Proceedings of I. Vekua Institute of Applied Mathematics Vol. 71, 2021

ON THE 3D STOKES FLOW IN THE INFINITE DOMAINS

Khatiashvili N., Janjgava D.

Abstract. In the paper the 3D problem for the non-stationary Stokes flow in the infinite cylindrical and prismatic areas is studied. We admit that the pressure can be controlled and depends on time exponentially. The linear Stokes system is considered with the appropriate initial-boundary conditions. By means of the Poisson formula and the integral equation method the system is reduced to the system of integral equations with the weakly singular kernel. The existence and uniqueness of solution is obtained, if the power at the exponent satisfies the certain conditions. The exact solutions are obtained by means of the stepwise approximation method. Several examples are given. The results have applications in technological processes and medicine.

Keywords and phrases: Stokes flow, Fredholm equation.

AMS subject classification (2010): 74G05, 74F10, 74F05, 74F99.

1. Introduction

For very viscous Newtonian fluids (creeping flows) and low Reynolds number the Navier-Stokes equations can be linearized and reduced to the linear Stokes equations [1, 14-22, 25, 27, 29, 30]. We study this equation in 3D case with the equation of continuity

$$\frac{\partial \vec{V}}{\partial t} + \frac{1}{\rho}\nabla P = \vec{F} + \nu \Delta \vec{V}, \tag{1}$$

$$div\vec{V} = 0, \tag{2}$$

where $\vec{V}(V_x, V_y, V_z)$ is the velocity, $\vec{F}(F_x, F_y, F_z)$ is the body force, P is the pressure, ρ is the density and ν is the viscosity of the fluid.

The examples of creeping flows are the flow of oils, flow of lava, flow of viscous polymers, etc.[1, 14-23, 25, 27, 29, 30].

In the stationary case in the different 2D and 3D areas the Stokes system was reduced to the biharmonic equation for the stream function with non-homogeneous boundary conditions [14-22, 24–27, 29, 30].

Lorentz and Hancock have introduced the fundamental solution for steady Stokes flow [3, 14-22, 25, 27, 29, 30].

The solutions of Stokes flow inside or outside the sphere was obtained by Lamb in terms of series of spherical harmonics [1, 14-22, 25, 27, 29, 30].

The case of the axial symmetry was investigated in [3, 6, 8-10, 14-22, 25, 27, 30].

The numerical treatment of (1), (2) by mixed finite element methods (FEM) was considered in [5–7, 21, 28, 30]. For rectangular channel and tubes of equilateral triangular cross-section and elliptical cross-section the velocity profile was found by J. Boussinesq [31].

We study the system (1), (2) in the cylindrical and prismatic areas with an arbitrary cross-section bounded by the piecewise smooth line (in the simply connected region) with the appropriate initial-boundary conditions. We suppose that the pressure can be regulated and depends on time exponentially. By means of Poissons formula the initial problem is reduced to the system of integral equations with the weakly singular kernel. Using the Fredholm theorem the sufficient conditions for the existence and uniqueness of solution of the system (1), (2) are obtained, the approximate solutions are constructed and hence the velocity components are defined.

Several examples are given and profiles of velocity are plotted by means of Maple. The results have applications in some industrial processes for pipelines, in the medical surgery and microfluidic devices (MEMS).

2. Statement of the problem

We study 3D Stokes flow in the infinite cylindrical or prismatic area $D = \{D_0 \times [-\infty, \infty]; -\infty < z < \infty\}$ with the cross-section D_0 , where D_0 is the simply connected region in $x0y$ plane bounded by a piecewise smooth line $\phi(x, y)$.

We rewrite system (1), (2) in terms of velocity components in the cartezian coordinates $xOyz$

$$\frac{\partial V_x}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial x} = F_x + \nu \Delta V_x, \tag{3}$$

$$\frac{\partial V_y}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial y} = F_y + \nu \Delta V_y, \tag{4}$$

$$\frac{\partial V_z}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial z} = F_z + \nu \Delta V_z, \tag{5}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0. \tag{6}$$

From (3), (4), (5), (6) we have

$$\Delta P = \rho div \vec{F}. \tag{7}$$

In the case when the body force is solenoidal ($div\vec{F} = 0$) one obtains $\Delta P = 0$ [1, 14-22, 25, 27, 29, 30].

We consider system (3), (4), (5), (6) with the following initial-boundary conditions

$$V_x|_S = V_y|_S = V_z|_S = 0, \tag{8}$$

$$V_x(x, y, z, 0) = V_x^0(x, y, z),\ V_y(x, y, z, 0) = V_y^0(x, y, z),\ V_z(x, y, z, 0) = V_z^0(x, y, z),$$

$$F_x(x, y, z, 0) = F_x^0(x, y, z),\ F_y(x, y, z, 0) = F_y^0(x, y, z),\ F_z(x, y, z, 0) = F_z^0(x, y, z),$$

$$P(x, y, z, 0) = P_0(x, y, z),$$

where S is the boundary of D, $V_x^0(x, y, z)$, $V_y^0(x, y, z)$, $V_z^0(x, y, z)$ are to be determined and

$F_x^0(x,y,z)$, $F_y^0(x,y,z)$, $F_z^0(x,y,z)$, $P(x,y,z,0)=P(x,y,z)$, are the given smooth functions.

Let us suppose that the pressure satisfies equation (7) and depends on time exponentially, besides we admit

$$V_x=\exp(-\alpha t)V_x^0,\ V_y=\exp(-\alpha t)V_y^0, V_z=\exp(-\alpha t)V_z^0, \tag{9}$$

$$\begin{aligned} &F_x=\exp(-\alpha t)F_x^0,\ F_y=\exp(-\alpha t)F_y^0,\ F_z=\exp(-\alpha t)F_z^0,\\ &P(x,y,z,t)=\exp(-\alpha t)P_0(x,y,z), \end{aligned} \tag{10}$$

$\alpha>0$ is the definite constant.

By (3), (4), (5), (6), (8), (9), (10) system (3), (4), (5) (6) will be reduced to the system

$$\Delta V_x^0+\frac{\alpha}{\nu}V_x^0=\frac{1}{\rho\nu}\frac{\partial P_0}{\partial x}-\frac{1}{\nu}F_x, \tag{11}$$

$$\Delta V_y^0+\frac{\alpha}{\nu}V_y^0=\frac{1}{\rho\nu}\frac{\partial P_0}{\partial y}-\frac{1}{\nu}F_y, \tag{12}$$

$$\Delta V_z^0+\frac{\alpha}{\nu}V_z^0=\frac{1}{\rho\nu}\frac{\partial P_0}{\partial z}-\frac{1}{\nu}F_z, \tag{13}$$

$$\frac{\partial V_x^0}{\partial x}+\frac{\partial V_y^0}{\partial y}+\frac{\partial V_z^0}{\partial z}=0, \tag{14}$$

with the following boundary conditions

$$V_x^0|_S=V_y^0|_S=V_y^0|_S=0. \tag{15}$$

We have to solve the following problem

Problem. For the given pressure satisfies equation (7) in the area D find the functions V_x^0, V_y^0, V_z^0 vanishing at infinity, having continuous second order derivatives, satisfying system (3), (4), (5), (6) and the boundary condition (15).

3. Solution of Problem 1

As $V_x^0(x,y,z)$, $V_y^0(x,y,z)$, $V_z^0(x,y,z)$, vanish at infinity, we suppose $V_x^0(x,y,z)$ ≈ 0, $V_y^0(x,y,z)\approx 0$, $V_z^0(x,y,z)\approx 0$, for $|z|>b$, $b=const>0$ b is a rather large number. We now consider Problem 1 for the area $D_0^*=\{D_0\times[-b,b]\}$. In the previous paragraph system (3), (4), (5), (6) was reduced to the system (11), (12), (13), (14). By using Poisson's formula for (11), (12), (13), (14), (15) one obtains the system of following integral equations [2]

$$\begin{aligned} &V_x^0-\frac{3\alpha}{4\pi\nu}\int_{D_0^*}G(x,y,z,x_1,y_1,z_1)\,V_x^0dx_1dy_1dz_1\\ &=-\tfrac{3}{4\pi}\textstyle\int_{D_0^*}G(x,y,z,x_1,y_1,z_1)\,\Phi_1dx_1dy_1dz_1, \end{aligned} \tag{16}$$

$$\begin{aligned} &V_y^0-\frac{3\alpha}{4\pi\nu}\int_{D_0^*}G(x,y,z,x_1,y_1,z_1)\,V_y^0dx_1dy_1dz_1\\ &=-\tfrac{3}{4\pi}\textstyle\int_{D_0^*}G(x,y,z,x_1,y_1,z_1)\,\Phi_2dx_1dy_1dz_1, \end{aligned} \tag{17}$$

$$\begin{aligned} &V_z^0 - \frac{3\alpha}{4\pi\nu}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1)\, V_z^0 dx_1dy_1dz_1 \\ &= -\tfrac{3}{4\pi}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1)\,\Phi_3 dx_1dy_1dz_1, \end{aligned} \tag{18}$$

where $G(x,y,z,x_1,y_1,z_1)$ is the Green function for the Laplace equation in the area D_0^*,

$$\Phi_1 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial x} - \frac{1}{\nu}F_x^0, \Phi_2 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial y} - \frac{1}{\nu}F_y^0, \Phi_3 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial z} - \frac{1}{\nu}F_z^0.$$

According to (7) and (14) the following integral equation should have only trivial solution

$$\begin{aligned} &\Big(\frac{\partial V_x^0}{\partial x} + \frac{\partial V_y^0}{\partial y} + \frac{\partial V_z^0}{\partial z}\Big) - \frac{3\alpha}{4\pi\nu}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1) \\ &\times\Big(\frac{\partial V_x^0}{\partial x} + \frac{\partial V_y^0}{\partial y} + \frac{\partial V_z^0}{\partial z}\Big) dx_1dy_1dz_1 = 0. \end{aligned} \tag{19}$$

Equations (16), (17), (18), (19) are Fredholm equations with the weakly-singular self-adjoint kernel $G(x,y,z,x_1,y_1,z_1)$. Hence, the Fredholm theorem is valid [2] and we conclude, that $\dfrac{3\alpha}{4\pi\nu}$ is not the eigenvalue of equation (19). Concequently, there exist the unique solutions of equations (16), (17), (18).

By means of the Banach theorem we obtain [2]:

If $\dfrac{3\alpha}{4\pi\nu} < \dfrac{1}{M}$, where

$$\int_{D_0^*} |G(x,y,z,x_1,y_1,z_1)| dx_1dy_1dz_1 \le M,\ (x,y,z) \in D_0^*,$$

then there exists the unique solutions of (16), (17), (18) which are given by the formulas

$$V_x^* = \lim_{n\to\infty} V_{x,n}; V_y^* = \lim_{n\to\infty} V_{y,n}; V_z^* = \lim_{n\to\infty} V_{z,n}, \tag{20}$$

where

$$V_{x,0} = -\frac{3}{4\pi}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1)\Phi_1\, dx_1dy_1dz_1,$$

$$V_{x,n} = V_{x,0} + \frac{3\alpha}{4\pi\nu}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1)V_{x,(n-1)} dx_1dy_1dz_1,$$

$$V_{y,0} = -\frac{3}{4\pi}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1)\Phi_2 dx_1dy_1dz_1,$$

$$V_{y,n} = V_{y,0} + \frac{3\alpha}{4\pi\nu}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1)V_{y,(n-1)} dx_1dy_1dz_1.$$

$$V_{z,0} = -\frac{3}{4\pi}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1)\Phi_3 dx_1dy_1dz_1,$$

$$V_{z,n} = V_{z,0} + \frac{3\alpha}{4\pi\nu}\int_{D_0^*} G(x,y,z,x_1,y_1,z_1)V_{z,(n-1)} dx_1dy_1dz_1.$$

For example, if $\frac{9\alpha}{16\pi^2\nu^2}$ is negligible, then the solutions of system (16), (17), (18) are $V_{x,0}$, $V_{y,0}$ $V_{z,0}$ respectively.

Hence, the following theorem is true

Theorem. *If the pressure $P(x,y,z,t)$ and the body force $\vec{F}(F_x,F_y,F_z)$ depend on time exponentially and are given by*

$$F_x = \exp(-\alpha t)F_x^0,\ F_y = \exp(-\alpha t)F_y^0,\ F_z = \exp(-\alpha t)F_z^0,$$

$$P(x,y,z,t) = \exp(-\alpha t)P_0(x,y,z),$$

where $\alpha > 0$ is the definite constant, the functions F_x^0, F_y^0, F_z^0 have first order derivatives and P_0 has second order derivatives in D, then :

1. If $\frac{3\alpha}{4\pi\nu}$ is not the eigenvalue of the homogeneous integral equation (19), there exists the unique solution of the Stokes system (3), (4), (5).

2. If $\frac{3\alpha}{4\pi\nu} < \frac{1}{M}$, where

$$\int_{D_0^*} |G(x,y,z,x_1,y_1,z_1)|dx_1dy_1dz_1 \le M,\ (x,y,z) \in D_0^*$$

then there exists the unique solution of Stokes system (3) , (4), (5) which is given by

$$V_x = \exp(-\alpha t)V_x^0,\ V_y = \exp(-\alpha t)V_y^0, V_z = \exp(-\alpha t)V_z^0,$$

where V_x^0, V_y^0, V_z^0 are given by formula (20).

Remark 1. The vortex $\vec{\Omega}(\Omega_1,\Omega_2,\Omega_3)$ for the Stokes flow in a pipe will be defined by formula [1, 14-22, 25, 27, 29, 30]

$$\Omega_1 = \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z},\ \Omega_2 = \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x},\ \Omega_3 = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}.$$

4. The case of the semi-infinite region

We now consider the Stokes flow in the area $D^* = \{D_0 \times [0,\infty]; 0 \le z < \infty\}\}$ with the cross-section D_0, where D_0, is the simply connected region bounded by a piecewise smooth line $\phi(x,y)$ and suppose

$$V_x^0|_{\phi(x,y)} = V_y^0|_{\phi(x,y)} = V_z^0|_{\phi(x,y)} = 0, \tag{21}$$

$$\begin{aligned}
V_x(x,y,z,t) &= \exp(-\alpha t)\exp(\beta z)V_x^0(x,y),\\
V_y(x,y,z,t) &= \exp(-\alpha t)\exp(\beta z)V_y^0(x,y),\\
V_z(x,y,z,t) &= \exp(-\alpha t)\exp(\beta z)V_z^0(x,y),\\
F_x(x,y,z,t) &= \exp(-\alpha t)\exp(\beta z)\,F_x^0(x,y),\\
F_y(x,y,z,t) &= \exp(-\alpha t)\exp(\beta z)\,F_y^0(x,y),\\
F_z(x,y,z,t) &= \exp(-\alpha t)\exp(\beta z)\,F_z^0(x,y),\\
P(x,y,z,t) &= \exp(-\alpha t)\exp(\beta z)\,P_0(x,y),
\end{aligned} \tag{22}$$

where β is some constant, $F_x^0(x,y)$, $F_y^0(x,y)$, $F_z^0(x,y)$, $P_0(x,y)$, are the given smooth functions, $V_x^0(x,y)$, $V_y^0(x,y)$, $V_z^0(x,y)$ are doubly differentiable functions

in D_0 to be determined. This case was considered in [12] and system (3), (4), (5), (6) was reduced to the system of Helmholtz equations

$$\Delta V_x^0 + \Big(\frac{\alpha}{\nu} + \beta^2\Big) V_x^0 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial x} - \frac{1}{\nu} F_x^0, \tag{23}$$

$$\Delta V_y^0 + \Big(\frac{\alpha}{\nu} + \beta^2\Big) V_y^0 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial y} - \frac{1}{\nu} F_y^0, \tag{24}$$

$$\Delta V_z^0 + \Big(\frac{\alpha}{\nu} + \beta^2\Big) V_z^0 = \frac{\beta}{\rho\nu} P_0 - \frac{1}{\nu} F_z^0, \tag{25}$$

$$\frac{\partial V_x^0}{\partial x} + \frac{\partial V_y^0}{\partial y} + \beta\, V_z^0 = 0, \tag{26}$$

with boundary conditions (21).

Equation (7) becomes

$$\Delta P_0 + \beta^2\, P_0 = \rho\frac{\partial F_x^0}{\partial x} + \rho\frac{\partial F_y^0}{\partial y} + \rho\beta\, F_z^0. \tag{27}$$

By (21), (22) and Poisson formula system (23), (24), (25), (26) will be reduced to the system of Fredholm integral equations [12]

$$\begin{aligned} &V_x^0 - \frac{1}{2\pi}\Big(\frac{\alpha}{\nu} + \beta^2\Big)\int_{D_0} G(x,y,x_1,y_1)\, V_x^0 dx_1 dy_1 \\ &= -\tfrac{1}{2\pi}\textstyle\int_{D_0} G(x,y,x_1,y_1)\, \Phi_1^0 dx_1 dy_1, \end{aligned} \tag{28}$$

$$\begin{aligned} &V_y^0 - \frac{1}{2\pi}\Big(\frac{\alpha}{\nu} + \beta^2\Big)\int_{D_0} G(x,y,x_1,y_1)\, V_y^0 dx_1 dy_1 \\ &= -\tfrac{1}{2\pi}\textstyle\int_{D_0} G(x,y,x_1,y_1)\, \Phi_2^0 dx_1 dy_1, \end{aligned} \tag{29}$$

$$\begin{aligned} &V_z^0 - \frac{1}{2\pi}\Big(\frac{\alpha}{\nu} + \beta^2\Big)\int_{D_0} G(x,y,x_1,y_1)\, V_z^0 dx_1 dy_1 \\ &= -\tfrac{1}{2\pi}\textstyle\int_{D_0} G(x,y,x_1,y_1)\, \Phi_3^0 dx_1 dy_1, \end{aligned} \tag{30}$$

where $G(x,y,x_1,y_1)$ is the Green function for the Laplace equation in the area D_0,

$$\Phi_1^0 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial x} - \frac{1}{\nu} F_x^0, \Phi_2^0 = \frac{1}{\rho\nu}\frac{\partial P_0}{\partial y} - \frac{1}{\nu} F_y^0, \Phi_3^0 = \frac{\beta}{\rho\nu} P_0 - \frac{1}{\nu} F_z^0.$$

In [12] it is shown that $\frac{1}{2\pi}(\frac{\alpha}{\nu} + \beta^2)$ is not the eigenvalue of the corresponding homogeneous integral equation and the system has the unique solution.

If $\frac{1}{2\pi}(\frac{\alpha}{\nu} + \beta^2)$ is rather small, the solution of system (28), (29), (30) is obtained by means of the stepwise approximation method and is given by

$$V_x^* = \lim_{n\to\infty} V_{x,n}; V_y^* = \lim_{n\to\infty} V_{y,n}, V_z^* = \lim_{n\to\infty} V_{z,n}, \tag{31}$$

where

$$V_{x,0} = -\frac{1}{2\pi}\int_{D_0} G(x,y,x_1,y_1)\Phi_1^0\, dx_1 dy_1,$$

$$V_{x,n} = V_{x,0} + \frac{1}{2\pi}\Big(\frac{\alpha}{\nu} + \beta^2\Big)\int_{D_0} G(x,y,x_1,y_1) V_{x,(n-1)} dx_1 dy_1,$$

$$V_{y,0} = -\frac{1}{2\pi}\int_{D_0} G(x,y,x_1,y_1)\Phi_2^0 dx_1 dy_1,$$

$$V_{y,n} = V_{y,0} + \frac{1}{2\pi}\Big(\frac{\alpha}{\nu}+\beta^2\Big)\int_{D_0} G(x,y,x_1,y_1)V_{y,(n-1)} dx_1 dy_1,$$

$$V_{z,0} = -\frac{1}{2\pi}\int_{D_0} G(x,y,x_1,y_1)\Phi_3^0 dx_1 dy_1,$$

$$V_{z,n} = V_{y,0} + \frac{1}{2\pi}\Big(\frac{\alpha}{\nu}+\beta^2\Big)\int_{D_0} G(x,y,x_1,y_1)V_{z,(n-1)} dx_1 dy_1.$$

If $\frac{1}{4\pi^2\nu}(\frac{\alpha}{\nu}+\beta^2)$ is negligible, then the solution of system (28), (29), (30) is $V_{x,0}$, $V_{y,0}\, V_{z,0}$.

Hence, the unique solution of system (3), (4), (5) in the area D^* in case of

$$F_x(x,y,z,t) = \exp(-\alpha t)\exp(\beta z)\, F_x^0(x,y),$$

$$F_y(x,y,z,t) = \exp(-\alpha t)\exp(\beta z)\, F_y^0(x,y),$$

$$F_z(x,y,z,) = \exp(-\alpha t)\exp(\beta z)\, F_z^0(x,y),$$

$$P(x,y,z,t) = \exp(-\alpha t)\exp(\beta z)\, P_0(x,y),$$

exists if $\frac{1}{2\pi}(\frac{\alpha}{\nu}+\beta^2)$ is not the eigenvalue of the homogeneous integral equation

$$V_x^0 - \frac{1}{2\pi}\Big(\frac{\alpha}{\nu}+\beta^2\Big)\int_{D_0} G(x,y,x_1,y_1)\, V_x^0 dx_1 dy_1 = 0,$$

and is given by

$$V_x(x,y,z,t) = \exp(-\alpha t)\exp(\beta z) V_x^0(x,y),$$

$$V_y(x,y,z,t) = \exp(-\alpha t)\exp(\beta z) V_y^0(x,y),$$

$$V_z(x,y,z,t) = \exp(-\alpha t)\exp(\beta z) V_z^0(x,y),$$

where $V_x^0(x,y)$, $V_y^0(x,y)$, $V_z^0(x,y)$ are the solutions of system (28), (29), (30).

Remark 2. For some areas the Green function can be constructed in the explicit form:

1. In case, when D_0 is the ellipse $x^2/a^2 + y^2/b^2 = 1$; a, $b > 0$,

$$G(x,y,x_0,y_0) = G(z,z_0) = -ln|w-w_0|;\ z = x+iy;\ z_0 = x_0+iy_0,$$

$$w = \frac{i}{dn(\frac{K}{\pi}arcosz)}$$

where $dnz = \sqrt{1-k^2 sn^2 z}$ is the Jakobi function, $K, k > 0$ are the definite constants [4, 20].

2. In case, when D_0 is the part of the lemniscate [4, 20]

$$(x^2+y^2)^2 = a^2(x^2-y^2);\ x \geq 0;\ \ a > 0, \tag{32}$$

($r = cos2\varphi$; $-\pi/4 \leq \varphi \leq \pi/4$ in polar coordinates), the Green function is

$$G(x,y,x_0,y_0) = G(z,z_0) = -ln|z-z_0|^2;\ z = x+iy;\ z_0 = x_0+iy_0.$$

3. In case, when D_0 is the hexagon with the one vertex at the point $x = y = 0$, the Green function will be given by

$$G(x, y, x_0, y_0) = G(z, z_0) = -ln|\frac{W - ih}{W + ih} - \frac{W_0 - ih}{W_0 + ih}|; W_0 = W(z_0),$$

where

$$W = f^{-1}(z); f(z) = C \int_0^z t^{-1/3}(t^2 - a^2)^{-1/3}(t^2 - b^2)^{-1/3}\, dt,$$

C is some parameter, $h, a, b > 0$ are the definite constants [4, 20].

Remark 3. The free boundary problem for the 2D Stokes flow has been studied in [11,13].

5. Example

As an example we consider the virtual oil pipeline. The oil pipeline is the complicated system consisting of chains of pipes of different slope and pump-stations (for the pressure regulation).

We consider the part of the pipeline− a pipe with the definite slope with the angle of inclination θ with respect to the horizontal line of the earth.

We consider the pipe of the length $L \approx 100km$,the diameter $D \approx 1m$, and admit that the axis Oz is the axis of symmetry of the pipe in the direction of the fluid flow. We will calculate the velocity of the flow for the pressure $P = P_0 + C_3 y$ and the following data: for $z = 0$,$P_0 \approx 2100KPa$, for $z = 100$, $P_0 \approx 780KPa$, $\nu \approx 1.5St$,

$$F_x^0 = 0; F_y^0 = (C_1 + C_2 y)\, cos\theta; F_z^0 = (C_1 + C_2 y)\, sin\theta;$$

$$\begin{aligned} &C_1 = C_2 = \beta = -0.01; \alpha = 0, P_0 \approx 2100KPa; \\ &\rho \approx 600kg/m^3; \theta = \pi/6; \nu \approx 1.5St. \end{aligned} \tag{33}$$

According to (27)

$$\beta(P_0 + C_3 y) = \rho(cos\theta + sin\theta) + \beta\rho y sin\theta; C_3 = \rho sin\theta.$$

In the circular pipe (D_0 is the circle $(x - 1)^2 + y^2 = 1$) and for the pipe of the lemniscate cross-section (formula (32) for $a = 2$) the velocity of the flow we calculate by formula (31) for data (33)

$$V_x(x, y, z, 0) = 0, \tag{34}$$

$$\begin{aligned} &V_y(x, y, z, t) = \exp(\beta z) V_{y,0} \\ &= -\exp(\beta z)\tfrac{1}{\nu} \textstyle\int_{D_0} (sin\theta + \beta(1 + y_1)\, cos\theta)\, G(x, y, x_1, y_1)\, dx_1 dy_1, \end{aligned} \tag{35}$$

$$\begin{aligned} &V_z(x, y, z, t) = \exp(\beta z) V_{z,0} \\ &= -\tfrac{\beta}{\nu} \exp(\beta z) \textstyle\int_{D_0} (\cos\theta)\, G(x, y, x_1, y_1)\, dx_1 dy_1, \end{aligned} \tag{36}$$

In Fig. 1 and Fig. 2 the profile of the velocity near the axis $x = 0$ is given for the pipe with the circular cross-section.

For the same data the flow velocity for the pipe with the lemniscates cross-section $D_0 : (x^2 + y^2)^2 = a^2(x^2 - y^2); x \geq 0\, a = 2$, will be calculated also by the formulas (34), (35), (36).

In Fig. 3 and Fig. 4 the profile of the velocity near the axis $x = 0$ is given for the pipe with the lemniscates cross-section.

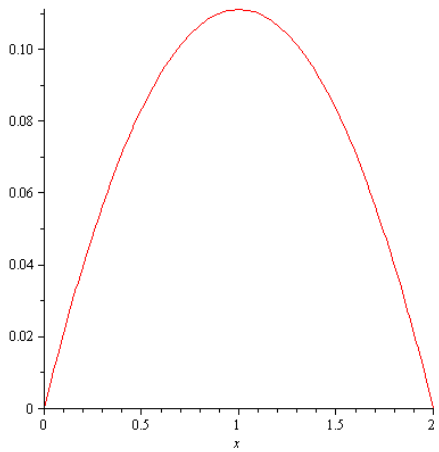

Fig. 1. The profile of the velocity near the axis $0x$ for the data (33) and $z = 0$ in case of the circular cross-section $(x-1)^2 + y^2 = 1$

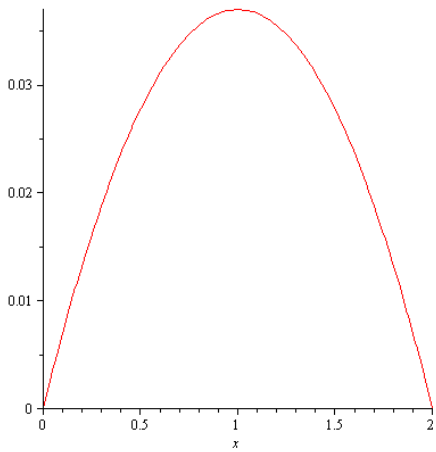

Fig. 2. The profile of the velocity near the axis $0x$ for the data (33) and $z = 100$ in case of the circular cross-section $(x-1)^2 + y^2 = 1$

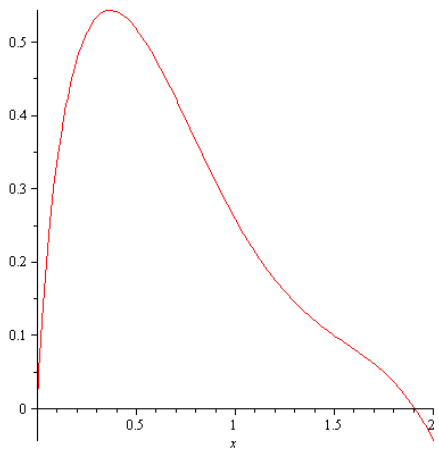

Fig. 3. The profile of the velocity near the axis $0x$ for the data (33) and $z = 0$ in case of the lemniscate (32) cross-section $a = 2$

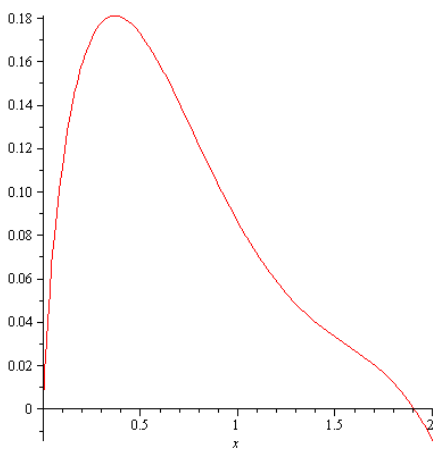

Fig. 4. The profile of the velocity near the axis $0x$ for the data (33) and $z = 100$ in case of the lemniscate (32) cross-section $a = 2$

R E F E R E N C E S

1. Bachelor G.K. An Introduction to Fluid Dynamics. *Cambridge Univ. Press, Cambridge*, 1967.

2. Bitsadze A. Some Classes of Partial Differential Equations. Translated from the Russian by H. Zahavi. Advanced Studies in Contemporary Mathematics. *Gordon and Breach Science Publishers, New-York*, 1988.

3. Chwang A., Wu T. Hydromechanics of low-Reynolds-number flow. Part 2. Singularity method for Stokes flows, part 4. *J. Fluid Mech.*, **62**, 6 (1974), 10-81.

4. Janke-Ende-Losch, Fafeenhonerer functionnen. *Stuttgart*, 1960.

5. Giga Y., Bounded H-calculus for the hydrostatic Stokes operator on L_p spaces and applications. *Proc. Am. Math. Soc.*, **145** (2017), 3865-3876.

6. Girault V., Riviere B., Wheeler M., A discontinuous Galerkin Method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems. *Math. Comp.*, **74** (2005), 53-84.

7. Hansbo P., Szepessy A. A velocity-pressure streamline diffusion finite element method for the incompressible Navier-Stokes equations. *Comput. Methods. Appl. Mech. Eng.*, **84** (1990), 175-192.

8. Khatiashvili N., Pirumova K., Janjgava D. On the Stokes flow over ellipsoidal type bodies, World Congress on Engineering 2013, 3-5 July, London, *Lecture Notes in Engineering and Computer Science*, **1** (2013), 148-151.

9. Khatiashvili N., Pirumova K., Janjgava D. On some Effective Solutions of Stokes Axisymmetric Equation for a Viscous Fluid. *Proceedings of World Academy of Science, Engineering and Technology*, **79** (2013), 690-694.

10. Khatiashvili N., Pirumova K., Akhobadze V. I.Khatiashvili, On the Influence of the Cancer Proteins on the Blood Flow, *Rep. Enlarged Sess. Sem. I. Vekua Inst. Appl. Math.*, **29** (2015), 60-63. [1, 14-22, 25, 27, 29, 30]

11. Khatiashvili N. On 2D Free Boundary Problem for the Stokes Flow, *Proc. of I. Vekua Inst. of Appl. Math.*, **70**, (2020), 61-71.

12. Khatiashvili N. On the Stokes Flow in a Pipes, *J. WayScience*, **2** (2020), 355-358.

13. Khatiashvili N. On the Free Boundary Problem for the crapping Flows, World Congress on Engineering, 2021, 3-5 July, London, UK. *Lecture Notes in Engineering and Computer Science*, **1**, 60-65.

14. Kim S., Karrila S.J. Microhydrodynamics: Principles and Selected Applications. *Dover, NY*, 2005.

15. Kirby B.J. Micro- and Nanoscale Fluid Mechanics: Transport in Microfluidic Devices. *Cambridge University Press, Cambridge*, 2010.

16. Lamb Horace. Hydrodynamics. *Sixth ed. Dover, NY*, 1945.

17. Landay L.D., Lifshitz E.M. Fluid Mechanics, Course of Theoretical Physics 6. *Pergamon Press, NY*, 1987.

18. Lautrup B. Physics of Continuous Matter, Second Edition: Exotic and Everyday Phenomena in the Macroscopic World. *CRC Press, NY*, 2011.

19. Lavrentiev M.A., Shabat B.V. Problems in hydrodynamics and their mathematical models (Russian). *Nauka, Moskow*, 1973.

20. Lavrentiev M.A., Shabat B.V. Methods of the theory of functions in a complex variable (Russian). *Nauka, Moskow*, 1987.

21. Lions P.L. Mathematical Topics in Fluid Mechanics. *The Clarendon Press, Oxford University Press, NY*, 1996.

22. Milne-Thompson L.M. Theoretical Hydrodynamics. *5-th ed, Macmillan*, 1968.

23. Necas J., Hlavacek I. Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction, in: Studies in Applied Mechanics, *Elsevier Scientific Publishing,Co., Amsterdam-NY*, **3**, 1980.

24. Neuber H. Einneuer Ansatzzur Losungtaumlicher Probleme der Elastizitatstheorie. *Journal of Appl. Math. And Mech Academy*, **14** (1934), 203-212.

25. Ockendon H., Ockendon J.R. Viscous Flow. *Cambridge University Press, Cambridge*, 1995.

26. Papkovich P.F. Solution Generale des equations differentials fondamentales delasticiteexprimee par trios fonctionsharmoniques, *Compt. Rend. Acad. Sci., Paris*, **195** (1932), 513-515.

27. Prandtl L. Essentials of Fluid Mechanics, third ed. In: Applied Mathematical Sciences, *Springer, NY*, 2010.

28. Schroeder P.W., Lehrenfeld C., Linke A., Lube G. Towards computable flows and robust estimates for inf-sup stable FEM applied to the time-dependent incompressible Navier-Stokes equations, *SEMA J.*, **75** (2018), 629-653.

29. Stokes G.G. On the steady motion of incompressible fluids. *Transactions of the Cambridge Philosophical Society: Mathematical and Physical Papers*, **7** (1880), 439-453.

30. Temam R. Navier-Stokes Equation, Theory and Numerical Analysis. *AMS Chelsea*, 2001.

31. Van Wylen, Gordon J., Sonntag R. Fundamentals of Classical Thermodynamics. *John Wiley and Sons, NY*, 1965.

Received 03.06.2021; accepted 23.11.2021.

Authors' addresses:

N. Khatiashvili
I. Vekua Institute of Applied Mathematics of
I. Javakhishvili Tbilisi State University
2, University str., Tbilisi 0186
Georgia
E-mail: ninakhatia@gmail.com

D. Janjgava
Georgian Pipeline Company
38, S. Tsintsadze str., Tbilisi 0160
Georgia
E-mail: janjgavd@bp.com