vec{w}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega) = = \{\vec{v}_{\mathbf{N}} \in [H^1_{loc}(\omega)]^{N_{1,m}}; \quad \|\vec{v}_{\mathbf{N}}\|_* < \infty, \overset{r_i}{v}_{\mathbf{N}i} = 0 \quad \text{on } \gamma_0, \ 0 \leq r_i \leq N_i, i = 1, ..., m\}$, such that

$$B_{\mathbf{N}}(\vec{w}_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) = L_{\mathbf{N}}(\vec{v}_{\mathbf{N}}), \qquad \forall \vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega), \tag{6}$$

where $B_{\mathbf{N}}(\vec{\hat{v}}_{\mathbf{N}}, \vec{v}_{\mathbf{N}})$ and $L_{\mathbf{N}}(\vec{v}_{\mathbf{N}})$ are the forms $B(\widehat{\boldsymbol{v}}_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}})$ and $L(\boldsymbol{v}_{\mathbf{N}})$ on the subspace $V_{\mathbf{N}}(\Omega)$ rewritten in terms of $\vec{\hat{v}}_{\mathbf{N}}$ and $\vec{v}_{\mathbf{N}}$, which correspond to $\widehat{\boldsymbol{v}}_{\mathbf{N}}$ and $\boldsymbol{v}_{\mathbf{N}}$, respectively.

Notice that in the definition of $\vec{V}_{\mathbf{N}}(\omega)$ condition $\overset{r_i}{v}_{\mathbf{N}i} = 0$ on γ_0 is understood in the trace sense, because for the vector-functions from the space $\vec{V}_{\mathbf{N}}(\omega)$ we can define the trace on γ_0. Indeed, if $\vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega)$, then the corresponding $\boldsymbol{v}_{\mathbf{N}} \in V_{\mathbf{N}}(\Omega) \subset \mathbf{H}^1(\Omega)$ and $tr(\boldsymbol{v}_{\mathbf{N}}) \in \mathbf{H}^{1/2}(\partial\Omega)$, where tr designates the trace operator. Therefore for any function $\overset{r_i}{v}_{\mathbf{N}i}$ the trace operator tr_{γ_0} we define by

$$tr_{\gamma_0}(\overset{r_i}{v}_{\mathbf{N}i}) = \int\limits_{h^-}^{h^+} tr(v_{\mathbf{N}i})|_{\Gamma_0} P_{r_i}(z) dx_3, \qquad 0 \leq r_i \leq N_i, \quad i = \overline{1, m}.$$

Thus, the $(n+1)$-dimensional boundary value problem (3) for elliptic system, in the case of prismatic body Ω with Lipschitz boundary, we have reduced to n-dimensional one (6). For the latter problem the following theorem is valid.

Theorem 1. *If* $\boldsymbol{M}_{pq}$, $\boldsymbol{M}_q$, $\boldsymbol{M}_0 \in [L^\infty(\Omega)]^{m\times m}$, $\boldsymbol{M}_{\Gamma_1} \in [L^\infty(\Gamma_1)]^{m\times m}$, $\boldsymbol{f} \in \in \widetilde{\mathbf{H}}^{-1}(\Omega)$, $\boldsymbol{g} \in \mathbf{H}^{-1/2}(\Gamma_1)$ *and condition (4) is fulfilled, then the reduced* n*-dimensional problem (6) has a unique solution* $\vec{w}_{\mathbf{N}}$.

Proof. Let us show that the space $\vec{V}_{\mathbf{N}}(\omega)$ is complete. Let $\{\vec{v}_{\mathbf{N}}^{(l)}\}_{l=1}^\infty$ be a Cauchy sequence in $\vec{V}_{\mathbf{N}}(\omega)$, i.e., $\|\vec{v}_{\mathbf{N}}^{(l)} - \vec{v}_{\mathbf{N}}^{(m)}\|_* \to 0$, as $l, m \to \infty$.

From the definition of the norm $\|.\|_*$ we deduce, that $\{\boldsymbol{v}_{\mathbf{N}}^{(l)}\}_{l=1}^\infty$ is a Cauchy sequence in the space $V_{\mathbf{N}}(\Omega)$, where $\boldsymbol{v}_{\mathbf{N}}^{(l)} = (v_{\mathbf{N}i}^{(l)})$, $v_{\mathbf{N}i}^{(l)} = \sum\limits_{r_i=0}^{N_i} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i(l)}{v}_{\mathbf{N}i} P_{r_i}(z)$, $0 \leq r_i \leq N_i$, $i = \overline{1,3}$. Hence, there exists $\boldsymbol{v}_{\mathbf{N}} \in \mathbf{H}^1(\Omega)$ such that $\boldsymbol{v}_{\mathbf{N}}^{(l)} \to \boldsymbol{v}_{\mathbf{N}}$, as $l \to \infty$ in $\mathbf{H}^1(\Omega)$. Therefore, when $l \to \infty$,

$$tr\boldsymbol{v}_{\mathbf{N}}^{(l)} \to tr\boldsymbol{v}_{\mathbf{N}} \qquad \text{in } \mathbf{H}^{1/2}(\partial\Omega),$$

$$\overset{r(l)}{v}_{\mathbf{N}i} = \int_{h^-}^{h^+} v_{\mathbf{N}i}^{(l)} P_r(z)dx_3 \to \overset{r}{v}_{\mathbf{N}i} = \int_{h^-}^{h^+} v_{\mathbf{N}i} P_r(z)dx_3 \qquad \text{in } L^2(\omega),\ \forall r \in \mathbb{N}.$$

Since $\boldsymbol{v}_{\mathbf{N}}^{(l)} \in V_{\mathbf{N}}(\Omega)$, we have that $tr\boldsymbol{v}_{\mathbf{N}}^{(l)} = \mathbf{0}$ on Γ_0 and $\overset{r_i(l)}{v}_{\mathbf{N}i} = 0$, for any $r_i > N_i$, from which we deduce $\boldsymbol{v}_{\mathbf{N}} = \mathbf{0}$ on Γ_0 and $\overset{r_i}{v}_{\mathbf{N}i} = 0$, for any $r_i > N_i$, $i = \overline{1,m}$. So,

$$\boldsymbol{v}_{\mathbf{N}} = (v_{\mathbf{N}i}), \quad v_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h}\left(r_i + \frac{1}{2}\right) \overset{r_i}{v}_{\mathbf{N}i} P_{r_i}(z), \qquad i = 1, ..., m,$$

and hence $\boldsymbol{v}_{\mathbf{N}} \in V_{\mathbf{N}}(\Omega)$. Consequently, corresponding vector-function $\vec{v}_{\mathbf{N}} = (\overset{0}{v}_{\mathbf{N}1}, ..., \overset{N_1}{v}_{\mathbf{N}1}, ..., \overset{0}{v}_{\mathbf{N}m}, ..., \overset{N_m}{v}_{\mathbf{N}m})^T \in \vec{V}_{\mathbf{N}}(\omega)$, since $\|\vec{v}_{\mathbf{N}}\|_* = \|\boldsymbol{v}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)} < \infty$, $\overset{r_i}{v}_{\mathbf{N}i} \in H^1_{loc}(\omega)$, $\overset{r_i}{v}_{\mathbf{N}i} = 0$ on γ_0, $0 \leq r_i \leq N_i$, $i = \overline{1,m}$. Moreover, $\vec{v}_{\mathbf{N}}^{(l)} \to \vec{v}_{\mathbf{N}}$ in $\vec{V}_{\mathbf{N}}(\omega)$, since

$$\|\vec{v}_{\mathbf{N}}^{(l)} - \vec{v}_{\mathbf{N}}\|_* = \|\boldsymbol{v}_{\mathbf{N}}^{(l)} - \boldsymbol{v}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)} \to 0, \qquad \text{as } l \to \infty.$$

Thus, $\vec{V}_{\mathbf{N}}(\omega)$ is a Hilbert space with scalar product defined by the norm $\|.\|_*$.

According to the inequality (4) the bilinear form $B(.,.)$ is coercive on $V(\Omega)$. Hence, B is coercive on the subspace $V_{\mathbf{N}}(\Omega) \subset V(\Omega)$, and, consequently, the bilinear form $B_{\mathbf{N}}(.,.)$ is coercive on the space $\vec{V}_{\mathbf{N}}(\omega)$,

$$B_{\mathbf{N}}(\vec{v}_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) = B(\boldsymbol{v}_{\mathbf{N}}, \boldsymbol{v}_{\mathbf{N}}) \geq c_B \|\boldsymbol{v}_{\mathbf{N}}\|^2_{\mathbf{H}^1(\Omega)} = c_B \|\vec{v}_{\mathbf{N}}\|^2_*, \qquad \forall \vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega).$$

Since $\boldsymbol{f} \in \widetilde{\mathbf{H}}^{-1}(\Omega)$, $\boldsymbol{g} \in \mathbf{H}^{-1/2}(\Gamma_1)$ we infer that the linear form L is continuous in $V(\Omega)$ and therefore $L_{\mathbf{N}}$ is also continuous in $\vec{V}_{\mathbf{N}}(\omega)$,

$$L_{\mathbf{N}}(\vec{v}_{\mathbf{N}}) = L(\boldsymbol{v}_{\mathbf{N}}) \leq \widetilde{c}\,\|\boldsymbol{v}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)} = \widetilde{c}\,\|\vec{v}_{\mathbf{N}}\|_*, \quad \forall \vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega),$$

and similarly the conditions on the matrices $\boldsymbol{M}_{pq}$, $\boldsymbol{M}_q$, $\boldsymbol{M}_0$, $\boldsymbol{M}_{\Gamma_1}$ imply continuity of the bilinear form $B_{\mathbf{N}}$. $\square$

So, we have constructed the sequence of n-dimensional problems, which can be considered as approximations of original problem (3) and prove that these problems has a unique solutions. In the sequel we investigate the convergence of the sequence of approximate solutions $\{\boldsymbol{w}_{\mathbf{N}}\}$, where $\boldsymbol{w}_{\mathbf{N}} \in V_{\mathbf{N}}(\Omega)$ corresponds to the solution $\vec{w}_{\mathbf{N}}$ of reduced problem, to the exact solution of the $(n+1)$-dimensional problem, but before we formulate the approximation theorem let us introduce the following anisotropic weighted Sobolev space

$$\mathbf{H}^{1,1,s}_{h^{\pm}}(\Omega) = \{\boldsymbol{v};\ \partial_3^{r-1}\boldsymbol{v} \in \mathbf{H}^1(\Omega),\ \partial_\alpha h^{\pm}\partial_3^r \boldsymbol{v} \in \mathbf{L}^2(\Omega),\ \alpha = \overline{1,n}, r = \overline{1,s}\}, \quad s \in \mathbb{N},$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\mathbf{H}^{1,1,s}_{h^{\pm}}(\Omega)} = \left(\sum_{r=1}^{s}\left[\left\|\partial_3^{r-1}\boldsymbol{v}\right\|^2_{\mathbf{H}^1(\Omega)} + \sum_{\alpha=1}^{n}\left(\left\|\partial_\alpha h^{+}\partial_3^r\boldsymbol{v}\right\|^2_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha h^{-}\partial_3^r\boldsymbol{v}\right\|^2_{\mathbf{L}^2(\Omega)}\right)\right]\right)^{1/2}.$$

Note, that $\mathbf{H}^{1,1,s}_{h^{\pm}}(\Omega)$ is a Hilbert space. Indeed, if $\{\boldsymbol{v}_k\}_{k\geq 1}$ is a Cauchy sequence in $\mathbf{H}^{1,1,s}_{h^{\pm}}(\Omega)$, then $\{\boldsymbol{v}_k\}_{k\geq 1}$ is a Cauchy sequence in the space $\mathbf{H}^1(\Omega)$ and, consequently, $\boldsymbol{v}_k \to \boldsymbol{v}$ in $\mathbf{H}^1(\Omega)$, as $k \to \infty$. Therefore, $\partial_3^r\boldsymbol{v}_k \to \partial_3^r\boldsymbol{v}$ in $\mathbf{H}^1(\Omega)$, as $k \to \infty$,$r = \overline{1, s-1}$. Since $h^{\pm} \in Lip(\omega)$ we have that $\partial_\alpha h^{\pm} \in L^\infty(\omega^*)$, for any subdomain ω^* of ω, $\overline{\omega^*} \subset \omega$, and hence

$$\partial_\alpha h^{\pm}\partial_3^r\boldsymbol{v}_k \to \partial_\alpha h^{\pm}\partial_3^r\boldsymbol{v} \qquad \text{in } \mathbf{L}^2(\Omega^*), \quad \text{as } k \to \infty, \tag{7}$$

where $r = \overline{1,s}$, $\alpha = \overline{1,n}$, Ω^* is a subdomain of Ω, $\overline{\Omega^*} \subset \Omega$. From (7), taking into acoount convergence of the sequence $\{\partial_\alpha h^{\pm}\partial_3^r\boldsymbol{v}_k\}_{k\geq 1}$ in $\mathbf{L}^2(\Omega)$, we infer that $\partial_\alpha h^{\pm}\partial_3^r\boldsymbol{v}_k \to \partial_\alpha h^{\pm}\partial_3^r\boldsymbol{v}$ in $\mathbf{L}^2(\Omega)$, as $k \to \infty$, $r = \overline{1,s}$, $\alpha = \overline{1,n}$, and, consequently, the space $\mathbf{H}^{1,1,s}_{h^{\pm}}(\Omega)$ is complete.

Theorem 2. *If all the conditions of Theorem 1 are fulfilled, then the vector-function* $\boldsymbol{w}_{\mathbf{N}} = (w_{\mathbf{N}i})$, $w_{\mathbf{N}i} = \sum\limits_{r_i=0}^{N_i} \dfrac{1}{h}\left(r_i + \dfrac{1}{2}\right) \overset{r_i}{w}_{\mathbf{N}i}\ P_{r_i}(z)$, $i = \overline{1,m}$, *restored from the solution* $\vec{w}_{\mathbf{N}} = (\overset{0}{w}_{\mathbf{N}1}, ..., \overset{N_1}{w}_{\mathbf{N}1}, \dots, \overset{0}{w}_{\mathbf{N}m}, ..., \overset{N_m}{w}_{\mathbf{N}m})^T$ *of n-dimensional problem (6) tends to the solution $\boldsymbol{u}$ of $(n+1)$-dimensional problem (3) in the space $\mathbf{H}^1(\Omega)$, as $N_1, ..., N_m \to \infty$. Moreover, if $\boldsymbol{u} \in \mathbf{H}^{1,1,s}_{h^{\pm}}(\Omega)$, $s \geq 2$, then*

$$\|\boldsymbol{u} - \boldsymbol{w}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)} \leq \frac{c_M}{c_B N^{s-1}}\theta(h^+, h^-, \mathbf{N}), \ \ \theta(h^+, h^-, \mathbf{N}) \to 0, \ \ \text{as } N = \min_{1\leq i\leq n}\{N_i\} \to \infty,$$

where $B(\boldsymbol{v}, \widetilde{\boldsymbol{v}}) \leq c_M\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)}\|\widetilde{\boldsymbol{v}}\|_{\mathbf{H}^1(\Omega)}$, for all $\boldsymbol{v}, \widetilde{\boldsymbol{v}} \in V(\Omega)$. If, in addition, $\|\boldsymbol{u}\|_{\mathbf{H}^{1,1,s}_{h^{\pm}}(\Omega)} \leq$ $\leq c$, c is independent of $h_{\max} = \max\limits_{(x_1,...,x_n)\in\overline{\omega}} h(x_1, ..., x_n)$, then

$$\|\boldsymbol{u} - \boldsymbol{w}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)} \leq \frac{c_M h_{\max}^{s-1}}{c_B N^{s-1}}\overline{\theta}(\mathbf{N}), \qquad \overline{\theta}(\mathbf{N}) \to 0, \ \ \text{as } N_1, ..., N_m \to \infty.$$

Proof. Note that since $\boldsymbol{u}$ satisfies the equation (3) for all $\boldsymbol{v} \in V(\Omega)$, then $B(\boldsymbol{u}, \boldsymbol{v}_{\mathbf{N}}) = L(\boldsymbol{v}_{\mathbf{N}})$, for all $\boldsymbol{v}_{\mathbf{N}} \in V_{\mathbf{N}}(\Omega) \subset V(\Omega)$. Hence, from the equation (6), taking into account definition of the forms $B_{\mathbf{N}}$ and $L_{\mathbf{N}}$, we infer that $B(\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}},\boldsymbol{v}_{\mathbf{N}})=0$, for all $\boldsymbol{v}_{\mathbf{N}}\in V_{\mathbf{N}}(\Omega)$, and, consequently, for all $\boldsymbol{v}_{\mathbf{N}}\in V_{\mathbf{N}}(\Omega)$,

$$B(\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}},\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}})=B(\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}},\boldsymbol{u}-\boldsymbol{v}_{\mathbf{N}})\leq c_M\|\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)}\|\boldsymbol{u}-\boldsymbol{v}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)}.$$

Applying the last inequality by the coerciveness of the bilinear form B we obtain

$$c_B\|\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)}\leq c_M\|\boldsymbol{u}-\boldsymbol{v}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)},\qquad \forall\boldsymbol{v}_{\mathbf{N}}\in V_{\mathbf{N}}(\Omega). \tag{8}$$

First, let us estimate the rate of approximation of $\boldsymbol{u}$ by $\boldsymbol{w}_{\mathbf{N}}$, when $\boldsymbol{u}$ satisfies additional regularity conditions. By means of the solution $\boldsymbol{u}$ we construct the vector-function $\boldsymbol{u}_{\mathbf{N}}$, which is an element of the space $V_{\mathbf{N}}(\Omega)$ and we can estimate the norm of the difference $\boldsymbol{u}-\boldsymbol{u}_{\mathbf{N}}$. Let us consider

$$\boldsymbol{u}_{\mathbf{N}}=(u_{\mathbf{N}i}),\quad u_{\mathbf{N}i}=\sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{u}_i\,P_{r_i}(z)+\sum_{r_i=N_i}^{N_i+1}\frac{1}{2}\,\overset{r_i}{\partial_3 u_i}\,P_{r_i-1}(z),$$

where $\overset{r}{v}=\int\limits_{h^-}^{h^+}vP_r(z)dx_3$, for any function $v\in L^2(\Omega)$, $r\in\mathbb{N}\cup\{0\}$. From the conditions of the theorem it follows, that $\boldsymbol{u}_{\mathbf{N}}\in V_{\mathbf{N}}(\Omega)$. Indeed, since $\boldsymbol{u}\in V(\Omega)$, then $\boldsymbol{u}_{\mathbf{N}}=\mathbf{0}$ on Γ_0. Hence, it suffices to prove that $\boldsymbol{u}_{\mathbf{N}}\in\mathbf{H}^1(\Omega)$.

Note that the Legendre polynomials satisfy the following equalities

$$\begin{aligned}P_r(t)&=\frac{1}{2r+1}(P'_{r+1}(t)-P'_{r-1}(t)), && r\geq 1,\\ tP'_r(t)&=P'_{r+1}(t)-(r+1)P_r(t), && r\geq 0,\end{aligned}$$

from which we have that for almost all $(x_1,...,x_n)\in\omega$,

$$\overset{r}{u}_i(x_1,...,x_n)=\frac{h}{2r+1}\left(\overset{r-1}{\partial_{n+1}u_i}(x_1,...,x_n)-\overset{r+1}{\partial_{n+1}u_i}(x_1,...,x_n)\right),\qquad r\geq 1, \tag{9}$$

$$\partial_\alpha(\overset{r}{u}_i)=\overset{r}{\partial_\alpha u_i}+\frac{\partial_\alpha h}{h}(r+1)\,\overset{r}{u}_i+\partial_\alpha\bar{h}\,\overset{r}{\partial_{n+1}u_i}+\partial_\alpha h\,\overset{r+1}{\partial_{n+1}u_i},\qquad r\geq 0,$$

where $i=\overline{1,m}$, $\alpha=\overline{1,n}$. Applying these formulas and expressions for derivatives of Legendre polynomials

$$\left.\begin{aligned}P'_r(t)&=\sum_{k=0}^{r-1}\left(k+\frac{1}{2}\right)(1-(-1)^{r+k})P_k(t),\\ tP'_r(t)&=rP_r(t)+\sum_{k=0}^{r-1}\left(k+\frac{1}{2}\right)(1+(-1)^{r+k})P_k(t),\end{aligned}\right\}\quad r\geq 1,$$

we obtain

$$\partial_{n+1}u_{\mathbf{N}i}=\sum_{r_i=0}^{N_i-1}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{\partial_{n+1}u_i}\,P_{r_i}(z),$$

$$\partial_\alpha u_{\mathbf{N}i} = \sum_{r_i=0}^{N_i} \frac{1}{h}\left(r_i+\frac{1}{2}\right) \overset{r_i}{\partial_\alpha u_i}\, P_{r_i}(z) + \frac{\partial_\alpha \bar{h}}{h}\left(N_i+\frac{1}{2}\right) \overset{N_i}{\partial_{n+1} u_i}\, P_{N_i}(z) +$$
$$+\sum_{r_i=N_i}^{N_i+1} \frac{\partial_\alpha h}{h}\left(r_i+\frac{1}{2}\right) \overset{r_i}{\partial_{n+1} u_i}\, P_{r_i-1}(z) + \sum_{r_i=N_i}^{N_i+1}\frac{1}{2}\Big(\partial_\alpha \overset{r_i}{\partial_{n+1} u_i} +$$
$$+\partial_\alpha \bar{h}(\partial_{n+1}\overset{r_i}{\partial_{n+1} u_i}) + \partial_\alpha h(\partial_{n+1}\overset{r_i+1}{\partial_{n+1} u_i})\Big) P_{r_i-1}(z), \quad i=\overline{1,m},\ \alpha=\overline{1,n}.$$

According to the conditions of the theorem $u_i, \partial_{n+1}u_i \in H^1(\Omega)$, $\partial_\alpha h^\pm \partial_{n+1}u_i$, $\partial_\alpha h^\pm \partial_{n+1} \partial_{n+1}u_i \in L^2(\Omega)$ and, consequently, taking into account expressions for $\partial_p u_{\mathbf{N}i}$ we deduce that $u_{\mathbf{N}i} \in H^1(\Omega)$, $i=\overline{1,m}$, $p=\overline{1,n+1}$.

In order to obtain the estimates of the theorem, let us consider the residue $\boldsymbol{\varepsilon}_{\mathbf{N}} = (\varepsilon_{\mathbf{N}i})$,

$$\varepsilon_{\mathbf{N}i} = u_i - u_{\mathbf{N}i} = \sum_{r_i=N_i+1}^{\infty} \frac{1}{h}\left(r_i+\frac{1}{2}\right) \overset{r_i}{u_i}\, P_{r_i}(z) - \sum_{r_i=N_i}^{N_i+1} \frac{1}{2}\, \overset{r_i}{\partial_{n+1} u_i}\, P_{r_i-1}(z), \quad i=\overline{1,m}.$$

By the orthogonality property of Legendre polynomials, taking into account expressions for $\partial_{n+1}u_{\mathbf{N}i}$, $\partial_\alpha u_{\mathbf{N}i}$ ($\alpha=\overline{1,n}, i=\overline{1,m}$) and Parseval equality, we obtain

$$\|\varepsilon_{\mathbf{N}i}\|^2_{L^2(\Omega)} = \sum_{r_i=N_i+1}^{\infty} \int_\omega \frac{1}{h}\left(r_i+\frac{1}{2}\right)(\overset{r_i}{u_i})^2 d\omega + \sum_{r_i=N_i-1}^{N_i} \int_\omega \frac{h}{2r_i+1}(\overset{r_i+1}{\partial_{n+1} u_i})^2 d\omega,$$
$$\|\partial_{n+1}\varepsilon_{\mathbf{N}i}\|^2_{L^2(\Omega)} = \sum_{r_i=N_i}^{\infty} \int_\omega \frac{1}{h}\left(r_i+\frac{1}{2}\right)(\overset{r_i}{\partial_{n+1} u_i})^2 d\omega,$$

$$\|\partial_\alpha\varepsilon_{\mathbf{N}i}\|^2_{L^2(\Omega)} \le \sum_{r_i=N_i+1}^{\infty} \int_\omega \frac{1}{h}\left(r_i+\frac{1}{2}\right)(\overset{r_i}{\partial_\alpha u_i})^2 d\omega +$$
$$+\frac{9}{2}\sum_{r_i=N_i}^{N_i+1} \int_\omega \frac{h}{2r_i-1}\left((\partial_\alpha \overset{r_i}{\partial_{n+1} u_i})^2 + (\partial_\alpha \bar{h})^2(\partial_{n+1}\overset{r_i}{\partial_{n+1} u_i})^2 + (\partial_\alpha h)^2(\partial_{n+1}\overset{r_i+1}{\partial_{n+1} u_i})^2\right) d\omega +$$
$$+9\sum_{r_i=N_i}^{N_i+1} \int_\omega \frac{(N_i+1-r_i)(\partial_\alpha \bar{h})^2 + (\partial_\alpha h)^2}{h}\left(r_i+\frac{1}{2}\right)(\overset{r_i}{\partial_{n+1} u_i})^2 d\omega,$$

where $\alpha=\overline{1,n}$, $i=\overline{1,m}$. From (9) we infer, that

$$\|(\overset{r}{\partial^{\widehat{\beta}}_{n+1}\partial^{\beta}_\alpha u_i})\|^2_{L^2(\omega)} \le \frac{c}{r^{2(s-\beta-\widehat{\beta})}} \sum_{k=r-s+\beta+\widehat{\beta}}^{r+s-\beta-\widehat{\beta}} \|h^{s-\beta-\widehat{\beta}}\, (\overset{k}{\partial^{s-\beta}_{n+1}\partial^{\beta}_\alpha u_i})\, \|^2_{L^2(\omega)}, \tag{10}$$

where $r \ge s-\beta-\widehat{\beta}$, $\beta, \widehat{\beta} = 0,1$, $i=\overline{1,m}$, $\alpha=\overline{1,n}$, $c = const > 0$ is independent of h^+, h^- and r. Therefore, from the last estimate, for all $i=\overline{1,m}$, $p=\overline{1,n+1}$, we obtain

$$\begin{aligned}&\|\varepsilon_{\mathbf{N}i}\|^2_{L^2(\Omega)} \leq \frac{1}{N_i^{2s}}\theta_i(h^+,h^-,N_i),\\ &\|\partial_p\varepsilon_{\mathbf{N}i}\|^2_{L^2(\Omega)} \leq \frac{1}{N_i^{2(s-1)}}\theta_i(h^+,h^-,N_i),\end{aligned} \qquad \theta_i(h^+,h^-,N_i)\to 0, \text{ as } N_i\to\infty. \tag{11}$$

Hence, the inequality (8) imply

$$\|\boldsymbol{u}-\boldsymbol{w}_{\mathbf{N}}\|_{\mathbf{H}^1(\Omega)} \leq \frac{c_M}{c_B N^{s-1}}\theta(h^+,h^-,\mathbf{N}), \qquad N=\min_{1\leq i\leq m} N_i,$$

where $\theta(h^+,h^-,\mathbf{N})\to 0$, as $N\to\infty$.

In addition, if the norm of $\boldsymbol{u}$ in the space $\mathbf{H}^{1,1,s}_{h^\pm}(\Omega)$ is independent of the maximum of the function $h(x_1,...,x_n)$ on the set $\overline{\omega}$, then applying (9) we have

$$\begin{aligned}&\|\varepsilon_{\mathbf{N}i}\|^2_{L^2(\Omega)} \leq \frac{h^{2s}_{\max}}{N_i^{2s}}\bar{\theta}_i(N_i),\\ &\|\partial_p\varepsilon_{\mathbf{N}i}\|^2_{L^2(\Omega)} \leq \frac{h^{2(s-1)}_{\max}}{N_i^{2(s-1)}}\bar{\theta}_i(N_i),\end{aligned} \qquad \bar{\theta}_i(N_i)\to 0, \text{as} N_i\to\infty, i=\overline{1,m},\ p=\overline{1,n+1},$$

from which, due to the inequality (8), follows the second estimate of the theorem.

Now let us prove the convergence result stated in the theorem. According to the trace theorems for Sobolev spaces [14], for any $\boldsymbol{v}\in\mathbf{H}^1(\Omega)$, $\boldsymbol{v}=\mathbf{0}$ on Γ_0, there exists continuation $\widetilde{\boldsymbol{v}}\in\mathbf{H}^1_0(\Omega_1)$ of $\boldsymbol{v}$, where Ω_1 is a Lipschitz domain, $\Omega\subset\Omega_1$, $\Gamma_0\subset\partial\Omega_1$. Note, that there exists a Lipschitz domain $\Omega_*\subset\Omega_1$, $\Omega_*=\{x\in\mathbb{R}^{n+1}; h^-_*(x_1,...,x_n)< < x_{n+1} < h^-_*(x_1,...,x_n),\ \ (x_1,...,x_n)\in\omega\}$, $h^-_*\leq h^-\leq h^+\leq h^+_*$, $h^-_*<h^+_*$ on $\overline{\omega}$, $h^\pm_*\in C^1(\overline{\omega})$. Since the set of infinitely differentiable functions $\mathcal{D}(\Omega_1)$ with compact support in Ω_1 is dense in $H^1_0(\Omega_1)$, then for any $\widetilde{\boldsymbol{w}}\in\mathbf{H}^1_0(\Omega_1)$ there exists a sequence of infinitely differentiable vector-functions $\{\widehat{\boldsymbol{v}}_k\}_{k\geq 1}$, $\widehat{\boldsymbol{v}}_k\in[C^\infty_{\Gamma^*_0}(\Omega_*)]^m$, which are defined on Ω_* and vanish on $\Gamma^*_0=\{x\in\partial\Omega_*;(x_1,...,x_n)\in\gamma_0\}$, such that $\widehat{\boldsymbol{v}}_k\to\widetilde{\boldsymbol{w}}|_{\Omega_*}$ in $\mathbf{H}^1(\Omega_*)$, as $k\to\infty$. Consequently, applying the inequalities (11) we deduce that the union of subspaces $[C^\infty_{\Gamma^*_0}(\Omega_*)]^m\cap V_{\mathbf{N}}(\Omega_*)$ for all $\mathbf{N}\in[\mathbb{N}\cup\{0\}]^m$ is dense in $[C^\infty_{\Gamma^*_0}(\Omega_*)]^m\subset V(\Omega_*)$, where $V_{\mathbf{N}}(\Omega_*)$ and $V(\Omega_*)$ are defined in the same way as $V_{\mathbf{N}}(\Omega)$ and $V(\Omega)$ in which Ω, Γ_0 is replaced by Ω_*, Γ^*_0. Hence $\bigcup\limits_{\mathbf{N}\geq\mathbf{0}}[C^\infty_{\Gamma^*_0}(\Omega_*)]^m\cap V_{\mathbf{N}}(\Omega_*)$ is dense in $V(\Omega_*)$ and, considering the restrictions of vector-functions from the spaces $V_{\mathbf{N}}(\Omega_*)$ on Ω, from (8) we obtain that $\boldsymbol{w}_{\mathbf{N}}\to\boldsymbol{u}$ in the space $\mathbf{H}^1(\Omega)$, as $N_1,...,N_m\to\infty$. $\square$

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