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SHEAVES OF PSEUDOHOLOMORPHIC FUNCTIONS AND DIFFERENTIAL 1-FORMS

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Abstract. Some sheaf-theoretic aspects of the pseudoholomorphic function theory are considered. Sheaves of pseudoholomorphic, or generalized analytic, functions and differential forms on a Riemann surface are defined; several propositions regarding them are proven; Čech cohomology groups of these sheaves are characterized. A proof of a Serre-type duality theorem that relates zeroth and first cohomology groups of the sheaf of generalized analytic functions and differential forms on a compact Riemann surface is given along with a proof of an analogue of the Riemann-Roch theorem for pseudoholomorphic functions; aforementioned proof utilizes facts about Čech cohomology with values in a sheaf. The way these sheaves are defined makes it possible to consider them on a complex manifold of any dimension.

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Introduction

The pseudoholomorphic functions are defined as solutions of generalised Cauchy-Riemann equations and as such they form a class of functions strictly broader than that of holomorphic functions, yet, as is shown below, they still retain some important properties of holomorphic functions, e.g., the Riemann-Roch theorem holds for them.

Pseudoholomorphic functions on a Riemann surface here are defined with reference to two global smooth sections of the antiholomorphic cotangent bundle of the surface, in order for them to satisfy the axioms of a sheaf, and sheaves of such functions together with the sheaf of pseudoholomorphic 1-forms, whose local representations have pseudoholomorphic coefficients, are considered.

A certain subtype of these sheaves, which is introduced in section 3, is especially close in regard to its properties to the sheaf of holomorphic functions, which makes it possible to apply to its study many of the same methods.

1. Sheaves of pseudoholomorphic functions and differential 1-forms Funtions

Let S denote a Riemann surface and let $\mathcal{O}_S$ be the sheaf of rings of holomorphic functions on it. Consider the following generalisation of this sheaf: $\mathcal{O}_S^{\alpha,\beta}$, whose section, $f \in \mathcal{O}_S^{\alpha,\beta}(U)$, over an open set $U \subseteq S$ is a smooth function $f : U \to \mathbb{C}$, satisfying the following condition:

$$\bar{\partial} f = f\alpha|_U + \bar{f}\beta|_U,$$

where α , β are smooth sections of the antiholomorphic cotangent bundle of S, i.e., $\alpha, \beta \in \Gamma(S, \mathcal{A}_S^{0,1})$; and the restriction homomorphisms are defined naturally:

$$r_{V,U} : \mathcal{O}_S^{\alpha,\beta}(V) \to \mathcal{O}_S^{\alpha,\beta}(U),$$

$$\mathcal{O}_S^{\alpha,\beta}(V) \ni f \mapsto f|_U,$$

for any pair of open sets, $U \subseteq V$. Obviously, for $\alpha = \beta = 0 \in \Gamma(S, \mathcal{A}_S^{0,1})$ one has $\mathcal{O}_S^{0,0} = \mathcal{O}_S$.

Proposition 1. *$\mathcal{O}_S^{\alpha,\beta}$ is a sheaf of $\underline{\mathbb{R}}$-modules, where $\underline{\mathbb{R}}$ is the sheaf of locally constant $\mathbb{R}$-valued functions on S.*

Proof. In order to show that $\mathcal{O}_S^{\alpha,\beta}$ is indeed a sheaf, it suffices to verify that $r_{V,U}(f) = f|_U \in \mathcal{O}_S^{\alpha,\beta}(U)$, where $U \subseteq V$ and $f \in \mathcal{O}_S^{\alpha,\beta}(V)$;

$$\begin{aligned}\bar{\partial}\,(f|_U) &= \left(\bar{\partial}(f)\right)\big|_U = \left(f\alpha|_V + \bar{f}\beta|_V\right)\big|_U = (f\alpha|_V)|_U + \left(\bar{f}\beta|_V\right)\big|_U \\ &= f|_U\alpha|_U + \bar{f}|_U\beta|_U,\end{aligned}$$

i.e., $f \in \mathcal{O}_S^{\alpha,\beta}(U)$.

Now let us consider the set of sections of this sheaf over an open set $U \subseteq S$, $\mathcal{O}_S^{\alpha,\beta}(U)$, and the ring of real-valued, locally constant functions on U, $\underline{\mathbb{R}}(U)$; $\mathcal{O}_S^{\alpha,\beta}(U)$ is an abelian group with respect to the ordinary addition of functions; indeed, for any $f_1, f_2 \in \mathcal{O}_S^{\alpha,\beta}(U)$

$$\begin{aligned}\bar{\partial}\,(f_1 + f_2) &= \bar{\partial} f_1 + \bar{\partial} f_2 = f_1\alpha|_U + \bar{f}_1\beta|_U + f_2\alpha|_U + \bar{f}_2\beta|_U \\ &= (f_1 + f_2)\,\alpha|_U + \overline{(f_1 + f_2)}\beta|_U,\end{aligned}$$

so that, $(f_1 + f_2) \in \mathcal{O}_S^{\alpha,\beta}(U)$; moreover, it is an $\underline{\mathbb{R}}(U)$-module: for any $\lambda \in \underline{\mathbb{R}}(U)$

$$\bar{\partial}\,(\lambda f) = \lambda\bar{\partial}\,(f) = \lambda\left(f\alpha|_U + \bar{f}\beta|_U\right) = \lambda f\alpha|_U + \lambda\bar{f}\beta|_U = \lambda f\alpha|_U + \overline{\lambda f}\beta|_U,$$

which means that $\lambda f \in \mathcal{O}_S^{\alpha,\beta}(U)$. Therefore, $\mathcal{O}_S^{\alpha,\beta}$ is a sheaf of $\underline{\mathbb{R}}$-modules.

This sheaf is called the sheaf of pseudoholomorphic, or pseudoanalytic, or generalised analytic, functions, and its sections are called pseudoholomorphic functions. The sheaf $\mathcal{O}_S^{\alpha,\beta}$ is a kernel of the following sheaf homomorphism: for any pair $\alpha, \beta \in \Gamma(S, \mathcal{A}_S^{0,1})$ of global 1-forms define the sheaf homomorphism $\bar{\partial}_{\alpha,\beta} : C_S^\infty \to \mathcal{A}_S^{0,1}$ as follows: for an open set $U \subseteq S$ and a section over it, $f \in C_S^\infty(U)$,

$$\bar{\partial}_{\alpha,\beta}(f) := \bar{\partial} f - f\alpha|_U - \bar{f}\beta|_U \in \mathcal{A}_S^{0,1}(U).$$

Clearly, $\mathcal{O}_S^{\alpha,\beta} = \operatorname{Ker} \bar{\partial}_{\alpha,\beta}$.

1-forms

Similarly, for $\alpha, \beta \in \Gamma(S, \mathcal{A}_S^{0,1})$ one can consider a sheaf-homomorphism $\bar{\partial}^\star_{\alpha,\beta} : \mathcal{A}_S^{1,0} \to \mathcal{A}_S^{1,1}$ defined as follows: for $U \subseteq S$ open and $\omega \in \mathcal{A}_S^{1,0}(U)$

$$\bar{\partial}^\star_{\alpha,\beta}\omega := -\bar{\partial}\omega - \alpha|_U \wedge \omega - \overline{\beta|_U \wedge \omega} \in \mathcal{A}_S^{1,1}(U).$$

Denote $\Omega_S^{\alpha,\beta} \equiv \operatorname{Ker} \bar{\partial}^\star_{\alpha,\beta}$; This is again a sheaf of $\underline{\mathbb{R}}$-modules; One has that $\Omega_S^{0,0} = \Omega_S^1$, where $\Omega_S^1 \equiv \Omega_S$ is the sheaf of holomorphic 1-forms on S. This sheaf is called the sheaf of pseudoholomorphic, or pseudoanalytic, or generalised analytic, 1-forms.

2. Čech cohomology groups of these sheaves

Let us consider the following sequence of sheaves on a Riemann surface S:

$$0 \to \mathcal{O}_S^{\alpha,\beta} \xrightarrow{i} C_S^\infty \xrightarrow{\bar{\partial}_{\alpha,\beta}} \mathcal{A}_S^{0,1} \to 0.$$

This sequence is exact; its exactness in the first two terms is obvious, and the fact that $\bar{\partial}_{\alpha,\beta} : C_S^\infty \to \mathcal{A}_S^{0,1}$ is a sheaf-epimorphism follows from the $\bar{\partial}$-Poincar lemma. The associated long exact sequence of Čech cohomology groups will be

$$\begin{aligned} 0 &\to H^0(S, \mathcal{O}_S^{\alpha,\beta}) \to H^0(S, C_S^\infty) \to H^0(S, \mathcal{A}_S^{0,1}) \\ &\to H^1(S, \mathcal{O}_S^{\alpha,\beta}) \to H^1(S, C_S^\infty) \to H^1(S, \mathcal{A}_S^{0,1}) \to \cdots \end{aligned}$$

Since C_S^∞ and $\mathcal{A}_S^{0,1}$ are fine sheaves, their cohomology groups are trivial:

$$H^p(S, C_S^\infty) = H^p(S, \mathcal{A}_S^{0,1}) = 0, \quad p > 0.$$

and so it follows that

$$H^p(S, \mathcal{O}_S^{\alpha,\beta}) = 0, \quad p > 1.$$

thus we have obtained the following exact sequence of cohomology groups

$$0 \to H^0(S, \mathcal{O}_S^{\alpha,\beta}) \to H^0(S, C_S^\infty) \to H^0(S, \mathcal{A}_S^{0,1}) \to H^1(S, \mathcal{O}_S^{\alpha,\beta}) \to 0$$

from whose exactness it follows that

$$H^1(S, \mathcal{O}_S^{\alpha,\beta}) \cong \frac{H^0(S, \mathcal{A}_S^{0,1})}{\bar{\partial}_{\alpha,\beta}(H^0(S, C_S^\infty))}.$$

Analogously, from the short exact sequence of sheaves

$$0 \to \Omega_S^{\alpha,\beta} \xrightarrow{i} \mathcal{A}_S^{1,0} \xrightarrow{\bar{\partial}^\star_{\alpha,\beta}} \mathcal{A}_S^{1,1} \to 0.$$

one obtains the associated long exact sequence of cohomology groups

$$\begin{aligned}0 &\to H^0(S, \Omega_S^{\alpha,\beta}) \to H^0(S, \mathcal{A}_S^{1,0}) \to H^0(S, \mathcal{A}_S^{1,1}) \\ &\to H^1(S, \Omega_S^{\alpha,\beta}) \to H^1(S, \mathcal{A}_S^{1,0}) \to H^1(S, \mathcal{A}_S^{1,1}) \to \cdots\end{aligned}$$

most of whose terms are trivial, since $\mathcal{A}_S^{1,0}$ and $\mathcal{A}_S^{1,1}$ are fine sheaves:

$$H^p(S, \mathcal{A}_S^{1,0}) = H^p(S, \mathcal{A}_S^{1,1}) = 0, \quad p > 0.$$

so that one gets

$$H^p(S, \Omega_S^{\alpha,\beta}) = 0, \quad p > 1.$$

and the exact sequence

$$0 \to H^0(S, \Omega_S^{\alpha,\beta}) \to H^0(S, \mathcal{A}_S^{1,0}) \to H^0(S, \mathcal{A}_S^{1,1}) \to H^1(S, \Omega_S^{\alpha,\beta}) \to 0$$

whence it follows that

$$H^1(S, \Omega_S^{\alpha,\beta}) \cong \frac{H^0(S, \mathcal{A}_S^{1,1})}{\bar{\partial}^{\star}_{\alpha,\beta}(H^0(S, \mathcal{A}_S^{1,0}))}.$$

3. Pseudoholomorphic functions and 1-forms that are multiple of a divisor

One can also consider pseuduholomorphic functions that have at most pole type isolated singularities, i.e., a function $f \in \mathcal{O}_S^{\alpha,\beta}(U \backslash U')$, where $U \subseteq S$ is open and U' is its discrete subset, at whose points the function has singularities. Denote the set of such functions by $\tilde{\mathcal{O}}_S^{\alpha,\beta}(U)$. Each such function defines a divisor

$$(f) := \sum_n \mathrm{ord_{p_n}(f) \cdot p_n},$$

where the sum is taken over the set of zeroes and poles of the function f, and $\mathrm{ord_{p_n}(f) = k}$, if p_n is a zero of degree k, and $\mathrm{ord_{p_n}(f) = -k}$, if p_n is a pole of degree k; f is a multiple of the divisor $-D$, if $(f) \geq -D$, i.e., the sum of the divisor of f and D is an effective divisor

$$(f) + D \geq 0.$$

The set of pseudoholomorphic 1-forms that may have poles is defined analogously $\tilde{\Omega}_S^{\alpha,\beta}(U)$; every $\omega \in \tilde{\Omega}_S^{\alpha,\beta}(U)$ defines the divisor

$$(\omega) := \sum_n \mathrm{ord_{p_n}}(\omega) \cdot \mathrm{p_n},$$

where $\mathrm{ord_{p_n}}(\omega) := \mathrm{ord_{p_n}(f)}$; $\omega = f\mathrm{dz}$ with respect to the local coordinate z; This notion is well-defined since it is invariant under holomorphic coordinate transformations; the sum is again taken over the set of zeroes and poles of ω.

Thus, to each divisor $D \in \mathrm{Div(S)}$ one can associate sheaves $\mathcal{O}_S^{\alpha,\beta}(D)$ and $\Omega_S^{\alpha,\beta}(D)$, whose sections over an open set $U \subseteq S$ are:

$$\Gamma(U, \mathcal{O}_S^{\alpha,\beta}(D)) = \{\, f \in \tilde{\mathcal{O}}_S^{\alpha,\beta}(U) \mid (f) + D \geq 0 \,\},$$

$$\Gamma(U, \Omega_S^{\alpha,\beta}(D)) = \{\, \omega \in \tilde{\Omega}_S^{\alpha,\beta}(U) \mid (\omega) + D \geq 0 \,\}.$$

The subtypes of these sheaves, $\mathcal{O}_S^{\alpha,0}(D)$ and $\Omega_S^{\alpha,0}(D)$, have a richer algebraic structure, than the general ones, namely:

Proposition 2. *$\mathcal{O}_S^{\alpha,0}(D)$ and $\Omega_S^{\alpha,0}(D)$ are sheaves of $\mathcal{O}_S$-modules.*

Proof. For each open set $U \subseteq S$ one has that $\Gamma(U, \mathcal{O}_S^{\alpha,0}(D))$ and $\Gamma(U, \Omega_S^{\alpha,0}(D))$ are abelian groups. For sections $f \in \Gamma(U, \mathcal{O}_S^{\alpha,0}(D))$ and $h \in \Gamma(U, \mathcal{O}_S)$

$$\begin{aligned}\bar{\partial}_{\alpha,0}(hf) &= \bar{\partial}(hf) - (hf)\alpha|_U = h\bar{\partial}(f) - h(f\alpha|_U) = h(\bar{\partial}(f) - (f\alpha|_U)) \\ &= h\bar{\partial}_{\alpha,0}(f) = 0,\end{aligned}$$

i.e., $hf \in \Gamma(U, \mathrm{Ker}\bar{\partial}_{\alpha,0}) = \Gamma(\mathrm{U}, \mathcal{O}_{\mathrm{S}}^{\alpha,0}(\mathrm{D}))$; Thus, $\Gamma(U, \mathcal{O}_S^{\alpha,0}(D))$ is a $\mathcal{O}_S(U)$-module..

Analogously, for a section $\omega \in \Gamma(U, \Omega_S^{\alpha,0}(D))$ one has

$$\bar{\partial}^{\star}_{\alpha,0}(h\omega) = -\bar{\partial}(h\omega) - \alpha|_U \wedge (h\omega) = h\bar{\partial}^{\star}_{\alpha,\beta}(\omega) = 0,$$

that is, $h\omega \in \Gamma(U, \mathrm{Ker}\bar{\partial}^{\star}_{\alpha,0}) = \Gamma(\mathrm{U}, \Omega_{\mathrm{S}}^{\alpha,0}(\mathrm{D}))$; therefore, $\Gamma(U, \Omega_S^{\alpha,0}(D))$ is a $\mathcal{O}_S(U)$-module; in particular, $\mathcal{O}_S^{\alpha,0}(D)$ and $\Omega_S^{\alpha,0}(D)$ are sheaves of vector spaces over the field of complex numbers $\mathbb{C}$.

4. Duality theorem

In this section a duality theorem similar to the Serre duality will be proven. Let us consider the sheaves $\mathcal{O}_S^{\alpha,0}(D)$ and $\Omega_S^{\alpha,0}(-D)$, and their sections $f \in \Gamma(U, \mathcal{O}_S^{\alpha,0}(D))$ and $\omega \in \Gamma(U, \Omega_S^{\alpha,0}(-D))$ over an open set $U \subseteq S$; though neither of these sections are holomorphic, we have the following

Proposition 3. *The product of $f \in \Gamma(U, \mathcal{O}_S^{\alpha,0}(D))$ and $\omega \in \Gamma(U, \Omega_S^{\alpha,0}(-D))$ is a holomorphic 1-form on $U \subseteq S$ without poles.*

Proof. It suffices to show that $\bar{\partial}\,(f\omega) = 0$ and $(f\omega) \geq 0$. Indeed,

$$\begin{aligned}\bar{\partial}\,(f\omega) &= (\bar{\partial} f) \wedge \omega + f\bar{\partial}\omega = f\alpha|_U \wedge \omega + f(-\alpha \wedge \omega) \\ &= f(\alpha \wedge \omega) - f(\alpha \wedge \omega) = 0,\end{aligned}$$

and

$$(f\omega) = (f) + (\omega) \geq D - D = 0.$$

Thus, $f\omega \in \Omega_S(U)$. Therefore, one can define a bilinear map

$$\Gamma(U, \mathcal{O}_S^{\alpha,0}(D)) \times \Gamma(U, \Omega_S^{\alpha,0}(-D)) \to \Gamma(U, \Omega_S),$$

$$(f,\omega)\mapsto f\omega,$$

which, for any open covering $\mathfrak{U}=\{U_i\}_{i\in I}$, induces the bilinear map

$$H^1(\mathfrak{U},\mathcal{O}_S^{\alpha,0}(D))\times H^0(\mathfrak{U},\Omega_S^{\alpha,0}(-D))\to H^1(\mathfrak{U},\Omega_S),$$

$$([f],\omega)\mapsto[f\omega]$$

Proposition 4. *The bilinear map*

$$H^1(\mathfrak{U},\mathcal{O}_S^{\alpha,0}(D))\times H^0(\mathfrak{U},\Omega_S^{\alpha,0}(-D))\to H^1(\mathfrak{U},\Omega_S),$$

$$([f],\omega)\mapsto[f\omega]$$

is well-defined.

Proof. We need to show that it doesn't depend on the choice of a representative of $[f]$. Consider a cocycle $f\in Z^1(\mathfrak{U},\mathcal{O}_S^{\alpha,0}(D))$ and a global section $\omega\in H^0(\mathfrak{U},\Omega_S^{\alpha,0}(-D))$. Let us define a cochain $f\omega\in C^1(\mathfrak{U},\Omega_S)$ by

$$(f\omega)_{ij}:=\{f_{ij}\omega|_{U_i\cap U_j}\}_{i,j\in I}\in\Omega_S(U_i\cap U_j);$$

and show that it is a cocycle:

$$\begin{aligned}(\delta(f\omega))_{ijk}&=(f_{ij}\omega|_{U_i\cap U_j})|_{U_i\cap U_j\cap U_k}-(f_{ik}\omega|_{U_i\cap U_k})|_{U_i\cap U_j\cap U_k}\\&+(f_{jk}\omega|_{U_j\cap U_k})|_{U_i\cap U_j\cap U_k}=(f_{ij}|_{U_i\cap U_j\cap U_k}-f_{ik}|_{U_i\cap U_j\cap U_k}\\&+f_{jk}|_{U_i\cap U_j\cap U_k})\omega|_{U_i\cap U_j\cap U_k}=(\delta f)_{ijk}\omega|_{U_i\cap U_j\cap U_k}=0,\end{aligned}$$

that is, $f\omega\in Z^1(\mathfrak{U},\Omega_S)$.

Consider any cocycle $f'\in Z^1(\mathfrak{U},\mathcal{O}_S^{\alpha,0}(D)$ cohomologous to f, which means, that there is a cochain $g\in C^0(\mathfrak{U},\mathcal{O}_S^{\alpha,0}(D))$, such that

$$f'_{ij}-f_{ij}=g_j-g_i.$$

$f\omega$ and $f'\omega$ are also cohomologous:

$$\begin{aligned}(f\omega)_{ij}-(f'\omega)_{ij}&=f'_{ij}\omega|_{U_i\cap U_j}-f_{ij}\omega|_{U_i\cap U_j}=(f'_{ij}-f_{ij})\omega|_{U_i\cap U_j}\\&=(g_j|_{U_i\cap U_j}-g_i|_{U_i\cap U_j})\omega|_{U_i\cap U_j}=g_j|_{U_i\cap U_j}\omega|_{U_i\cap U_j}-g_i|_{U_i\cap U_j}\omega|_{U_i\cap U_j}\\&=(g_j\omega|_{U_j})|_{U_i\cap U_j}-(g_i\omega|_{U_i})|_{U_i\cap U_j}=(\delta(g\omega))_{ij},;\end{aligned}$$

where $g\omega\equiv\{g_i\omega|_{U_i}\}_{i\in I}\in C^0(\mathfrak{U},\Omega_S^{\alpha,0})$; which means that

$$f'\omega-f\omega\in\delta(C^0(\mathfrak{U},\Omega_S^{\alpha,0}).$$

Therefore $[f\omega]=[f'\omega]\in H^1(\mathfrak{U},\Omega_S)$ so that the bilinear map $([f],\omega)\mapsto[f\omega]$ is well-defined.

By passing to the direct, or inductive, limit over the set of open coverings of S ordered by fineness, we get the following bilinear map

$$H^1(S,\mathcal{O}_S^{\alpha,0}(D))\times H^0(S,\Omega_S^{\alpha,0}(-D))\to H^1(S,\Omega_S),$$

$$([f],\omega) \mapsto [f\omega],$$

where f now denotes an element of $H^1(\mathfrak{U}, \mathcal{O}_S^{\alpha,0}(D))$ for some open covering $\mathfrak{U}$, so that it itself is already an equivalence class, and $[f]$ is its corresponding equivalence class in $H^1(S, \mathcal{O}_S^{\alpha,0}(D))$.

By considering the short exact sequence of sheaves

$$0 \to \Omega_S \to \mathcal{A}^{1,0} \stackrel{\mathrm{d}}{\to} \mathcal{A}^{1,1} \to 0$$

and its associated long exact sequence of cohomology groups, one arrives, in virtue of the fact that $\mathcal{A}^{1,0}$ and $\mathcal{A}^{1,1}$ are fine sheaves, to the following exact sequence of Čech cohomology groups:

$$0 \to H^0(S, \Omega_S) \to H^0(S, \mathcal{A}^{1,0}) \to H^0(S, \mathcal{A}^{1,1}) \to H^1(S, \Omega_S) \to 0,$$

from which it follows that

$$H^1(S, \Omega_S) \cong \frac{\mathcal{A}_S^{1,1}(S)}{\mathrm{d}(\mathcal{A}_{\mathrm{S}}^{1,0}(\mathrm{S}))}.$$

Let us denote some map establishing this isomorphism by ϕ; and by means of it define a linear map $H^1(S, \Omega_S) \to \mathbb{C}$:

$$H^1(S, \Omega_S) \ni \omega \mapsto \frac{1}{2\pi i}\int_S \eta \in \mathbb{C},$$

where $\eta \in \mathcal{A}_S^{1,1}(S)$ is a representative of the equivalence class $\phi(\omega)$, that is $\phi(\omega) = [\eta]$; this map is well-defined, since for each 1-form $\xi \in \mathcal{A}_S^{1,0}(S)$

$$\frac{1}{2\pi i}\int_S (\eta + \mathrm{d}\xi) = \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{S}} \eta + \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{S}} \mathrm{d}\xi = \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{S}} \eta.$$

Composing the maps

$$H^1(S, \mathcal{O}_S^{\alpha,0}(D)) \times H^0(S, \Omega_S^{\alpha,0}(-D)) \to H^1(S, \Omega_S) \to \mathbb{C}$$

yields the following bilinear map

$$H^1(S, \mathcal{O}_S^{\alpha,0}(D)) \times H^0(S, \Omega_S^{\alpha,0}(-D)) \to \mathbb{C},$$

$$([f],\omega) \mapsto \langle [f],\omega\rangle,$$

where

$$\langle [f],\omega\rangle := \frac{1}{2\pi i}\int_S \eta \in \mathbb{C}, \quad \eta \in \phi([f\omega])$$

Lemma 1. *The bilinear map* $([f],\omega) \mapsto \langle [f],\omega\rangle$ *in non-degenerate.*

Proof. Let us take $[f] \in H^1(S, \mathcal{O}_S^{\alpha,0}(D))$ and $\omega \in H^0(S, \Omega_S^{\alpha,0}(-D))$ and consider the image

$$\langle [f],\omega\rangle = \frac{1}{2\pi i}\int_S \eta = 0, \quad \eta \in \phi([f\omega]).$$

Since S is a two-dimensional orientable smooth (real) manifold and $\eta \in \mathcal{A}_S^{1,1}(S)$ is a 2-form on it, the exactness of η follows from this integral being equal to zero, so that $\eta \in \mathrm{d}(\mathcal{A}_\mathrm{S}^{1,0}(\mathrm{S}))$; therefore

$$\phi([f\omega]) = 0.$$

ϕ is an isomorphism, and so its kernel is trivial, and thus

$$[f\omega] = 0 \in H^1(S, \Omega_S),$$

which, in virtue of the fact that ω is global, is possible only if either $[f] = 0 \in H^1(S, \mathcal{O}_S^{\alpha,0}(D))$ or $\omega = 0 \in H^0(S, \Omega_S^{\alpha,0}(-D))$; which means that this bilinear map is non-degenerate.

Using this lemma we can prove the main theorem of this section:

Theorem 1. (Duality) $H^0(S, \Omega_S^{\alpha,0}(-D))$ *is isomorphic to the space of linear functionals on* $H^1(S, \mathcal{O}_S^{\alpha,0}(D))$*, that is,* $H^0(S, \Omega_S^{\alpha,0}(-D)) \cong (H^1(S, \mathcal{O}_S^{\alpha,0}(D)))^*$.

Proof. It follows from Lemma 1 that $\dim_\mathbb{C} \mathrm{H}^1(\mathrm{S}, \mathcal{O}_\mathrm{S}^{\alpha,0}(\mathrm{D})) = \dim_\mathbb{C} \mathrm{H}^0(\mathrm{S}, \Omega_\mathrm{S}^{\alpha,0}(-\mathrm{D}))$ and that the kernel of the linear map

$$\begin{aligned} H^0(S, \Omega_S^{\alpha,0}(-D)) &\to (H^1(S, \mathcal{O}_S^{\alpha,0}(D)))^*, \\ \omega &\mapsto ([f] \mapsto \langle [f], \omega \rangle), \end{aligned}$$

is trivial, so that it is injective, besides,

$$\dim_\mathbb{C} \mathrm{H}^1(\mathrm{S}, \mathcal{O}_\mathrm{S}^{\alpha,0}(\mathrm{D})) = \dim_\mathbb{C} \mathrm{H}^0(\mathrm{S}, \Omega_\mathrm{S}^{\alpha,0}(-\mathrm{D})) = \dim_\mathbb{C} (\mathrm{H}^1(\mathrm{S}, \mathcal{O}_\mathrm{S}^{\alpha,0}(\mathrm{D})))^*,$$

which means that this map is also surjective; therefore, it is bijective and is an isomorphism and thus the spaces are indeed isomorphic:

$$H^0(S, \Omega_S^{\alpha,0}(-D)) \cong (H^1(S, \mathcal{O}_S^{\alpha,0}(D)))^*.$$

5. The Riemann-Roch theorem for pseudoholomorphic functions

Let us consider a point, $P \in S$ (capital letters have been chosen to denote points, in order to highlight that they are also viewed as divisors with a single term), and a coordinate chart around it, (U, φ), that is, $P \in U$ and $\varphi(P) = 0$. With respect to this coordinate system the solution of $\bar{\partial}_{\alpha,0} f = 0$ in a neighbourhood of P can be represented, according to the similarity principle, as follows

$$\hat{f}(z) = \phi(z) \exp\left\{ -\frac{1}{2\pi i} \int_{\varphi(U)} a(w) \, \frac{\mathrm{dz} \wedge \mathrm{d\bar{z}}}{w - z} \right\},$$

where $\hat{f} = f \circ \varphi^{-1}$ is the coordinate representation of f with respect to the given coordinate system, ϕ is a meromorphic function that determines zeroes and poles of f, and $a\mathrm{d}\bar{\mathrm{z}} = \varphi^*\alpha$. If $f \in \Gamma(U, \mathcal{O}_S^{\alpha,0}(D+P))$, then ϕ may have a pole at $z = \varphi(P) = 0$ whose degree can't exceed the constraint put on it by the divisor, so that its Laurent series will be

$$\phi(z) = \sum_{n=-k-1}^{\infty} c_n z^n,$$

where k is the coefficient of P in the divisor D.

Let us define a map $\epsilon_U : \Gamma(U, \mathcal{O}_S^{\alpha,0}(D+P)) \to \mathbb{C}_P(U)$, where $\mathbb{C}_P$ is the skyscraper sheaf associated with P, as follows: if $P \in U$, then

$$\epsilon_U(f) := c_{-k-1}.$$

this map is a homomorphism: for the sum of $f_1, f_2 \in \Gamma(U, \mathcal{O}_S^{\alpha,0}(D+P))$, $f = f_1 + f_2 \in \Gamma(U, \mathcal{O}_S^{\alpha,0}(D+P))$, we have

$$\begin{aligned}
\hat{f}(z) &= \hat{f}_1(z) + \hat{f}_2(z) = \phi_1(z) \exp\left\{-\frac{1}{2\pi i}\int_{\varphi(U)} a(w)\,\frac{\mathrm{dz}\wedge\mathrm{d\bar{z}}}{w-z}\right\} \\
&+ \phi_2(z) \exp\left\{-\frac{1}{2\pi i}\int_{\varphi(U)} a(w)\,\frac{\mathrm{dz}\wedge\mathrm{d\bar{z}}}{w-z}\right\} \\
&= (\phi_1(z) + \phi_2(z)) \exp\left\{-\frac{1}{2\pi i}\int_{\varphi(U)} a(w)\,\frac{\mathrm{dz}\wedge\mathrm{d\bar{z}}}{w-z}\right\},
\end{aligned}$$

so that,

$$\epsilon_U(f) = \epsilon_U(f_1) + \epsilon_U(f_2).$$

it is also surjective; Therefore, the set of homomorphisms $\{\epsilon_U\}_{U\in\mathfrak{Opn}_S}$ is a sheaf epimorphism, $\mathcal{O}_S^{\alpha,\beta}(D+P) \xrightarrow{\epsilon} \mathbb{C}_P$. Let us consider the following short exact sequence of sheaves

$$0 \to \mathcal{O}_S^{\alpha,0}(D) \to \mathcal{O}_S^{\alpha,0}(D+P) \xrightarrow{\epsilon} \mathbb{C}_P \to 0$$

and its associated exact sequence of cohomology groups

$$\begin{aligned}
0 &\to H^0(S, \mathcal{O}_S^{\alpha,0}(D)) \to H^0(S, \mathcal{O}_S^{\alpha,0}(D+P)) \to \mathbb{C} \\
&\to H^1(S, \mathcal{O}_S^{\alpha,0}(D)) \to H^1(S, \mathcal{O}_S^{\alpha,0}(D+P)) \to 0
\end{aligned}$$

Theorem 2. *For a divisor $D \in \mathrm{Div(S)}$ on a compact Riemann surface S of genus g the complex vector spaces $H^0(S, \mathcal{O}_S^{\alpha,0}(D))$ and $H^1(S, \mathcal{O}_S^{\alpha,0}(D))$ have finite dimensions which are related by the following equality:*

$$\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D)) - \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D)) = 1 - g + \deg D.$$

Proof. Note that the genus of a compact Riemann surface S can be defined as follows:

$$g(S) := \dim_{\mathbb{C}} \mathrm{H}^1(\mathrm{S}, \mathcal{O}_{\mathrm{S}}),$$

which coincides with the usual topological definition of the genus:

$$g(S) = \frac{b_1(S)}{2} = \frac{-\chi(S)+2}{2},$$

where $b_1(S)$ is the first Betti number of S, and $\chi(S)$ is its Euler characteristic. Let us prove the theorem by induction on the number of (non-zero) terms in the divisor. When $D = 0$, the left-hand side of the above equation is: $\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D)) - \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D))$; as is known, [9], one has

$$\dim_{\mathbb{R}} H^0(S, \mathcal{O}_S^{\alpha,0}(D)) - \dim_{\mathbb{R}} H^0(S, \Omega_S^{\alpha,0}(D)) = 2 - 2g,$$

so that,

$$\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D)) - \dim_{\mathbb{C}} H^0(S, \Omega_S^{\alpha,0}(D)) = 1 - g;$$

by the duality theorem of the previous section, we have $\dim_{\mathbb{C}} H^0(S, \Omega_S^{\alpha,0}(D)) = \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D))$; thus,

$$\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D)) - \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D)) = 1 - g.$$

Suppose that the theorem holds for an arbitrary divisor, $D \in \mathrm{Div(S)}$. Consider a point $P \in S$ and the divisor $D - P$. As was shown above, we have the following short exact sequence of sheaves:

$$0 \to \mathcal{O}_S^{\alpha,0}(D-P) \to \mathcal{O}_S^{\alpha,0}(D) \xrightarrow{\epsilon} \mathbb{C}_P \to 0,$$

whose associated exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H^0(S, \mathcal{O}_S^{\alpha,0}(D-P)) \to H^0(S, \mathcal{O}_S^{\alpha,0}(D)) \to \mathbb{C} \\ &\to H^1(S, \mathcal{O}_S^{\alpha,0}(D-P)) \to H^1(S, \mathcal{O}_S^{\alpha,0}(D)) \to 0 \end{aligned}$$

can be split into two short exact sequences:

$$0 \to H^0(S, \mathcal{O}_S^{\alpha,0}(D-P)) \to H^0(S, \mathcal{O}_S^{\alpha,0}(D)) \to \mathrm{Im(H^0(S, \mathcal{O}_S^{\alpha,0}(D)) \to \mathbb{C})} \to 0$$

and

$$0 \to \frac{\mathbb{C}}{\mathrm{Im(H^0(\mathcal{O}_S^{\alpha,0}(D)) \to \mathbb{C})}} \to H^1(S, \mathcal{O}_S^{\alpha,0}(D-P)) \to H^1(S, \mathcal{O}_S^{\alpha,0}(D)) \to 0.$$

therefore we have,

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{Im(H^0(S, \mathcal{O}_S^{\alpha,0}(D)) \to \mathbb{C})} + \dim_{\mathbb{C}} \frac{\mathbb{C}}{\mathrm{Im(H^0(\mathcal{O}_S^{\alpha,0}(D))}} = 1 \\ &= \deg(D) - \deg(D-P). \end{aligned} \qquad (*)$$

and, since a short exact sequence of vector spaces splits, one has:

$$\begin{aligned} H^0(S, \mathcal{O}_S^{\alpha,0}(D)) &\cong H^0(S, \mathcal{O}_S^{\alpha,0}(D-P)) \oplus \mathrm{Im(H^0(S, \mathcal{O}_S^{\alpha,0}(D)) \to \mathbb{C})}, \\ H^1(S, \mathcal{O}_S^{\alpha,0}(D-P)) &\cong \frac{\mathbb{C}}{\mathrm{Im(H^0(\mathcal{O}_S^{\alpha,0}(D)) \to \mathbb{C})}} \oplus H^1(S, \mathcal{O}_S^{\alpha,0}(D)), \end{aligned}$$

from which it follows that:

$$\begin{aligned}
&\dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D) = \dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D-P))\\
&+\dim_{\mathbb{C}} \mathrm{Im(H^0(S,\mathcal{O}_S^{\alpha,0}(D))\to\mathbb{C})},\\
&\dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D-P)) = \dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D))\\
&+\dim_{\mathbb{C}} \frac{\mathbb{C}}{\mathrm{Im(H^0(\mathcal{O}_S^{\alpha,0}(D))\to\mathbb{C})}}.
\end{aligned}$$

Summing the above equations, one gets:

$$\begin{aligned}
&\dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D-P)) - \dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D-P))\\
&+\dim_{\mathbb{C}} \mathrm{Im}(H^0(S,\mathcal{O}_S^{\alpha,0}(D))\to\mathbb{C}) = \dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D))\\
&-\dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D)) - \dim_{\mathbb{C}} \frac{\mathbb{C}}{\mathrm{Im(H^0(\mathcal{O}_S^{\alpha,0}(D))\to\mathbb{C})}}.
\end{aligned}$$

by $(*)$:

$$\begin{aligned}
&\dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D-P)) - \dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D-P))\\
&= \dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D)) - \dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D)) - 1.
\end{aligned}$$

By assumption, the theorem holds for D, i.e.,

$$\dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D)) - \dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D)) = 1-g+\deg D;$$

so, we have:

$$\dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D-P)) - \dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D-P)) = 1-g+\deg(D)-1$$

But $\deg(D)-1=\deg(D-P)$. Thus,

$$\dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D-P)) - \dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D-P)) = 1-g+\deg(D-P).$$

So, if the theorem holds for a divisor D, it also holds for $D-P$, for any point $P\in S$.

Analogously, assume that the theorem holds for D and consider $D+P$ and, similarly to the above case, by considering the short exact sequence

$$0\to\mathcal{O}_S^{\alpha,0}(D)\to\mathcal{O}_S^{\alpha,0}(D+P)\xrightarrow{\epsilon}\mathbb{C}_P\to 0$$

and its associated exact sequence of cohomology groups,

$$\begin{aligned}
0&\to H^0(S,\mathcal{O}_S^{\alpha,0}(D))\to H^0(S,\mathcal{O}_S^{\alpha,0}(D+P))\to\mathbb{C}\\
&\to H^1(S,\mathcal{O}_S^{\alpha,0}(D))\to H^1(S,\mathcal{O}_S^{\alpha,0}(D+P))\to 0
\end{aligned}$$

one gets the following equality:

$$\begin{aligned}
&\dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D+P)) - \dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D+P))\\
&-\dim_{\mathbb{C}} \frac{\mathbb{C}}{\mathrm{Im(H^0(\mathcal{O}_S^{\alpha,0}(D+P))\to\mathbb{C})}} = \dim_{\mathbb{C}} H^0(S,\mathcal{O}_S^{\alpha,0}(D))\\
&-\dim_{\mathbb{C}} H^1(S,\mathcal{O}_S^{\alpha,0}(D)) + \dim_{\mathbb{C}} \mathrm{Im(H^0(S,\mathcal{O}_S^{\alpha,0}(D+P))\to\mathbb{C})},
\end{aligned}$$

i.e.,

$$\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D+P)) - \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D+P))$$
$$- \dim_{\mathbb{C}} \frac{\mathbb{C}}{\mathrm{Im}(\mathrm{H}^0(\mathcal{O}_{\mathrm{S}}^{\alpha,0}(\mathrm{D}+\mathrm{P})) \to \mathbb{C})} = \dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D))$$
$$\dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D)) + \dim_{\mathbb{C}} \mathrm{Im}(\mathrm{H}^0(\mathrm{S}, \mathcal{O}_{\mathrm{S}}^{\alpha,0}(\mathrm{D}+\mathrm{P})) \to \mathbb{C}),$$

from which we get

$$\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D+P)) - \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D+P)) = \dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D))$$
$$- \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D)) + 1 = 1 - g + \deg(D) + 1,$$

thus, since $\deg(D) + 1 = \deg(D+P)$,

$$\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S^{\alpha,0}(D+P)) - \dim_{\mathbb{C}} H^1(S, \mathcal{O}_S^{\alpha,0}(D+P)) = 1 - g + \deg(D+P).$$

Therefore, if the theorem holds for D, then it also holds for $D+P$. On a compact Riemann surface any divisor $D \in \mathrm{Div(S)}$ has the form:

$$D = \sum_{n=1}^{N} k_n P_N, \quad \forall n \in \{1, \dots, N\},\ k_n \in \mathbb{Z}.$$

Thus, the theorem holds for any divisor. $D \in \mathrm{Div(S)}$.

6. Concluding remarks

Thus, two important theorems, namely, the duality theorem and the Riemann-Roch theorem, that hold for the sheaf of holomorphic functions (on a given Riemann surface) also hold for this subtype of the sheaf of pseudoholomorphic functions (on a given Riemann surface).

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