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ON SINGULAR INTEGRAL APPROXIMATION METHOD IN NUMERICAL CONFORMAL MAPPINGS

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Application of the boundary integral equiations method to the conformal mapping problems is well known (see e.g.[1], [2], [3]). The mentioned method is one of the effective methods for mapping of both simply and multiply connected domains. At this it is sufficiently convenient namely for numerical solution of similar problems.

Usually in classical literature, speaking about solution of conformal mapping problems by integral equations, application of the Fredholm integral equations is meant. By that realization of numerical solution of the initial problem in such cases is possible by using this or that approximate method of solution of analogous integral equations. However, we can say that generally the kernels of mentioned Fredholm integral equations depend in a not rather easy way on the boundary of the domain (see e.g. [1],[2],[4]), what sometimes may be a reason of some difficulties in calculations of their values, especially when the boundary is given in tabular or graphical way. In such cases most of all it is necessary to use the numerical differentiation with respect to the approximate data, which can lead to the decrease of reliability of calculation results.

The influence of the similar factors on the accuracy of the schemes constructed on such basis is more pereceptible when besides the problem of conformal mapping of any domain to a disk also the inverse problem — the problem of conformal mapping of a disk to the given domain is being solved.

As further considerations show, from the mentioned point of view, it is significantly more convenient to represent the corresponding boundary integral equations by means of singular integrals (in principle value sense) with their approximate schemes.

The boundary integral equations for conformal mapping problems (generally for the multiply connected domains) may be constructed starting from the Dirichlet modified problem [1,4,5] (which for simply connected domains represents the classical Dirichlet problem). In this note, using one concrete approximation of singular integrals with Cauchy type kernel, a certain simplified numerical scheme for solution of conformal mapping problem is given.

For the sake of definiteness we will consider in more details a question of mapping of multiply connected domain. Let us note also that the application of the considered below scheme is supposed in the case of finite domains.

Thus, let D be a finite simply connected domain in a plane of complex variable z with a boundary L which represents an arbitrary (closed) Liapunov contour.

Assuming that the origin is inside the domain D, we should find an analytical function $\omega = \omega(z)$ realizing one-to-one conformal mapping of the domain D to a unit disk $|\omega| < 1$ under the conditions $\omega(0) = 0$, $\omega'(0) > 0$. Usually the unknown function may be represented in the form

$$\omega(z) = ze^{\Phi(z)} \quad (z \in D) \tag{1}$$

where $\Phi(z)$ is an analytical function in D and $\mathrm{Im}\ \Phi(0) = 0$. Under the condition that $Re\Phi(z)$ is continuously continuable up to the contour L, the function $\Phi(z)$ may be represented [4] as

$$\Phi(z) = \Phi(\varphi; z) = \frac{1}{\pi i}\int\limits_L \frac{\varphi(t)}{t-z}dt + iC, \tag{2}$$

in which φ is a real (continuous) function and C is a real constant. From $Im\ \Phi(0) = 0$

$$C = -Im\frac{1}{\pi i}\int\limits_L \frac{\varphi}{t}dt.$$

Further, since $\omega^+(t_0) = \lim\limits_{z\to t_0}\omega(z) \quad (t_0 \in L)$, $\omega^+(t_0) \in \gamma \quad (|\gamma| = 1)$ we have $|\omega^+(t_0)| = 1$. According to this from (1) we can easily get

$$Re\Phi^+(\varphi; t_0) = -\ln|t_0|. \tag{3}$$

Meaning under ϕ^+ the boundary value of the represented by the formula (2) function and taking into account the given condition on the contour L, on the basis of Sokhotski-Plemelj formula the boundary integral equation

$$\varphi(t_0) + Re\frac{1}{\pi i}\int\limits_L \frac{\varphi(t)}{t-t_0}dt = -\ln|t_0| \;\; (t_0 \in L) \tag{4}$$

follows from (3). The singular integral in (4) is considered in Cauchy principal value sense. It is known (see e.g. [4]) that equation (4) is uniquely solvable. At this the unknown (real) function φ belongs to the Holder class.

The equation (4) may be numerically solved by the scheme which means an approximate change of the singular integral in (4). One of such schemes is shown in [6]. Namely, the mentioned scheme bases on the quadrature formula

$$\frac{1}{\pi i}\int\limits_L \frac{\varphi(t)}{t-t_0}dt \approx \varphi(t_0) + (p_{\nu-1} + 2p_\nu + p_{\nu+1})\frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} +$$

$$+\sum_{\sigma=1}^{n-2}(p_{\nu+\sigma} + p_{\nu+\sigma+1})\frac{\varphi(\tau_{\nu+\sigma+1}) - \varphi(\tau_\nu)}{\tau_{\nu+\sigma+1} - t_0} \quad (t_0 \in \tau_\nu\tau_{\nu+1}, \;\; \nu = 0, 1, , ..., n-1), \tag{5}$$

where the knots system $\{\tau_j\} \subset L$, $\tau_\nu\tau_{\nu+1}(0 \le \nu \le n-1)$ is the least arc of the contour L with ends τ_ν, $\tau_{\nu+1}$, $p_j = \frac{1}{2\pi i}(\tau_{j+1} - \tau_j)$. At the same time in [6] it is meant that the knots $\{\tau_j\}_{j=1}^n$ divide the contour into equal parts. However we can see that all the proofs will remain true if the contour L is divided into equivalent arcs (it means that the ratio of the lengths of two arbitrary arcs is bounded from both sides by absolute constants). For the function $\varphi(t)$ satisfying the Holder condition with index α $(0 < \alpha \le 1)$ the shown quadrature formula realizes the approximation of the singular integral with accuracy $O(n^{-\alpha}\ln n)$. For sufficiently smooth functions φ the accuracy $O(n^{-2}\ln n)$ is accessible. [1]

Assuming $t_0 = \tau_\nu$ $(\nu = 0, 1, ..., n-1)$ in formula (5) after its application to approximation of singular integral in equation (4), we will have a linear algebraic system. The solving of the system is one of the steps in realization of the offered numerical scheme.

It is evident that the calculation of the coefficients of linear algebraic system approximating the initial integral equation (4) on the basis of formula (5) can be realized by easy scheme independently of the configuration of the contour L. At this, as is shown in [6] the calculation scheme can be founded (in Holder space). It is important that the accuracy of approximation of the equation (4) depends exclusively (within the limits of accuracy order of the used quadrature formula) on differential properties of the density $\varphi(t)$ which in its turn is determined by the properties of L.

After calculation of the approximate values of the function $\varphi(t)$ the construction of the approximate conformally mapping function on the basis of the formula (2) is reduced to approximate calculation of the Cauchy type integral

$$\frac{1}{\pi i}\int_L \frac{\varphi(t)}{t-z}dt \tag{6}$$

for arbitrary $z \in D$. The cases $z \to t^* \in L$ are especially important. From this point of view it seems sufficiently effective to use the following quadrature formula

$$\frac{1}{\pi i}\int_L \frac{\varphi(t)}{t-z}dt \approx Q_n(\varphi; z, \tau), \tag{7}$$

$$\begin{gathered}
Q_n(\varphi; z, \tau) = 2L_\mu(\varphi;\tau) + \Bigg\{p_{\mu-1} + 2p_\mu + p_{\mu+1} + \frac{1}{\pi i}(z-\tau)\bigg[\frac{z-\tau_{\mu-1}}{\tau_\mu - \tau_{\mu-1}}\ln\frac{\tau_\mu - z}{\tau_{\mu-1} - z} + \\
+ \ln\frac{\tau_{\mu+1} - z}{\tau_\mu - z} + \frac{z - \tau_{\mu+2}}{\tau_{\mu+1} - \tau_{\mu+2}}\ln\frac{\tau_{\mu+2} - z}{\tau_{\mu+1} - z}\bigg]\Bigg\}\frac{\varphi(\tau_{\mu+1}) - \varphi(\tau_\mu)}{\tau_{\mu+1} - \tau_\mu} + \\
+ \sum_{\sigma=1}^{n-2}\Bigg\{p_{\mu+\sigma} + p_{\mu+\sigma+1} + \frac{1}{\pi i}(z-\tau)\bigg[\frac{z - \tau_{\mu+\sigma}}{z_{\mu+\sigma+1} - \tau_{\mu+\sigma}}\ln\frac{\tau_{\mu+\sigma+1} - z}{\tau_{\mu+\sigma} - z} + \\
+ \frac{z - \tau_{\mu+\sigma+2}}{\tau_{\mu+\sigma+1} - \tau_{\mu+\sigma+2}}\ln\frac{\tau_{\mu+\sigma+2} - z}{\tau_{\mu+\sigma+1} - z}\bigg]\Bigg\}\frac{\varphi(\tau_{\mu+\sigma+1}) - L_\mu(\varphi;\tau)}{\tau_{\mu+\sigma+1} - \tau}, \quad \tau \in L
\end{gathered}$$

[1] about similar formulas of higher order see in [7].

$$(\tau \in \tau_\mu \tau_{\mu+1},\ \mu = 0, 1, ..., n-1)$$

(with known chosen branch of logarithmic function) based on the approximation

$$\varphi(t) \approx L_\mu(\varphi;\tau) + \sum_{\sigma=0}^{n-1} \sum_{k=0}^{1} l_{\sigma k}(t) \alpha_{\sigma k}(\varphi; t, \tau),$$

$$\alpha_{\sigma k}(\varphi;\tau) = \begin{cases} \dfrac{\varphi(\tau_{\sigma+k}) - L_\mu(\varphi;\tau)}{\tau_{\sigma+k} - \tau}, & \sigma + k \neq \mu, \mu+1; \\ \dfrac{\varphi(\tau_{\mu+1}) - \varphi(\tau_\mu)}{\tau_{\mu+1} - \tau_\mu}, & \sigma = \mu, \mu+1. \end{cases}$$

Here $\{\tau_j\}_{j=1}^n$ represents the mentioned above points system, $L_\mu(\varphi;\tau)$ is a linear interpolator with knots $\tau_m,\ \tau_{m+1}$, and $l_{mo}(t),\ l_{m1}(t)$ are the corresponding Lagrange fundamental polynomials. Let us note that the formula (7) for arbitrary $\tau \in L$ and fixed $z \in D$ has the same accuracy (in the known sense) as the formula (5). It may be shown that if the parameter $\tau \in L$ is chosen so that the distance between z and values of τ is the least among all distances between z and boundary points, the appropriate estimate holds whatever the way of approaching of z to the boundary may be.

For practical construction of approximate conformally mapping function let us note that really we have no values of function $\varphi(t)$ but only their approximate values $\{\varphi_n(\tau_j)\}$, obtained by solution of linear algebraic system approximating the integral equation (4). In connection with this, taking into account (7), for approximation of function $\Phi(z) = \Phi(\varphi; z)$ in (1) we get the expression

$$\Phi_n(\varphi_n; z, \tau) = Q_n(\varphi_n; z, \tau) - i\ Im Q_n(\varphi_n; 0, \tau).$$

At this it is not difficult to get the following estimate $(z \to \tau)$

$$|\Phi(\varphi; z) - \Phi_n(\varphi_n; z, \tau)| \leq c_0 \max_{1 \leq j \leq n} |\varphi(\tau_j) - \varphi_n(\tau_j)| \ln n \ \ (n > 1),$$

where the constant c_0 is independent of z and τ. On this basis we can consider that for sufficiently large n the function

$$\omega_n(z) = z e^{\Phi_n(\varphi_n; z, \tau)} \tag{8}$$

is realizing the approximate conformal mapping of the domain D to the unit disk for the mentioned z.

As it was particularly noted above, the calculation formulas in the given scheme sufficiently simply depend on used knots of contour L and do not put on their choice any strong limitations. It can be observed that for equivalent division of the contour L we have an appropriate similar division of the circle γ $(|\gamma| = 1)$. Using this we can realize and found the scheme of appropriate solution problem of conformal mapping of unit disk to domain D. Assuming in these case for function $z = w(\zeta)$ $(|\zeta| < 1)$

$$w(\zeta) = \zeta e^{\Phi_0(\zeta)},$$

let us use the representation

$$\Phi_0(\zeta) = \frac{1}{\pi i} \int\limits_{\gamma} \frac{\varphi_0(\xi)}{\xi - \zeta} d\xi + ib_0 \quad (Im\, \varphi_0(\xi) \equiv 0),$$

where analogously to above

$$b_0 = -Im \int\limits_{\gamma} \frac{\varphi_0(\xi)}{\xi} d\xi.$$

Considering that the point $\xi_0 \in \gamma$ corresponds to the point $t_0 \in L$, from $|\xi_0| = 1$ and $w^+(\xi_0) = t_0$ we get $Re\Phi_0^+(t_0) = \ln|t_0|$, which leads to the following integral equation

$$\varphi_0(\xi_0) + Re\frac{1}{\pi i} \int\limits_{\gamma} \frac{\varphi_0(\xi)}{\xi - \xi_0} d\xi = \ln|t_0| \quad (\xi_0 = w^+(t_0)).$$

For properly chosen knots $\{t_j\}_{j=1}^N$ on the contour L we have certain system of knots $\xi_j = \omega_n^+(t_j) \quad (j = \overline{1,n})$ for approximate solution of the inverse conformal mapping problem. In particular, we can take for $\{t_j\}_{j=1}^N$ the system $\{\tau_j\}_{j=1}^n$, which is used in approximate solution of equation (4). Probably it seems that the specifics of the singular integral approximation method makes this method more convenient, namely, for solution of the inverse mapping problem — conformal mapping of the disk to the given domain.

As it was noted above, the most effective error estimates of the approximate formula (7) for the points z close to the boundary L are obtained for certain choice of the parameter $\tau \in L$. In practical calculations, while solving numerically the problem of conformal mapping to the disk, in the role of τ the nearest (to z) knot from the set $\{\tau_j\}_{j=1}^n$ is taken. For such choice of τ satisfactory results are anticipated when n is sufficiently large. This fact is confirmed by numerical experiment too. Similar fact takes place for corresponding inverse problem.

At calculating boundary values of conformal mapping functions (on L or on the unit circle γ) the formulas for approximate calculation of Cauchy type integrals turn into quadrature formulas for singular integrals in the principle value sense. In particular, this happens to the formula (7) when $z \to \tau$, $\tau \in L$ in the expression (8) on the basis of which for the point t_0, given on the boundary, its image on the circle γ may be found (approximately). In this case, as mentioned above, it is reasonable to put $\tau = t_0$ in the used quadrature formula. Choice of the number ν, for which in the corresponding quadrature formula is assumed $t_0 \in \tau_\nu \tau_{\nu+1}$, is automatically simplified when $t_0 \in \{\tau_j\}_{j=1}^n$. The similar refers to the case of the inverse problem of conformal mapping too.

For z (as well as for ζ), being relatively far from the boundary points, for approximate calculation of the Cauchy type integrals instead of formulas of the form (7), a usual (more simple than (7)) quadrature formula with values of corresponding integrands (naturally, with the same knots and coefficients) may be used.

Let us illustrate the stated scheme on concrete examples.

Example 1. Let D be a domain bounded by the Pascal's limacon whose parametric equation is

$$x = \cos\theta + m\cos 2\theta$$

$$y = \sin\theta + m\sin 2\theta$$

$$(0 \le \theta \le 2\pi;\ \ 0 < m < 1/2).$$

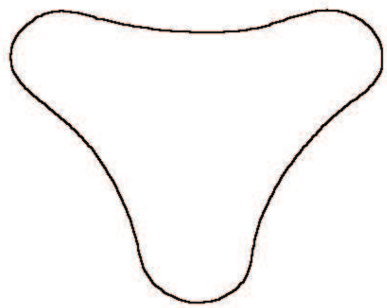
The function, conformally mapping a unit disk to the given domain, has the form $w(\zeta) = \zeta + m\zeta^2$ (see, e.g., [8]).

In the Table 1 are given approximate values of conformally mapping function $\zeta = = \omega(z)$ and $z = w(\zeta)$ $(|\zeta| < 1)$, found by the scheme, in some inner and boundary points (the second column) for different numbers of division (in our case $m = 0.2$). For comparison the values of the function $\zeta + m\zeta^2$ are given also. Values of $|\zeta|$ in the third column indicate the closeness of the points ζ to the unit circle. In the case when the points are sufficiently close to the boundary (we consider so if the distance between them and the circle is less than 0.1) the formula (7) is applied.

Table 1

n	ζ	$\lvert\zeta\rvert$	$z = w(\zeta)$	$z = \zeta + m\zeta^2$
100	0.86516-0.49950i	0.99900	0.96510-0.67223i	0.96496-0.67236i
200			0.96503-0.67233i	
100	0.86169-0.49750i	0.99500	0.96085-0.66885i	0.96070-0.66898i
200			0.96077-0.66899i	
100	-0.85592+0.41219i	0.95000	-0.74330+0.27119i	-0.74338+0.27107i
200			-0.74336+0.27110i	
100	0.17321-0.1000i	0.20000	0.17721-0.10692i	0.177205-0.10698i
200			0.17721-0.10693i	
100	0.21694+0.45048i	0.50000	0.18580+0.48951i	0.18577+0.48958i
200			0.18578+0.48956i	
100	-0.76582+0.36880i	0.85000	-0.67581+0.25600i	-0.67573+0.25583i
200			-0.67575+0.25587i	
100	0.99803+0.06279i	1.00000	1.19643+0.08788i	1.19645+0.08786i
200			1.19645+0.08786i	
100	-0.70711+0.70711i	1.00000	-0.71539+0.51314i	-0.70711+0.50711i
200			-0.70706+0.50712i	
100	-0.50000-0.86603i	1.00000	-0.60502-0.69917i	-0.60000-0.69282i
200			-0.59741-0.68990i	

Example 2. Let D be a domain bounded by the contour, given below graphycally:



The results of calculation (for $n = 100$) are presented in the Table 2. In this case we have no possibility to compare the obtained (approximate) values $w(\zeta)$ with the values of the exact solution. But, taking into account that $\{\tau_k\}_{k=1}^{n}$ are the knots taken[2] on the boundary of the given domain, which are firstly mapped to the unit circle, and the values of the function $t = w(\xi)$ ($\xi \in \gamma$, $t \in L$) are conformal images of the knots on the circle, corresponding to τ_k, by their closeness we may judge about the certain reliability of the obtained results.

Table 2

$w(\xi)$	$t = \tau_k$
0.237614-1.840094i	0.22500-1.84500i
2.45422+1.46120i	2.47863+1.43820i
-1.11792+2.16323i	-1.12429+2.16004i
-0.54668-0.83743i	-0.54739-0.82911i
1.21493-0.19125i	1.21220-0.19250i
0.91804+1.97434i	0.92885+1.97163i
-1.63406+0.89161i	-1.62392+0.91009i
1.56459+2.21466i	1.59059+2.19457i
-0.63106-0.56854i	-0.62762-0.56440i
-1.95545+1.16876i	-1.91774+1.21306i

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