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STABLE EQUILIBRIA OF THREE CONSTRAINED UNIT CHARGES

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Abstract. We discuss stable equilibrium points of electrostatic potential of three constrained unit point charges with Coulomb interaction. The main aim is to describe all triples of points on a unit circle in the Euclidean plane such that unit charges placed at those points form a trapping configuration, i.e. possess a stable equilibrium. We show that trapping configurations exist and give an explicit description of the set of all trapping configurations in the reduced configuration space (Theorem 1). We also prove that the set of all stable equilibria arising in this way is a circle of radius approximately equal to 0.4 (Theorem 2).

We also consider the case where three equal charges are constrained to two concentric circles. Here one has to distinguish two cases: two charges on the inner circle, and two charges on the outer circle. In the first case we show that trapping configurations always exist and describe the set of all stable equilibria arising in this way (Theorem 3). In the second case, we give a criterion of existence of trapping configurations in terms of the ratio of the two radii (Theorem 4). In conclusion several possible generalizations are outlined and relations to other mathematical models of constrained point charges are briefly discussed.

Keywords and phrases: Coulomb interaction, point charges, stable equilibrium, Maxwell conjecture.

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1. Introduction

We deal with the equilibria (critical points) of the electrostatic (Coulomb) potential of three equal point charges. The problems discussed in this paper are motivated by the Maxwell conjecture for three point charges [3], [8], Bohr's 1913 atomic model [2] and the inverse problem considered in [5], [4].

Recall that for a triple of positive real numbers $Q = (q_1, q_2, q_3)$ and three points P_1, P_2, P_3 in the Euclidean plane, the Coulomb potential E_S of system $S = Q@T$ of charges q_i placed at points P_i is defined as a function on the plane given by

$$E_S(P) = \frac{q_1}{d_1} + \frac{q_2}{d_2} + \frac{q_3}{d_3}, \tag{1}$$

where P is a point in the plane and d_i is the distance between points P_i and P.

In this setting, *equilibria* (or *equilibrium points*) of system S are defined as critical points of Coulomb potential E_S considered as a function of point P. If potential E_S has an isolated minimum at $P \in T$ then P is called a *stable equilibrium* of system S. We are interested in two problems concerned with the stable equilibria of three concentrically constrained unit charges. Constraints in the form of concentric circles appear, e.g., in Bohr's 1913 atomic model [2].

Remark 1. Since in this setting Coulomb potential depends on two variables one can determine the type of its critical point by calculating the hessian of E_S at this critical point. This observation plays an important role in Section 3.

In the sequel we repeatedly refer to results of [8], [9]. In particular, if T is a regular triangle then there are four equilibria of unit charges placed at vertices one of which is stable (minimum) and the three others are non-degenerate saddle-points.

We will be concerned with possible locations of unit charges possessing stable equilibria and the sets of arising stable equilibria. Our approach is motivated by [4] and heavily relies on some results of [9] which are presented in the next section.

2. Notation and auxiliary results

Following [9] let us assume that unit point charges are placed at points $P_1 = (-1, 0), P_2 = (1, 0)$ and $P_3 = (z, w)$ with positive z, w. Note that multiplication of all charges by the same positive number does not change the number and local properties of equilibria so one can speak of any three equal point charges. The following results have been established in [9].

Let us take $z = 0, w = \sqrt{3}$ and denote by I the center $(0, \frac{\sqrt{3}}{3})$ of the arising regular triangle T_*. Then by symmetry, I is an equilibrium of equal charges placed at vertices of T_*. Putting all vertex charges equal to 1 the Coulomb potential E of unit charges takes the form

$$E = \frac{1}{\sqrt{(x+1)^2 + y^2}} + \frac{1}{\sqrt{(x-1)^2 + y^2}} + \frac{1}{\sqrt{(x-z)^2 + (y-w)^2}} \tag{2}$$

It is well known and easy to verify that there are four equilibria of E: the center I itself and a single point on the segment of each median between I and the midpoint of the corresponding side. From the equilibrium condition it easily follows that ordinates of equilibria lying on Oy are the positive roots of the algebraic equation

$$(y^2 + 1)^3 - 4y^2(y - \sqrt{3})^4 = 0.$$

This equation has four real roots which equal approximately

$$-0.1463, 0.2489, \frac{\sqrt{3}}{3} = 0.5774, 6.2044.$$

The first and the last root do not fit the situation so there are two lacunas I, L_1 on the axis of symmetry with the ordinates $\frac{\sqrt{3}}{3}$ and approximately 0.249 respectively. It is instructive to investigate the behaviour of the Coulomb potential near these points. To this end we compute the hessian h_E and evaluate it at these points using computer, which gives $h_E(I) = 27/256 > 0$ and $h_E(L_1) = -0.105 < 0$. Thus I is a non-degenerate minimum of E while L_1 and the remaining two equilibria L_2, L_3 are non-degenerate saddles of E. In other words, I is a stable equilibrium point of E.

Consider now the triangle $T(z, w)$ with variable third vertex (z, w). A standard use of implicit function theorem shows that, for (z, w) sufficiently close to

$(0, \sqrt{3})$, the equally charged vertices of $T(z, w)$ have a stable equilibrium point near to I. Moreover, a standard argument of non-linear analysis shows that the stable equilibrium points arising in this way fill in a certain neighbourhood of I. On the other hand, it is known that for (z, w) tending to $(0, \infty)$ there are no stable equilibria and there remain only two saddle-points.

A natural problem now is to characterize the set $D(1)$ of those pairs (z, w) for which $T(z, w)$ has stable equilibria and the whole set $V(1)$ of stable equilibria arising in this way. As it was shown in [9] the set $D = D(1)$ is a union of two connected domains D_+ and D_- which symmetric to each other with respect to the Ox axis. The "upper" domain D_+ has the form of a symmetric deltoid with piecewise differentiable boundary ∂D_+ having three cusps at the points with approximate coordinates

$$(0, 2), (-0.134, 1.616), (0.134, 1.616).$$

The second point of intersection of ∂D_+ with Oy has an approximate ordinate 1.65. An explicit equation of ∂D_+ is given on page 92 of [9]. The plot of $D(1)$ given in [9] shows that the arcs between the cusps are concave and have minimal curvature at point $Q_1(0, 1.65)$ and two other points Q_2, Q_3 obtained from Q_1 by rotation through $2\pi/3$ about $(0, \sqrt{3})$.

Let us denote by $D(a)$ an analogous domain for a regular triangle with the base $(-a, 0), (a, 0)$ with some $a > 0$. The following lemma generalizes the above results by rescaling and plays a crucial role in the sequel.

Lemma 1. *For a regular triangle with vertices* $(-a, 0), (a, 0), (0, a\sqrt{3})$*, the deltoid* $D(a)$ *consists of three points* $(0, 2a), (-0.134a, 1.616a), (0.134a, 1.616a)$ *connected by three concave arcs with central points* $(0, 1.65a), (0.29a, 2.17a)$, $(-0.29a, 2.17a)$.

In the next section we apply these results in the case of three unit charges on a circle. We will say that configuration of three charges is a *trapping configuration* if it possesses a stable equilibrium.

3. Three unit charges on the circle

Let S^1 be the unit circle in the complex plane and $\{P_i\}$ a triple of points in S^1. We aim at characterizing the triples of points such that equal charges q_i, say $q_i = 1$, placed at those points have a stable equilibrium in the unit disc U. Without loss of generality we can assume that the first two given points on S^1 are always symmetric with respect to the y-axis, i.e. are of the form $(\pm a, \pm\sqrt{1 - a^2})$. Clearly, it is sufficient to consider only pairs of points of the form $(\pm a, -\sqrt{1 - a^2})$ with $0 < a < 1$. Let us analyze what happens for variable third point P_3.

From the results presented in the introduction follows that, for given $a > 0$, stable equilibrium exists if and only if P_3 belongs to the region $V(a)$. This means that the intersection $V(a) \cap S^1$ should be non-empty. Since we know the extremal points and exact shape of $V(a)$ for any a we can explicitly express the property of non-empty intersection. To this end let us consider each of the two components $V_\pm(a)$. Taking into account the known shape of $\partial V_+(a)$ it is clear that we should consider three specific values of a:

(1) a_-, for which the point $(0, 2a - \sqrt{1 - a^2})$ lies on the unit circle

(2) a_0, for which the point $(0, 1.65a - \sqrt{1 - a^2})$ lies on the unit circle

(3) a_+, for which the two symmetric points with coordinates $(\pm 0.232, 1.59 - \sqrt{1-a^2})$ lie on the unit circle.

Each of these conditions reduces to a simple equation, namely:

(1) $2x - \sqrt{1-x^2} = 1$

(2) $1.65x - \sqrt{1-x^2} = 1$

(3) $0.232^2 + (1.59x - \sqrt{1-x^2})^2 = 1$.

Solving these equations one obtains $a_{=}0.800, a_0 = 0.887, a_+ = 0.912$. Writing down similar equations for intersection points of $\partial V_-(a)$ with S^1 one finds out that they do not have solutions in the interval $(0,1)$. Thus for the two points $[(-a, -\sqrt{1-a^2})$ and $[(a, -\sqrt{1-a^2})$ the intersection of $V(a)$ with S^1 is non-empty if and only if $0.8 < a < 0.912$. Notice that for $a \in [0.8, 0887]$ this intersection is connected, while for $a \in (0.887, 0.891]$ it consists of two arcs symmetric with respect to Oy. The union of all such arcs is denoted by $V(P_1, P_2)$.

Theorem 1. *Three points P_1, P_2, P_3 on the unit circle yield stable positions of unit charges if and only if for one pair of points, say P_1, P_2, the third point belongs to the set $V(P_1, P_2)$.*

Remark 2. An analogous result for a circle of arbitrary radius follows by obvious rescaling.

Remark 3. From the symmetry of data it follows that if this condition is fulfilled for one pair of given points, then it is fulfilled for any of the three possible pairs.

Remark 4. From the symmetry of data it also follows that this criterion depends only on the shape of $\triangle P_1P_2P_3$. Thus it defines a certain subset on the Kendall sphere of triangular shapes [6], which is rotationally invariant about the axis containing the classes of regular triangle with two possible orientations. It would be interesting to find an invariant reformulation of this criterion, for example, in terms of the *irregularity measure* of $\triangle P_1P_2P_3$ defined as the sum of absolute values of pairwise differences of angles of the triangle. Most likely, this measure should be less than certain threshold which can be found by computer experiments.t Alternatively, one can fix position of one point on the circle, reformulate this criterion in terms of the two polar angles of remaining two points, and estimate the area of corresponding domain on the two-torus.

Let us now describe the set of all stable equilibria of three unit charges sliding on the unit circle. First of all, we show that any point inside S^1 is an equilibrium of a certain configuration of three unit charges on S^1. Without loss of generality we can assume that the given point has coordinates $(0, -r)$ with some $r \in (0,1)$.

Lemma 2. *For any $r \in (0,1)$, the point $P(0,-r)$ is an equilibrium of a certain configuration of three unit charges on the unit circle.*

Proof. Let us denote by P_i the sought positions of unit charges in S^1. For symmetry reasons it is sufficient to take the first two points of the form $P_1(-a, -\sqrt{1-a^2}), P_2(a, -\sqrt{1-a^2})$. Since the resultant of two forces acting on P from P_1, P_2 is collinear with the positive ray $0y_+$ of Oy-axis it follows that there always exists a unique third point $P_3(0,z) \in 0y_+$ such that P is an equilibrium of arising configurations of unit charges. Assuming that this is the case let us write down the equilibrium equation as vanishing of resultant force at point $P(0,-r)$. Distances between P_1, P_2 and P are equal to $\sqrt{a^2 + (\sqrt{1-a^2} - r^2)^2}$ and the sum of projections of the first two forces is equal to

$$\frac{2(\sqrt{1-a^2}-r)}{(a^2+(\sqrt{1-a^2}-r^2))^{3/2}}.$$

Since $z > 0$ the distance between P_3 and P is simply $z + r$ the equilibrium equation takes the form

$$\frac{2(\sqrt{1-a^2}-r)}{(a^2+(\sqrt{1-a^2}-r^2))^{3/2}} = \frac{1}{(z+r)^2}.$$

Clearly, for given r and a there exits a unique positive value of z satisfying the latter equation. We are interested in the case where $P_3 \in S^1$, i.e. $z = 1$. In other words we should substitute $z = 1$ and look for values of a which are solutions to the equation:

$$\frac{2(\sqrt{1-a^2}-r)}{(a^2+(\sqrt{1-a^2}-r^2))^{3/2}} = \frac{1}{(1+r)^2}.$$

The latter equation can be solved in an explicit form. Putting $u = \sqrt{1-a^2}-r$ we have that $a^2 = 1-(u+r)^2$ and rewrite it as

$$\frac{2u}{(1-(u+r)^2+u^2)^{3/2}} = \frac{1}{(1+r)^2}.$$

Squaring both sides and clearing denominators we arrive at the equation

$$4u^2(1+r)^4-(1-2ru+r^2)^3 = 0$$

which is equivalent to a cubic equation for u:

$$8r^3u^3+4((1+r)^4-3r^2(1+r^2))u^2+6r(1+r^2)^2u-(1+r^2)^3 = 0.$$

It turns out that this equation has exactly one positive root u_* and the sought value of a is $\sqrt{1-(u+r)^2}$. This means that, for any $r > 0$, a trapping configuration for $(0, -r)$ exists and is uniquely defined by the above formulas. The lemma is proven.

Notice that we can now get an explicit formula for the Coulomb potential of the trapping triple and compute its hessian h_r, which leads to the second main result of this paper.

Theorem 2. *The set of all stable equilibria of three point charges on a unit circle is a circle centered at the origin with the radius equal approximately to* 0.4.

The fact that this set is a circle follows by the symmetry of all steps in the proof of Lemma 2. Its radius can be computed using the hessian $h(r)$ of the Coulomb energy of the trapping configuration $E(r)$ constructed in the proof of Lemma 2.

4. Three unit charges on two concentric circles

The following setting is motivated by Bohr's 1913 model of lithium [2]. Consider three unit charges distributed on two concentric circles of radii

r and R with the common center at the origin and $0 < r < R$. One has to distinguish two cases: (a) two charges on the outer circle and (b) two charges are on the inner circle. It turns out that the trapping positions of three unit charges always exist in the first case.

Theorem 3. *For any $0 < r < R$, there exist points $P_1, P_2 \in S_R$ and $P_3 \in S_r$ such that unit charges placed at these points possess a stable equilibrium. The set of all stable equilibria arising in this way is an open centered circular ring in the outer circle.*

The situation in the second case is more complicated.

Theorem 4. *The set of trapping configurations with $P_1, P_2 \in S_r$ and $P_3 \in S_R$ is non-empty if and only if $R \leq r\sqrt{5}$. If this is fulfilled then the set of stable equilibria is an open disc centered at the origin.*

Both these results are proven using the same considerations as in Theorems 1 and 2.

Remark 5. It is also possible to get estimates for the size the set of stable equilibria in both cases. We omit the details since it is a quite technical issue.

5. Concluding remarks

A natural next step would be to consider three unit charges on three distinct concentric circles. However here a criterion of existence of stable triples is more complicated so we delay it for the future. It is also interesting to consider the case of three circles with disjoint interiors. For example, this can be easily done for "kissing" equal circles.

Another perspective is to consider unit charges with other constraints, for example on an ellipse. Actually, certain qualitative results can be obtained for the case of a smooth convex curve considered in [7].

Notice also that potential E_P has degenerate critical points at $\partial V(a)$. In the case of regular triangle, bifurcation points are of pitchfork (saddle-node) type [9]. It would be interesting to elaborate upon this observation in our setting.

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