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ON THE SET OF INTEGER PARTITION AND CLOSED FORM FOR ITS LENGTH IN SPECIAL CASES

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Abstract. In this article we study a set of integer partitions and its length. The purpose of this study is better understand how partition set is constructed and develop an algorithm, which will make computations in acceptable time. Here we introduce an algorithm to construct the set and give explicit forms for a number of n partitions into k parts for some values of k.

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1. Introdaction

A partition of a non-negative integer n into k parts is a set of vectors with the k length and non-negative integer coordinates whose sum is n. Let's denote this set as $\mathcal{P}_k(n)$ and write it as follows

$$\begin{aligned}\mathcal{P}_k(n) = \{(n_1, n_2, \dots, n_k) :& \\ & n_1 + n_2 + \cdots + n_k = n \;\&\; n_1 \geq n_2 \geq \cdots \geq n_k \geq 0\},\end{aligned}$$

here we add restriction $n_i \geq n_{i+1}$, since other cases are permutations of elements in the set.

For example, if we partition 5 into 3 parts, we get,

$$5 + 0 + 0 = 4 + 1 + 0 = 3 + 2 + 0 = 3 + 1 + 1 = 2 + 2 + 1,$$

so our set looks like

$$\mathcal{P}_3(5) = \{(5, 0, 0), (4, 1, 0), (3, 2, 0), (3, 1, 1), (2, 2, 1)\}.$$

Efficiently constracting the set $\mathcal{P}_k(n)$ is important for many problems of mathematics and physics. For example in Series Reversion [1] and Invert power series [2], which is computation of the coefficients of the inverse function. There have been made many works on the unrestricted partition function $p(n)$, which is length of n partition into n parts

$$p(n) = \|\mathcal{P}_n(n)\|.$$

One of the most famous results were given by Euler [3] by showing a generator function for $p(n)$

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}.$$

and by pentagonal number theorem [4] we have recursion for this function

$$p(n) = \sum_{k=1}^{\infty}(-1)^{k+1}\left(p\left(n - \frac{k(3k+1)}{2}\right) + p\left(n - \frac{k(3k-1)}{2}\right)\right).$$

An asymptotic expression for $p(n)$ function

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \ n \to +\infty$$

was first obtained by G. H. Hardy and Ramanujan in 1918. Later, in 1937, Hans Rademacher [3] was able to write exact formula for $p(n)$.

Bell [5] in his work proved that the length of restricted partitions of non-negative n into k parts is a quasi polynomial of degree $k-1$. In this work we will prove the similar result, which will allow us to better understand how partition lengths are related to each other and find closed form specific k.

2. Constracting set of integer partition

In this section we will define $\mathcal{D}_k(n)$ restricted partitions of n into k parts. This will allow us to algorithmically construct the set of unrestricted partitions and find its length.

Definition 1. $\mathcal{D}_k(n)$ restricted partitions of non-negative integer n into k parts is

$$\mathcal{D}_k(n) = \{(n_1, n_2, \dots, n_k) : \ n_1 + 2n_2 + \cdots + kn_k = n \ \& \ n_i \geq 0\}.$$

Lemma 1. *There exists a bijection between $\mathcal{P}_k(n)$ and $\mathcal{D}_k(n)$.*

Proof. Let us define f and g mappings as

$$f(n_1, n_2, \dots, n_k) = (n_1 - n_2, n_2 - n_3, \dots, n_{k-1} - n_k, n_k),$$

$$g(n_1, n_2, \dots, n_k) = (n_1 + \cdots + n_k, n_2 + \cdots + n_k, \dots, n_{k-1} + n_k, n_k),$$

then we can see that g is inverse of f

$$(f \circ g)(n_1, n_2, \dots, n_k) = (g \circ f)(n_1, n_2, \dots, n_k) = (n_1, n_2, \dots, n_k).$$

This asserts that we have a bijection between two finite sets

$$f : \mathcal{P}_k(n) \longrightarrow \mathcal{D}_k(n) \quad \text{and} \quad g : \mathcal{D}_k(n) \longrightarrow \mathcal{P}_k(n).$$

This completes the proof.

Theorem 1. *Set of restricted $\mathcal{D}_k(n)$ for $k > 1$ can be written constructed by the following algorithm*

$$\mathcal{D}_k(n) = \left\{\left(n - \sum_{s=1}^{k-1}(k-s+1)u_s, u_{k-1}, u_{k-2}, \dots, u_2, u_1\right) : \right. \\ \left. 0 \leq u_1 \leq d_k(n), 0 \leq u_{j+1} \leq d_k(n; u_1, \dots, u_j)\right\}, \tag{1}$$

where

$$d_k(n) = \left\lfloor \frac{n}{k} \right\rfloor, \quad d_k(n; u_1, \ldots, u_j) = \left\lfloor \frac{n}{k-j} - \frac{1}{k-j} \sum_{s=1}^{j} (k-s+1)u_s \right\rfloor.$$

Proof. Let's start by cheking first few cases to see the pattern. For $k = 2$ it is trivial

$$\mathcal{D}_2(n) = \{(n_1, n_2) : \ n_1 + 2n_2 = n\} = \left\{(n-2u, u) : \ 0 \le u \le \left\lfloor \frac{n}{2} \right\rfloor \right\},$$

since if u is bigger than $\lfloor n/2 \rfloor$, then $n - 2u < 0$.

For $k = 3$ by the theorem, we have

$$\mathcal{D}_3(n) = \left\{(n - 3u_1 - 2u_2, u_2, u_1) : \ 0 \le u_1 \le \left\lfloor \frac{n}{3} \right\rfloor, 0 \le u_2 \le \left\lfloor \frac{n}{2} - \frac{3u_1}{2} \right\rfloor \right\},$$

we can see that for any u_1 and u_2 it satisfies the main condition

$$(n - 3u_1 - 2u_2) + 2u_2 + 3u_1 = n.$$

Therefore we need to find boundaries for u_1 and u_2. Suppose $u_1 > n/3$, then $3u_1 > n$ and we get contradiction, so $u_1 \le n/3$ and sine u_1 is a non-negative integer, its maximum value is $\lfloor n/3 \rfloor$. To find the boundary for u_2 we need to study the following inequality

$$n - 3u_1 - 2u_2 \ge 0 \Rightarrow u_2 \le \frac{n}{2} - \frac{3u_1}{2} \Rightarrow 0 \le u_2 \le \left\lfloor \frac{n}{2} - \frac{3u_1}{2} \right\rfloor,$$

so we showed that the theorem is true when $k \in \{2, 3\}$.

Now we will prove the general result when $k > 1$. Without loss of generality we can parametries elements of $\mathcal{D}_k(n)$ as

$$p_0 = \left(n - \sum_{s=1}^{k-1} (k-s+1)u_s, u_{k-1}, u_{k-2}, \ldots, u_2, u_1 \right),$$

since it has the length of k and the identity

$$\left(n - \sum_{s=1}^{k-1} (k-s+1)u_s \right) + 2u_{k-1} + 3u_{k-2} + \cdots + (k-1)u_2 + ku_1 = n$$

is satisfied. Now we estimate boundaries of parameters. If $u_1 > \lfloor n/k \rfloor$ then $ku_1 > n$ and $p_0 \notin \mathcal{D}_k(n)$, so

$$0 \le u_1 \le \left\lfloor \frac{n}{k} \right\rfloor.$$

We take u_1 to right side of the equation

$$\left(n - \sum_{s=1}^{k-1} (k-s+1)u_s \right) + 2u_{k-1} + 3u_{k-2} + \cdots + (k-1)u_2 = n - ku_1,$$

and similarly if $u_2 > \lfloor (n-ku_1)/(k-1) \rfloor$, then

$$(k-1)u_2 > n - ku_1 \Rightarrow (k-1)u_2 + ku_1 > n$$

and $p_0 \notin \mathcal{D}_k(n)$, so

$$0 \le u_2 \le \left\lfloor \frac{n-ku_1}{k-1} \right\rfloor .$$

So we can make assertion that for $j > 1$

$$0 \le u_j \le \left\lfloor \frac{n - (ku_1 + (k-1)u_2 + \cdots + (k-j+2)u_{j-1})}{k-j+1} \right\rfloor .$$

We will prove this statement using a proof by contradiction. Suppose

$$u_j > \left\lfloor \frac{n - (ku_1 + (k-1)u_2 + \cdots + (k-j+2)u_{j-1})}{k-j+1} \right\rfloor$$

and $p_0 \in \mathcal{D}_k(n)$, then

$$\begin{aligned}(k-j+1)u_j &> n - (ku_1 + (k-1)u_2 + \cdots + (k-j+2)u_{j-1}) \\ &\Rightarrow (k-j+1)u_j + (k-j+2)u_{j-1} + \cdots + (k-1)u_2 + ku_1 > n,\end{aligned}$$

which is a contradiction. This completes the proof.

Corollary 1. For any non-negative integer n length of $\mathcal{D}_2(n)$ is

$$\|\mathcal{D}_2(n)\| = \left\lfloor 1 + \frac{n}{2} \right\rfloor . \tag{2}$$

Proof. From Theorem 1 it follows that

$$\mathcal{D}_2(n) = \left\{ (n-2u, u) : \ 0 \le u \le \left\lfloor \frac{n}{2} \right\rfloor \right\}$$

and

$$\|\mathcal{D}_2(n)\| = \left\| \left\{ (n-2u, u) : \ 0 \le u \le \left\lfloor \frac{n}{2} \right\rfloor \right\} \right\| = 1 + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor 1 + \frac{n}{2} \right\rfloor .$$

This completes the proof.

Corollary 2. For any non-negative integer n the length of $\mathcal{D}_3(n)$ is

$$\|\mathcal{D}_3(n)\| = \left\lfloor 1 + \frac{n}{2} + \frac{n^2}{12} \right\rfloor . \tag{3}$$

Proof. Let's write $\mathcal{D}_3(n)$

$$\mathcal{D}_3(n) = \left\{ (n - 3u_1 - 2u_2, u_2, u_1) : \ 0 \le u_1 \le \left\lfloor \frac{n}{3} \right\rfloor, 0 \le u_2 \le \left\lfloor \frac{n}{2} - \frac{3u_1}{2} \right\rfloor \right\}$$

and so its length is

$$\begin{aligned}
\|\mathcal{D}_3(n)\| &= \sum_{u_1=0}^{d_3(n)} \sum_{u_2=0}^{d_3(n;u_1)} 1 = \sum_{u_1=0}^{d_3(n)} (1 + d_3(n;u_1)) \\
&= 1 + d_3(n) + \sum_{u_1=0}^{d_3(n)} d_3(n;u_1) \\
&= 1 + \left\lfloor \frac{n}{3} \right\rfloor + \sum_{u_1=0}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n}{2} - \frac{3u_1}{2} \right\rfloor = 1 + \left\lfloor \frac{n}{3} \right\rfloor \\
&+ \sum_{u=0}^{\lfloor n/6 \rfloor} \left\lfloor \frac{n}{2} - \frac{3(2u)}{2} \right\rfloor + \sum_{u=0}^{\lfloor (n-3)/6 \rfloor} \left\lfloor \frac{n}{2} - \frac{3(2u+1)}{2} \right\rfloor \\
&= 1 + \left\lfloor \frac{n}{3} \right\rfloor + \sum_{u=0}^{\lfloor n/6 \rfloor} \left\lfloor \frac{n}{2} - 3u \right\rfloor + \sum_{u=0}^{\lfloor (n-3)/6 \rfloor} \left\lfloor \frac{n-3}{2} - 3u \right\rfloor \\
&= 1 + \left\lfloor \frac{n}{3} \right\rfloor + \left(\left\lfloor \frac{n}{2} \right\rfloor - \frac{3}{2} \left\lfloor \frac{n}{6} \right\rfloor \right) \left\lfloor \frac{n+6}{6} \right\rfloor \\
&+ \left(\left\lfloor \frac{n-3}{2} \right\rfloor - \frac{3}{2} \left\lfloor \frac{n-3}{6} \right\rfloor \right) \left\lfloor \frac{n+3}{6} \right\rfloor .
\end{aligned}$$

To prove equation (3) we need to check it for the following six cases:

$$n = 6k + j,\ k = 0, 1, 2, \ldots,\ j \in \{0, 1, \ldots, 5\}.$$

These cases are

$$\begin{aligned}
\|\mathcal{D}_3(6k+0)\| &= 3k^2 + 3k + 1, \\
\|\mathcal{D}_3(6k+1)\| &= 3k^2 + 4k + 1, \\
\|\mathcal{D}_3(6k+2)\| &= 3k^2 + 5k + 2, \\
\|\mathcal{D}_3(6k+3)\| &= 3k^2 + 6k + 3, \\
\|\mathcal{D}_3(6k+4)\| &= 3k^2 + 7k + 4, \\
\|\mathcal{D}_3(6k+5)\| &= 3k^2 + 8k + 5
\end{aligned}$$

and we compare each with (3). This completes the proof.

Example 1. For example, when $n = 6k + 3$, then

$$\|\mathcal{D}_3(n)\| = \left\lfloor 1 + \frac{6k+3}{2} + \frac{(6k+3)^2}{12} \right\rfloor = \left\lfloor \frac{13}{4} + 6k + 3k^2 \right\rfloor = 3 + 6k + 3k^2.$$

Theorem 2. *For any non-negative integer n and $k > 1$ we have the following recursion*

$$\|\mathcal{D}_k(n)\| = \sum_{j=0}^{\lfloor n/k \rfloor} \|\mathcal{D}_{k-1}(n - kj)\|,$$

where $\|\mathcal{D}_1(n)\| = 1$.

Proof. From Theorem 1 it follows that the length of $\mathcal{D}_k(n)$ is

$$\|\mathcal{D}_k(n)\| = \sum_{ku_1+(k-1)u_2+\cdots+u_{k-1}\le n} 1 = \sum_{u_1=0}^{d_k(n)} \sum_{u_2=0}^{d_k(n;u_1)} \cdots \sum_{u_{k-1}=0}^{d_k(n;u_1,\ldots,u_{k-2})} 1 \qquad (*)$$

and from this we have

$$\|\mathcal{D}_{k-1}(n-ku_1)\| = \sum_{u_2=0}^{d_{k-1}(n-ku_1)} \sum_{u_3=0}^{d_{k-1}(n-ku_1;u_2)} \cdots \sum_{u_{k-1}=0}^{d_{k-1}(n-ku_1;u_2,u_3,\ldots,u_{k-2})} 1.$$

By definition of the function d_k we have the following equality

$$\begin{aligned} d_k(n;u_1,\ldots,u_j) &= \left\lfloor \frac{n}{k-j} - \frac{1}{k-j}\sum_{s=1}^{j}(k-s+1)u_s \right\rfloor \\ &= \left\lfloor \frac{n-ku_1}{k-j} - \frac{1}{k-j}\sum_{s=2}^{j}(k-s+1)u_s \right\rfloor \\ &= \left\lfloor \frac{n-ku_1}{k-j} - \frac{1}{k-j}\sum_{s=1}^{j-1}((k-1)-s+1)u_{s+1} \right\rfloor \\ &= d_{k-1}(n-ku_1;u_2,\ldots,u_j), \end{aligned}$$

so

$$\|\mathcal{D}_{k-1}(n-ku_1)\| = \sum_{u_2=0}^{d_k(n;u_1)} \sum_{u_3=0}^{d_k(n;u_1,u_2)} \cdots \sum_{u_{k-1}=0}^{d_k(n;u_1,u_2,u_3,\ldots,u_{k-2})} 1,$$

which is a sub summation in (*)

$$\|\mathcal{D}_k(n)\| = \sum_{u_1=0}^{d_k(n)} \|\mathcal{D}_{k-1}(n-ku_1)\|.$$

This completes the proof.

Corollary 3. For any non-negative integer n the length of $\mathcal{D}_4(n)$ is

$$\|\mathcal{D}_4(n)\| = \left\lfloor 1 + \frac{n}{2} + \frac{n}{8}\left\lfloor\frac{n}{2}\right\rfloor + \frac{n^2}{24} + \frac{n^3}{144} \right\rfloor \qquad (4)$$

Proof. From Theorem 2 we have

$$\begin{aligned} \|\mathcal{D}_4(12k+0)\| &= 12k^3+15k^2+6k+1, \\ \|\mathcal{D}_4(12k+1)\| &= 12k^3+18k^2+8k+1, \\ \|\mathcal{D}_4(12k+2)\| &= 12k^3+21k^2+12k+2, \end{aligned}$$

$$\begin{aligned} \|\mathcal{D}_4(12k+6)\| &= 12k^3+33k^2+30k+9, \\ \|\mathcal{D}_4(12k+7)\| &= 12k^3+36k^2+35k+11, \\ \|\mathcal{D}_4(12k+8)\| &= 12k^3+39k^2+42k+15, \end{aligned}$$

$$
\begin{aligned}
\|\mathcal{D}_4(12k+3)\| &= 12k^3 + 24k^2 + 15k + 3,\\
\|\mathcal{D}_4(12k+4)\| &= 12k^3 + 27k^2 + 20k + 5,\\
\|\mathcal{D}_4(12k+5)\| &= 12k^3 + 30k^2 + 24k + 6,
\end{aligned}
$$

$$
\begin{aligned}
\|\mathcal{D}_4(12k+9)\| &= 12k^3 + 42k^2 + 48k + 18,\\
\|\mathcal{D}_4(12k+10)\| &= 12k^3 + 45k^2 + 56k + 23,\\
\|\mathcal{D}_4(12k+11)\| &= 12k^3 + 48k^2 + 63k + 27
\end{aligned}
$$

and by checking each case $n = 12k+j$, with (4) we can justify the equation. This completes the proof.

Example 2. For example when $n = 12k$, then

$$
\begin{aligned}
\|\mathcal{D}_4(n)\| &= \left\lfloor 1 + \frac{12k}{2} + \frac{12k}{8}\left\lfloor\frac{12k}{2}\right\rfloor + \frac{(12k)^2}{24} + \frac{(12k)^3}{144}\right\rfloor\\
&= \left\lfloor 1 + 6k + 9k^2 + 6k^2 + 12k^3 \right\rfloor\\
&= 1 + 6k + 15k^2 + 12k^3.
\end{aligned}
$$

Corollary 4. For any non-negative integer n the length of $\mathcal{D}_5(n)$ is

$$
\|\mathcal{D}_5(n)\| = \left\lfloor 1 + \frac{11n}{24} + \frac{n}{16}\left\lfloor\frac{n}{2}\right\rfloor + \frac{11n^2}{144} + \frac{n^3}{96} + \frac{n^4}{2880}\right\rfloor. \tag{5}
$$

Proof. To prove equation (5) we need to check 60 cases. To derive the formula we need to analise the first few cases

$$
\begin{aligned}
\|\mathcal{D}_5(60k+0)\| &= 4500k^4 + 2250k^3 + \frac{775}{2}k^2 + \frac{55}{2}k + 1,\\
\|\mathcal{D}_5(60k+1)\| &= 4500k^4 + 2550k^3 + \frac{1015}{2}k^2 + \frac{81}{2}k + 1,\\
\|\mathcal{D}_5(60k+2)\| &= 4500k^4 + 2850k^3 + \frac{1285}{2}k^2 + \frac{123}{2}k + 2,\\
\|\mathcal{D}_5(60k+3)\| &= 4500k^4 + 3150k^3 + \frac{1585}{2}k^2 + \frac{167}{2}k + 3,\\
\|\mathcal{D}_5(60k+4)\| &= 4500k^4 + 3450k^3 + \frac{1915}{2}k^2 + \frac{229}{2}k + 5.
\end{aligned}
$$

We obtained this expressions by Theorem 2. Now we can do this in revese. If $n = 60k$, then

$$
\|\mathcal{D}_5(n)\| = \frac{n^4}{2880} + \frac{n^3}{96} + \frac{31n^2}{288} + \frac{11n}{24} + 1,
$$

and if $n = 60k+1$, then

$$
\|\mathcal{D}_5(n)\| = \frac{n^4}{2880} + \frac{n^3}{96} + \frac{31n^2}{288} + \frac{41n}{96} + \frac{1309}{2880}.
$$

If we continue, we will see that the first three coefficients stay same, while the coefficient of n is $11/24$ if n is even and $41/96$ if n is odd. To merge these two expressions, we use simple obsession

$$\frac{41}{96} = \frac{44-3}{96} = \frac{11}{24} - \frac{3}{96}$$

and so we get

$$\frac{n^4}{2880} + \frac{n^3}{96} + \frac{31n^2}{288} + \frac{(44 - 3(n - 2\lfloor n/2 \rfloor))n}{96} + 1$$

and we take the *floor part* of it. After writing our candidate we manually check it for all $n = 60k + j$, where $j \in \{0, \dots, 59\}$. This completes the proof.

Theorem 3. *The set of restricted partitions $\mathcal{D}_k(n)$ satisfy the following recursion*

$$\mathcal{D}_k(n) = \bigcup_{j=0}^{\lfloor n/k \rfloor} \mathcal{D}_{k-1}(n - kj) \times \{(j)\}, \tag{6}$$

where $\mathcal{D}_1(n) = \{(n)\}$.

Proof. By Theorem 1, we have a closed form of $\mathcal{D}_k(n)$, which is a union of disjoint sets. Let's take $p \in \mathcal{D}_k(n)$ and rewrite it as

$$\begin{aligned} p &= \left(n - \sum_{s=1}^{k-1}(k-s+1)u_s, u_{k-1}, u_{k-2}, \dots, u_2, u_1\right) \\ &= \left((n - ku_1) - \sum_{s=2}^{k-1}(k-s+1)u_s, u_{k-1}, u_{k-2}, \dots, u_2, u_1\right) \\ &= \left((n - ku_1) - \sum_{s=1}^{k-2}(k-s)u_{s+1}, u_{k-1}, u_{k-2}, \dots, u_3, u_2\right) \times \{(u_1)\}, \end{aligned}$$

so $p \in \mathcal{D}_{k-1}(n - ku_1) \times \{(u_1)\}$. Therefore, we broke the original set into the following subsets

$$\mathcal{D}_{k-1}(n) \times \{(0)\},\ \mathcal{D}_{k-1}(n-k) \times \{(1)\}, \dots,\ \mathcal{D}_{k-1}(n - k\lfloor n/k \rfloor) \times \{(\lfloor n/k \rfloor)\}.$$

Because of the Cartesian product, it is clear that subsets are disjoint the Theorem is proved.

Theorem 4. *For any non-negative integer n and $k > 1$ the length of $\mathcal{D}_k(n)$ is a polynomial of degree $k-1$ and is same for each remainder class*

$$\|\mathcal{D}_k(l_k b + r)\| = \sum_{s=0}^{k-1} a_{r,s} b^s, \quad 0 \le r < l_k, \tag{7}$$

where l_k *which is a least common multiple of* $1, 2, 3, \ldots, k$,

$$l_k = \operatorname{lcm}(1, 2, 3, \ldots, k)$$

and $n = l_k b + r$.

Proof. We use induction to prove this result. From equation (2) it follows that for $k = 2$

$$\|\mathcal{D}_2(2b)\| = 1 + b, \quad \|\mathcal{D}_2(2b+1)\| = 1 + b$$

the statement holds. Suppose that this is true for some k, then from Theorem 2 we have

$$\|\mathcal{D}_{k+1}(n)\| = \sum_{j=0}^{\lfloor n/(k+1) \rfloor} \|\mathcal{D}_k(n - (k+1)j)\|,$$

which is a sum of polynomials with degree k. Consider the representation n as $l_{k+1}b + r$, where $0 \leq r < l_{k+1}$, so the summation number becomes

$$\left\lfloor \frac{n}{k+1} \right\rfloor = \left\lfloor \frac{l_{k+1}b + r}{k+1} \right\rfloor = \frac{l_{k+1}}{k+1} b + \left\lfloor \frac{r}{k+1} \right\rfloor,$$

where b is outside the $floor$ function and since the finite sum of type

$$\sum_{j=0}^{c_1 b + c_2} j^{k-1},$$

where this is k degree polynomial with respect to b, so $\|\mathcal{D}_{k+1}(n)\|$ is a polynomial of degree k. This completes the proof.

3. Conclusion

By Theorem 2 we developed an efficient algorithm to construct the set of restricted partition. Since we have bijection between $\mathcal{D}_k(n)$ and $\mathcal{P}_k(n)$ and they are finite, we get that their lengths are also same. Therefore, all corollaries and theorems we proved for the length of $\mathcal{D}_k(n)$ also apply for $\mathcal{P}_k(n)$. In the article we showed an explicit form for some values of k :

$$\begin{aligned}
\|\mathcal{P}_1(n)\| &= 1, \\
\|\mathcal{P}_2(n)\| &= \left\lfloor 1 + \frac{n}{2} \right\rfloor, \\
\|\mathcal{P}_3(n)\| &= \left\lfloor 1 + \frac{n}{2} + \frac{n^2}{12} \right\rfloor, \\
\|\mathcal{P}_4(n)\| &= \left\lfloor 1 + \frac{n}{2} + \frac{n}{8} \left\lfloor \frac{n}{2} \right\rfloor + \frac{n^2}{24} + \frac{n^3}{144} \right\rfloor, \\
\|\mathcal{P}_5(n)\| &= \left\lfloor 1 + \frac{11n}{24} + \frac{n}{16} \left\lfloor \frac{n}{2} \right\rfloor + \frac{11n^2}{144} + \frac{n^3}{96} + \frac{n^4}{2880} \right\rfloor.
\end{aligned}$$

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