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ONE MIXED PROBLEM OF THE THEORY OF ANALYTICAL FUNCTIONS FOR CIRCULAR RING CUT ALONGAN ARC OF CIRCLE

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Let us consider the mixed boundary value problem of the theory of analytical functions. The circular ring cut along the arc of circle. The considered problem leads to various problems of plane theory of elasticity and bending of a plate for double-connected domain with a partly unknown boundary. (see [1],[2]) Analogous problem for infinite plane cut along arc of circle is studied in [3], [4] etc., and problem of circular ring cut along of arc is studied in [5].

According to the analytic functions method the solution of considered problem is constructed effectivly.

Let D be a circular ring $\{\frac{1}{R} < |z| < R\}$ cut by separetely lying arc $\delta_k\gamma_k$ $(k = = 1,\dots,r)$ of the circle l_1 $(|z| = 1)$. The points $\delta_1,\gamma_1,\dots$ lie one after another in positive direction, i.e. the directions that leave on the left area $\{1 < |\zeta| < R\}$.The totality of arcs $\{\delta_k\gamma_k\}$ we shall arbitrarily divide into two parts and introduce symbols $l_1' = = \bigcup\limits_{k=1}^{r'} \delta_k'\gamma_k'$; $l_1'' = \bigcup\limits_{k=1}^{r''} \delta_k''\gamma_k''$; $l_1''' = l_1 - (l_1' \bigcup l_1'')$. Analogously consider a_kb_k the are $(k = = 1,\dots,p)$ separetely lying on the circle l_0 $(|z| = R)$, their divide into two totality and introduce symbols $l_0' = \bigcup\limits_{k=1}^{p'} a_k'b_k'$; $l_0'' = \bigcup\limits_{k=1}^{p''} a_k''b_k''$; $l_0''' = l_0 - (l_1' \bigcup l_0'')$. We shall denote by $l_0^*,\dots,l_0^{*\prime\prime\prime}$, accordingly the map of $l_0,\dots,l_0'''$ in mapping $z_1 = z/R^2$.

Let us consider the problem:

Find holomorpic function $\Phi(z) = u + iv$ with boundary condition in D.

$$\begin{aligned} &u(\tau) = f_{10}(\tau), \quad \tau \in l_0'; \quad v(\tau) = g_{10}(\tau), \quad \tau \in l_0''; \\ &\Phi(\tau) - \Phi(\tau/R^2) = h_{10}(\tau), \quad \tau \in l_0''. \end{aligned} \tag{1}$$

$$u(\tau) = f_{11}(\tau), \quad \tau \in l_0^{*\prime}; \quad v(\tau) = g_{11}(\tau), \quad \tau \in l_0^{*\prime\prime}. \tag{2}$$

$$\begin{aligned} &u^{\pm}(\sigma) = f^{\pm}(\sigma), \quad \sigma \in l_1'; \quad v^{\pm}(\sigma) = g^{\pm}(\sigma), \quad \sigma \in l_1''; \\ &\Phi^+(\sigma) - \Phi^-(\sigma) = f_1(\sigma) + if_2(\sigma), \quad \sigma \in l_1''. \end{aligned} \tag{3}$$

where $f_{10}(\tau),\dots,f_1(\sigma)$ are H, class functions, which are given respectively on $l_0',\dots,l_1'''$.

We require that sought-for function $\Phi(z)$ be continously on the bound of domain D with the exception of points $\delta_1', \ldots, b''_{p''}$, near that it has integable singularity.

The desigration is mentioned in monograph [3].

Let us consider piecewise-holomorphic functions $\Omega(z)$ and $\Psi(z)$, defined following form:

$$\Omega(z) = \frac{1}{2}\left[\Phi(z) + \Phi_*(z)\right]; \quad \Psi(z) = \frac{i}{2}\left[\Phi(z) - \Phi_*(z)\right], \tag{4}$$

where $\Phi_*(z) = \overline{\Phi(1/\overline{z})}$. It is evident these functions satisfies following condition:

$$\Omega_*(z) = \Omega(z); \quad \Psi_*(z) = \Psi(z), \tag{5}$$

And the function $\Phi(z)$ is defined by these functions

$$\Phi(z) = \Omega(z) - i\Psi(z), \tag{6}$$

On the basis of these formulas, from boundary condition (1)-(3) with respect to functions $\Omega(z)$ and $\Psi(z)$ we have boundary value problem.

$$\begin{aligned} &\Omega(\tau) + \overline{\Omega(\tau)} = f_{10}(\tau) + f_{11}(\tau/R^2), \quad \tau \in l_0'; \\ &\Omega(\tau) - \overline{\Omega(\tau)} = i\left[g_{10}(\tau) - g_{11}(\tau/R^2)\right], \quad \tau \in l_0''; \\ &\Omega(\tau) - \overline{\Omega(\tau)} = i\, Im\, h_{10}(\tau), \quad \tau \in l_0'''. \end{aligned} \tag{7}$$

$$\begin{aligned} &\Omega(\sigma) + \overline{\Omega(\sigma)} = f_{10}(R^2\sigma) + f_{11}(\sigma), \quad \sigma \in l_0^{*\prime}; \\ &\Omega(\sigma) - \overline{\Omega(\sigma)} = -i\left[g_{10}(R^2\sigma) - g_{11}(\sigma)\right], \quad \sigma \in l_0^{*\prime\prime}; \\ &\Omega(\sigma) - \overline{\Omega(\sigma)} = -i\, Im\, h_{10}(R^2\sigma), \quad \sigma \in l_0^{*\prime\prime\prime}. \end{aligned} \tag{8}$$

$$\begin{aligned} &\Omega^+(t) + \Omega^-(t) = f^+(t) + f^-(t), \quad t \in l_1'; \\ &\Omega^+(t) - \Omega^-(t) = i\left[g^+(t) - g^-(t)\right], \quad t \in l_1''; \\ &\Omega^+(t) - \Omega^-(t) = i f_2(t), \quad t \in l_1'''. \end{aligned} \tag{9}$$

$$\begin{aligned} &\Psi(\tau) - \overline{\Psi(\tau)} = i\left[f_{10}(\tau) - f_{11}(\tau/R^2)\right], \quad \tau \in l_0'; \\ &\Psi(\tau) + \overline{\Psi(\tau)} = -\left[g_{10}(\tau) + g_{11}(\tau/R^2)\right], \quad \tau \in l_0''; \\ &\Psi(\tau) - \overline{\Psi(\tau)} = i\, Re\, h_{10}(\tau), \quad \tau \in l_0'''. \end{aligned} \tag{10}$$

$$\begin{aligned} &\Psi(\sigma) - \overline{\Psi(\sigma)} = -i\left[f_{10}(R^2\sigma) - f_{11}(\sigma)\right], \quad \sigma \in l_0^{*\prime}; \\ &\Psi(\sigma) + \overline{\Psi(\sigma)} = -\left[g_{10}(r^2\sigma) + g_{11}(\sigma)\right], \quad \sigma \in l_0^{*\prime\prime}; \\ &\Psi(\sigma) - \overline{\Psi(\sigma)} = -i\, Re\, h_{10}(R^2\sigma), \quad \sigma \in l_0^{*\prime\prime\prime}. \end{aligned} \tag{11}$$

$$\begin{aligned} &\Psi^+(t) - \Psi^-(t) = i[f^+(t) - f^-(t)], \quad t \in l_1'; \\ &\Psi^+(t) + \overline{\Psi^+(t)} = -\left[g^+(t) + g^-(t)\right], \quad t \in l_1''; \\ &\Psi^+(t) - \Psi^-(t) = i f_1(t), \quad t \in l_1'''. \end{aligned} \tag{12}$$

Let's define the following problem (9).

For simple expression the general solution of this problem we should take is partial solution corresponded zero index. The following class will be for example $h(\gamma_1', \ldots, \gamma_{r'}')$ (see [3]),moreover solution of this class imply solution, almost-bounded near terminal of $\gamma'_1, \dots, \gamma'_{r'}$. According to this the canonical functions $R_1(z)$ of the problem (9), that is satisfies condition $R_1(z) = R_{1*}(z)$ has the form

$$R_1(z) = \prod_{j=1}^{r'} \left(\frac{\delta'_j}{\gamma'_j}\right)^{\frac{1}{4}} \left(\frac{z-\gamma'_j}{z-\delta'_j}\right)^{\frac{1}{2}}, \tag{13}$$

the general solution of this problem has the following form:

$$\Omega(z) = I_1(z) + R_1(z)\omega_1(z), \tag{14}$$

where

$$I_1(z) = \frac{R_1(z)}{2\pi i}\left[\int_{l_1} \frac{h_1(t)dt}{R_1(t)(t-z)} - \frac{1}{2}\int_{l_1} \frac{h_1(t)dt}{R_1(t)t}\right],$$
$$h_1(t) = \begin{cases} f^+(t) + f^-(t), & t \in l'_1, \\ i[g^+(t) - g^-(t)], & t \in l''_1, \\ if_2(t), & t \in l'''_1, \end{cases} \tag{15}$$

$\omega_1(z)$ is any holomorphic function in the ring $D\left\{\frac{1}{R} < |z| < R\right\}$ satistyng following condition

$$\omega_{1*}(z) = \omega_1(z). \tag{16}$$

Analogously, $h(\gamma''_1, \dots, \gamma''_{r''})$ class general solution of problem (12) has the following form:

$$\Psi(z) = I_2(z) + R_2(z)\omega_2(z), \tag{17}$$

where

$$I_2(z) = \frac{R_2(z)}{2\pi i}\left[\int_{l_1} \frac{h_2(t)dt}{R_2(t)(t-z)} - \frac{1}{2}\int_{l_1} \frac{h_2(t)dt}{R_2(t)t}\right],$$
$$R_2(z) = \prod_{j=1}^{r''} \left(\frac{\delta''_j}{\gamma''_j}\right)^{\frac{1}{4}} \left(\frac{z-\gamma''_j}{z-\delta''_j}\right)^{\frac{1}{2}}; \quad h_2(t) = \begin{cases} i[f^+(t) - f^-(t), & t \in l'_1, \\ -[g^+(t) - g^-(t)], & t \in l''_1, \\ if_1(t), & t \in l'''_1, \end{cases} \tag{18}$$

$\omega_2(z)$ is holomorphic function in ring D satisfying the following condition:

$$\omega_{2*}(z) = \omega_2(z). \tag{19}$$

Thus sougft-for function $\Phi(z)$ is represented according to the following form:

$$\Phi(z) = I_1(z) + R_1(z)\omega_1(z) - i[I_2(z) + R_2(z)\omega_2(z)]. \tag{20}$$

On the basis of (14) and (17) from boundary conditions (9) and (12) with respect to functions $\omega_1(z)$ and $\omega_2(z)$ we obtain Riemann-Hilbert boundary value problem for circular ring D.

$$\omega_1(\tau)+\overline{R_1(\tau)}\,[R_1(\tau)]^{-1}\,\overline{\omega_1(\tau)}=F_1(\tau),\quad \tau\in l_0'\bigcup l_0^{*\prime},$$
$$\omega_1(\tau)-\overline{R_1(\tau)}\,[R_1(\tau)]^{-1}\,\overline{\omega_1(\tau)}=F_2(\tau),\quad \tau\in l_0''\bigcup l_0^{*\prime\prime},\tag{21}$$
$$\omega_1(\tau)-\overline{R_1(\tau)}\,[R_1(\tau)]^{-1}\,\overline{\omega_1(\tau)}=F_3(\tau),\quad \tau\in l_0'''\bigcup l_0^{*\prime\prime\prime}.$$

$$\omega_2(\tau)-\overline{R_2(\tau)}\,[R_2(\tau)]^{-1}\,\overline{\omega_2(\tau)}=G_1(\tau),\quad \tau\in l_0'\bigcup l_0^{*\prime},$$
$$\omega_2(\tau)+\overline{R_2(\tau)}\,[R_2(\tau)]^{-1}\,\overline{\omega_2(\tau)}=G_2(\tau),\quad \tau\in l_0''\bigcup l_0^{*\prime\prime},\tag{22}$$
$$\omega_2(\tau)-\overline{R_2(\tau)}\,[R_2(\tau)]^{-1}\,\overline{\omega_2(\tau)}=G_3(\tau),\quad \tau\in l_0'''\bigcup l_0^{*\prime\prime\prime}.$$

where $F_j(\tau),\ G_j(\tau\ (j=1,2,3)$ defined functions, has simple expression and satisfy following conditions:

$$F_j(\sigma)=\overline{F_j(\tau)};\quad G_j(\sigma)=\overline{G_j(\tau)}\quad (j=1,2,3),\quad \sigma\in l_0^*,\quad \tau\in l_0.\tag{23}$$

The problems (21) (22) are one and the same type. Let's consider the problem (21).

If we passage the value of complex conjugate, taking account of (23) it will be to prove, that if $\omega^0(z)$ satisfies the codition (21),then $\omega_*^0(z)$ also satisfies same condition imply, that the function $\omega_1(z)=\frac{1}{2}[\omega^0(z)+\omega_*^0(z)]$ will be a solution of the problem (21), satisfing the condition (16).

With imediate verification have:

$$Ind\,\overline{R_1(\tau)}\,[R_1(\tau)]^{-1}\Big|_{l_0}=Ind\,\overline{R_1(\sigma)}\,[R_1(\sigma)]^{-1}\Big|_{l_0^*}=0,$$

This gives possibility of presenting the functions $\left\{\overline{R_1(\tau)}\,[R_1(\tau)]^{-1}\right\}^n$ (n is defined number) of the following form:

$$\left\{\overline{R_1(\tau)}\,[R_1(\tau)]^{-1}\right\}^n=M_1(\tau)\left[\overline{M_1(\tau)}\right]^{-1},\quad \tau\in l_0\bigcup l_0^*,$$

where $M_1(\tau)$ is holomorphic function in the ring D,that satisfies the following form:

$$ln\,M_1(\tau)-ln\,\overline{M_1(\tau)}=ln\left\{\overline{R_1(\tau)}\,[R_1(\tau)]^{-1}\right\}^n,\quad \tau\in l_0,$$
$$ln\,M_1(\sigma)-ln\,\overline{M_1(\sigma)}=ln\left\{\overline{R_1(\sigma)}\,[R_1(\sigma)]^{-1}\right\}^n,\quad \sigma\in l_0^*.\tag{24}$$

The problem (24) presents Dirichlet boundary problem for the circular ring. For solving given problem we should use the method given in [6]. The necessary and sufficient condition for problem solving (24) has such a form:

$$\prod_{j=1}^{r'}\left(\frac{\delta_j'}{\gamma_j'}\right)^{\frac{n}{4}}=1.\tag{25}$$

If we choose number n this way $n=\dfrac{8\pi}{\theta'}$; where $\theta' = arg\prod\limits_{j=1}^{r'}\left(\dfrac{\delta'_j}{\gamma'_j}\right)$, the condition (25) will be satisfied, and the solution of the problem (24), will have such a form:

$$M_1(z)=k_1^0\cdot\prod_{j=-\infty}^{\infty}\prod_{k=1}^{r'}\left\{\frac{(z-R^{4j}R^2\gamma'_k)(z-R^{4j}\gamma'_k)}{(z-R^{4j}R^2\delta'_k)(z-R^{4j}\delta'_k)}\right\}^{\frac{n}{2}}, \tag{26}$$

where k_1^0 is any real function.

The function $M_1(z)$ satisfies the condition $M_{1*}(z)=M_1(z)$.

Let us consider functions:

$$\chi_1(z)=\chi_{10}(z)+\left[\chi_{10}(z)\right]^{-1}, \tag{27}$$

where

$$\chi_{10}(z)=\prod_{k=1}^{p'}\left(\frac{a'_k}{b'_k}\right)^{\frac{1}{4}}\cdot\prod_{j=-\infty}^{\infty}\prod_{k=1}^{p'}\left\{\frac{(b'_k-R^{4j}z)(a'_k-R^{4j}R^2z)}{(b'_k-R^{4j}R^2z)(a'_k-R^{4j}z)}\right\}^{\frac{1}{2}}.$$

The function $\chi_1(z)$ satisfies the following conditions:

$$\chi_1(z)=[\chi_1(z)]_*;\;\; \chi_1(\tau)=-\overline{\chi_1(\tau)},\;\; \tau\in l'_0\bigcup l_0^{*\prime},\;\; \chi(\tau)=\overline{\chi(\tau)},\;\; \tau\in l''_0\bigcup l_0^{*\prime\prime}\bigcup l'''_0\bigcup l_0^{*\prime\prime\prime}.$$

On the basis of this result, the boundary condition (21) has the following form:

$$\Omega_1(\tau)-\overline{\Omega_1(\tau)}=F(\tau),\quad \tau\in l_0\bigcup l_0^*, \tag{28}$$

where

$$\Omega_1(z)=\omega_1(z)\left[\chi_1(z)\cdot M_1(z)\right]^{-1};\;\; F(\tau)=\begin{cases} F_1(\tau)\left[\chi_1(\tau)\cdot M_1(\tau)\right]^{-1},\;\tau\in l'_0\bigcup l_0^{*\prime},\\ F_2(\tau)\left[\chi_1(\tau)\cdot M_1(\tau)\right]^{-1},\;\tau\in l''_0\bigcup l_0^{*\prime\prime},\\ F_3(\tau)\left[\chi_1(\tau)\cdot M_1(\tau)\right]^{-1},\;\tau\in l'''_0\bigcup l_0^{*\prime\prime\prime}.\end{cases} \tag{29}$$

Taking into account that function $F(\tau)$ satisfies condition $F(\tau)=\overline{F(\tau/R^2)}$, $\tau\in l_0$, than necessary and sufficient condition of posible solving will be written in the followings form (see [6]).

$$\oint\limits_{l_0}\frac{F(\tau)d\tau}{\tau}=0, \tag{30}$$

The solution of this problem has the following form:

$$\Omega_1(z)=\frac{1}{2\pi i}\sum_{j=-\infty}^{\infty}\left[\oint\limits_{l_0}\frac{F(\tau)d\tau}{\tau-R^{4j}z}-\oint\limits_{l_0^*}\frac{F(\sigma)d\sigma}{\sigma-R^{4j}z}\right]+k_1, \tag{31}$$

where k_1 is any real constants. According to that on the basis of (31) the function $\omega(z)$ has the form of:

$$\omega_1(z)=\Omega_1(z)\cdot\chi_1(z)\cdot M_1(z). \tag{32}$$

As the problems (21) and (22) are one and the same type let's present terminal results with respect to function $\omega_2(z)$.

Necessary and sufficient condition of problem solving (22) has such a form:

$$\oint_{l_0} \frac{G(\tau)d\tau}{\tau} = 0, \tag{33}$$

where

$$G(\tau) = \begin{cases} G_1(\tau)\left[\chi_2(\tau)\cdot M_2(\tau)\right]^{-1}, \ \tau \in l_0' \bigcup l_0^{*\prime}, \\ G_2(\tau)\left[\chi_2(\tau)\cdot M_2(\tau)\right]^{-1}, \ \tau \in l_0'' \bigcup l_0^{*\prime\prime}, \\ G_2(\tau)\left[\chi_2(\tau)\cdot M_2(\tau)\right]^{-1}, \ \tau \in l_0''' \bigcup l_0^{*\prime\prime\prime}. \end{cases}$$

$\chi_2(z) = \chi_{20}(z) + [\chi_{20}(z)]^{-1}$; $\chi_{20}(z) = \prod\limits_{j=-\infty}^{\infty}\prod\limits_{k=1}^{r''}\left\{\frac{(b_k'' - R^{4j}z)(a_k'' - R^{4j}R^2z)}{(b_k'' - R^{4j}R^2z)(a_k'' - R^{4j}z)}\right\}^{\frac{1}{2}}$, $M_2(z) =$ $= k_2^0 \cdot \prod\limits_{j=-\infty}^{\infty}\prod\limits_{k=1}^{r''}\left\{\frac{(z - R^{4j}R^2\gamma_k'')(z - R^{4j}\gamma_k'')}{(z - R^{4j}R^2\delta_k'')(z - R^{4j}\delta_k'')}\right\}^{\frac{m}{2}}$; k_2^0 is any real constants $m = \frac{8\pi}{\theta''}$, $\theta'' =$ $= arg\prod\limits_{j=1}^{r''}\left(\frac{\delta_j''}{\gamma_j''}\right)$.

In case of condition (33) the function $\omega_2(z)$ taken according the formula:

$$\omega_2(z) = \chi_2(z)\cdot M_2(z)\cdot \Omega_2(z), \tag{34}$$

where

$$\Omega_2(z) = \frac{1}{2\pi i}\sum_{j=-\infty}^{\infty}\left[\oint_{l_0}\frac{D(\tau)d\tau}{\tau - R^{4j}z} - \oint_{l_0^*}\frac{D(\sigma)d\sigma}{\sigma - R^{4j}z}\right] + k_2,$$

$k_2 -$ is any real constants

Obtained results may be formulated as the following theorems:

Theorem. The necessary and sufficient condition of solvability of the problem (1)-(3) has the form (30), (33) and its very solution is attained according to the formula (20), where $I_1(z)$, $R_1(z)$, $\omega_1(z)$, $I_2(z)$, $R_2(z)$ and $w_2(z)$ are defined accordingly (15), (13), (32), (18) and (34).

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