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ON STABLE POLYNOMIAL MAPPINGS OF THE PLANE

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Abstract. In the paper we study algebraic criterion of stability for polynomial endomorphism of the plane and we obtain estimation of number of cusps for such mappings.

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1. Algebraic criterion of stability for endomorphism of the plane

In this paper we'll try to give mentioned criterion for the general case, despite the fact that for our aim it's enough to consider the two-dimensional case. At first recall the definition.

Definition 1.1. The map $f \in C^{\infty}(X, Y)$ is called stable mapping if there exists a neighbourhood W_f of $f \in C^{\infty}(X, Y)$ such that any $f' \in W_f$ is diffeomorphically equivalent to f [5].

By Whitney theorem stable germs of mappings of plane into plane are regular, or folds, or cusps.

Theorem 1.1. [3] *By Whitney theorem there exist only three equivalence classes of stable map germs* $f : \mathbb{R}^2 \to \mathbb{R}^2$. 1. *The neighbourhood of point* $w \in \mathbb{R}^2$ *is mapped regularly:* $(x, y) \longmapsto (x, y)$ *(regular point).*

2. *Point w lies on a fold, if in some system of coordinates the map looks so:* $(x, y) \longmapsto (x, y^2) = (x, z)$ *(fold).*

3. *A point w lies where the fold begins or ends; if in some system of coordinate the map looks so:* $(x, y) \longmapsto (x, y^3 - xy) = (x, z)$*; in this case w is called a point of cusp.*

Further iterating the well known Splitting lemma one obtains.

Proposition 1.1. *Let a map* $f = (f_1, \ldots, f_m) : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ *have locally constant rank* $rk_0 f = k$*, then there exist* $r_{k+1}, \ldots, r_k \in N$ *and* $\varphi, \psi \in \mathrm{Diff}(\mathbb{R}^n, 0)$ *such, that*

$$\psi \circ f \circ \varphi^{-1}(x) = \Big(x_1, \ldots, x_k, \ldots, \sum_{j=k+l}^{n} a_s^j x_j^2 + \sum_{j,l=l,j\neq l}^{n} b_s^{jl} x_j x_l + g_s(x), \ldots\Big)$$

where $s = k+1, \ldots, m)$, $g \in \mathfrak{m}^3$. *For* $m = 2$ *and* $\mathrm{rk}_0 f = 1$*, the needed form of the splitting lemma is obtained.*

In our case when $n = 2$ too,we have

$$(x, y) \longmapsto (x, bxy + cy^2 + g(x, y)).$$

We'll show that it is easy to distinguish folds and cusps in the sense of Definition 1.1, by canonical form given by the Splitting lemma.

Consider several cases.

1. The first case where $b = c = 0$, i.e., $\mathrm{rk}\, Hess f_2(0) = 0$, as it's easily seen, is unstable.

Indeed, the existence common zero of three functions $\dfrac{\partial^2 f_2}{\partial x^2}$, $\dfrac{\partial^2 f_2}{\partial x\,\partial y}$ and $\dfrac{\partial^2 f_2}{\partial y^2}$ of two variables, which of course is unstable, because open and dense set consists of Morse functions, i.e.,

2. If $c \neq 0$ then $\dfrac{\partial f_2}{\partial y}(0) = 0$ and $\dfrac{\partial^2 f_2}{\partial y^2}(0) = 0$.

According [3] we'll show that this is a point of the fold.

Recall that by standard use of locally finite covering one obtains an open and dense set $U \subset C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, such that every map $f \in U$ has local coordinate representation $(x, y) \longmapsto (x, f_2(x, y))$ in each point, where the function f_2 satisfies one of the conditions

$$\begin{aligned}
&\text{I)} && \frac{\partial f_2}{\partial y}(0) \neq 0, \\
&\text{II)} && \frac{\partial f_2}{\partial y}(0) = 0, \quad \text{and} \quad \frac{\partial^2 f_2}{\partial y^2}(0) \neq 0, \\
&\text{III)} && \frac{\partial f_2}{\partial y}(0) = \frac{\partial^2 f_2}{\partial y^2}(0) = 0, \quad \text{and} \quad \frac{\partial^2 f_2}{\partial x\,\partial y}(0) \neq 0, \quad \frac{\partial f_3}{\partial y^3}(0) \neq 0.
\end{aligned} \tag{1}$$

Case I) corresponds to a regular point.

Let's consider in detail the remaining two cases.

In case II) f_2 can be expressed as

$$f_2(0, y) = y^2 \cdot q(y), \quad \text{where} \quad q(0) \neq 0,$$

so the ideals $\langle x, f_2\rangle_{\varepsilon(2)}$ and $\langle x, y^2\rangle_{\varepsilon(2)}$ coincide with each other.

By Malgrange preparation theorem, if $\widetilde{f} : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a differential germ and induces homomorphism $\widetilde{f}^* : E(p) \to E(n)$, the following statements are equivalent.

a) $\varphi_1, \ldots, \varphi_n \in E(n)$ generates $E(n)$ as $E(p)$-module,

b) the classes of germs $\varphi_1, \ldots, \varphi_n \in E(n)$ generate vector space $E(n)/f^*\mathfrak{m}(\mathfrak{p}) \cdot \mathfrak{E}(\mathfrak{n})$.

So the functions f and g generate $E(2)$ as $E(2)$-module with the module structure defined with the help of map f^*.

Particularly, germ y^2 can be written as:

$$y^2 = \Phi(x, f_2(x, y)) + 2\Psi(x, f_2(x, y))y$$

where $\Phi(c, z)$ and $\Psi(x, z)$ are certain germs.

Taking Taylor's expansion up to the members of second order both parts of the equality we get two polynomials of second order with coinciding coefficients:

$$\Phi(0, 0) = \Psi(0, 0) = 0,$$
$$\partial\Psi(x, f(x, y))/\partial y(0) = \frac{\partial \Psi}{\partial z}(0) \cdot \frac{\partial f_2}{\partial y}(0) = 0,$$

It means that the maps

$$h(x,y) = \big(x, y - \Psi(x, f_2(x,y))\big),$$
$$k(x,y) = \big(x, y - \Phi(x,z) + \Psi^2(x,y)\big)$$

are changes of coordinates. So the following diagram is commutative

$$\begin{array}{ccc} (x,y) & \overset{h}{\longmapsto} & \big(x, y - \Psi(x, f_2(x,y))\big) \\ f \downarrow & & \downarrow (id, id^2) \\ (x, f_2(x,y)) & \overset{k}{\longmapsto} & (x, \Phi(x, f_2(x,y)) + \Psi^2(x, f_2(x,y)) \end{array}$$

since,

$$y^2 + \Psi^2 - 2\Psi y = \Phi + 2\Psi y + \Psi^2 - 2\Psi y = \Phi + \Psi^2.$$

Hence, f is equivalent to the map

$$(x,y) \longmapsto (x, y^2).$$

3. If $c = 0$, then by the above said, certainly $b \neq 0$. Notice that according to (1) in our conditions we have equivalence: stability $f \Longleftrightarrow f_2(0,y) \in \mathfrak{m}^3/\mathfrak{m}^4$.

Actually, simple addition to f_2 a term εy^3 gives deformation changing the type of germ. So we have

$$\frac{\partial^2 f_2}{\partial x\, \partial y}(0) \neq 0, \quad \frac{\partial^3 f_3}{\partial y^3}(0) \neq 0,$$

i.e., we consider situation when in (1) takes place case III) . In this case

$$\frac{\partial f_2}{\partial y}(0) = \frac{\partial^2 f_2}{\partial y^2}(0) = 0,$$

and

$$\frac{\partial^2 f_2}{\partial x\, \partial y}(0) \neq 0, \quad \frac{\partial^3 f_2}{\partial y^3}(0) \neq 0.$$

As above, we conclude from preparation theorem that there exist $\widetilde{\Phi}, \widetilde{\Psi}, \widetilde{\Theta}$ such that

$$y^3 = \widetilde{\Phi}(x, \widetilde{f}_2(x,y)) + \widetilde{\Psi}(x, \widetilde{f}(x,y)) \cdot y + 3 \cdot \widetilde{\Theta}(x, \widetilde{f}(x,y)) \cdot y^2.$$

From this equality there follows an equality between the jets at the origin. Comparing the coefficients of f, y, y^2 we obtain $\widetilde{\Phi}(0) = \widetilde{\Psi}(0) = \widetilde{\Theta}(0) = 0$ and function $\widetilde{\Theta}(0, f(0,y))$ has zero at least third order in y (recall that it means that this function lies into $(1)^3$). Consequently, the transformation

$$(x,y) \longmapsto \big(x, y - \widetilde{\Theta}(x, \widetilde{f}_2(x,y))\big)$$

is the desirable change of coordinate.

This change of coordinate transforms the function f_2 into the function f_T, defined with the help of the following diagram

$$\begin{array}{ccccc} (x,y) & \longrightarrow & \big(x, y - \widetilde{\Theta}(x, \widetilde{f}_2(x,y))\big) & = (x, \overline{y}) \\ & f_2 \searrow & \swarrow f_T & \\ f_2(x,y) & = & f_T(x,y) & \end{array}$$

Function f_T satisfies (1) in III) case, the same condition as f: function

$$f_t(0,\overline{y}) = f\big(0, y - \Theta(0, f_2(0,y))\big)$$

has zero of exactly first order, and $\dfrac{\partial f_T}{\partial x(0,\overline{y})}$ has zero of exactly first order.

We can proceed by comparing coefficients 2-jet:

$$\begin{aligned} \widehat{f}(x,y) &= a_1x + a_2x^2 + a_3xy \mod \widehat{\mathfrak{m}}(2)^3, \\ \widehat{y}(x,y) &= y + c_1x + c_2x^2 + c_3xy \mod \widehat{\mathfrak{m}}(2)^3, \end{aligned}$$

and

$$\widehat{y}(x,\overline{y}) = \overline{y} - c_1x - (c_2 - c_1c_3)^2x - x_3x\overline{y} \mod \widehat{\mathfrak{m}}(2)^3,$$

consequently,

$$\widehat{f_T}(x,\overline{y}) = \widehat{f}(x, \widehat{y}(x,\overline{y})) = f_1x + (f_2 - f_3c_1)x^2 + f_3x\overline{y} \mod \widehat{\mathfrak{m}}(2)^3,$$

where f_3 is different from zero by assumption.

Notice that in new coordinates

$$\overline{y}^3 = (y-\Theta)^3 = y^3 - 3y^2\Theta + 3y\Theta^2 - \Theta^3 = \Phi + \Psi y + 3\Theta y^2$$

$$-3y^2\Theta + 3y\Theta^2 - \Theta^3 = (\Psi + 3\Theta^2)(y - \Theta) + (\Phi + 2\cdot\Theta^3 + \Psi\cdot\Theta)$$

$$= \Psi_1\overline{y} + \Phi,$$

where Ψ_1 and Φ are certain germs. This shows that we could assume from the very beginning that $\Theta = 0$, i.e.,

$$y^3 = \Phi(x, f_2) + \Psi(x, f_2)y; \quad \Phi(0,0) = \Psi(0,0) = 0.$$

Now change the coordinates in preimage $\{(x_1,x_2)\}$ and in image $\{z_1,z_2)\}$ by formulae

$$(A) \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \Psi(x_1, f(x_1,x_2)) \\ x_2 \end{pmatrix},$$

$$(B) \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} \Psi(z_1,z_2) \\ \Phi(z_1,z_2) \end{pmatrix}.$$

First of all we must check, that this formulae really give admissible change of coordinates. Let's do several calculations in 3-jets.

$$\widehat{f_2}(x_1,x_2) = a_1x_1 + a_2x_1^2 + a_3x_1x_2 + a_4x_2^3 + x_1\cdot 0(2) \mod \widehat{\mathfrak{m}}^4,$$

where $a_3, a_4 \neq 0$ (note $O(2)$ is a zero of order 2.

$$\begin{aligned} &\widehat{\Phi}(x,z) = b_1x_1 + b_2z \mod \widehat{\mathfrak{m}}^2, \\ &\widetilde{\Psi}(x_1,z) = c_1x_1 + c_1z \mod \widehat{\mathfrak{m}}^2, \\ (C)\quad &x_2^3 = \Phi(x_1, \widehat{f}(x_1,x_2)) + \widehat{\Psi}(x_1, \widehat{f}(x_1,x_2))x_2. \end{aligned}$$

In order to change (A) we must show that $c_1+c_2a_1 \neq 0$ and for changing (B)

$$\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \neq 0.$$

From formula (C) which represents equality between power series follows that $b_2 \neq 0$, because b_2a_4 is coefficient of x_2^3 in right part. On the other hand, coefficient by x_1x_2, namely $(c_1 + c_2a_1) + b_2a_3$, equals zero, and as $a_3 \neq 0$, $b_2 \neq 0$, we have $c_1 + c_2a_1 \neq 0$. Coefficient to x_1 equals $b_1 + b_2a_1$, i.e., zero, so

$$b_1c_2 - b_2c_1 = -b_2(a_1c_2 + c_1) \neq 0.$$

For completing the proof we must check commutativeness of the following diagram:

$$\begin{array}{ccc} (x_1, x_2) & \overset{(A)}{\longmapsto} & \big(\Psi(x_1, f_2(x_1, x_2)), x_2\big) = (\overline{x}, \overline{z}) \\ \downarrow f & & \downarrow \\ (x, f_2(x_1, x_2)) & \overset{(B)}{\longmapsto} & \big(\Psi(x_1, f_2(x_1, x_2)), \Phi(x_1, f_2(x_1, x_2))\big) = (\overline{x}, \overline{z}^3 - \overline{x}, \overline{z}) \end{array}$$

which is equivalent to our equation

$$x_2^3 - \Psi(x_1, f_2(x_1, x_2))x_2 = \Phi(x_1, f_2(x_1, x_2)).$$

In order to receive an effective criterion it is useful to use another more geometric form of definition of fold and cusp offered by H. Brodersen [4], which is equivalent to classical definition.

At first we introduce notations for more general case, when $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$, then we'll reduce these to the case when $n = 2$.

Let f be a stable map of rank 1 and with jet j^1f. $\sum^{n-1} \subset J^1(n, 2)$ ($n{-}1$ upper index denotes decrease of rank), then $\sum(f) = (j^1f)^{-1}(\sum^{n-1})$ is the smooth manifold of dimensional 1. So we can parametrise $\sum(f)$ is a smooth curve $\rho(t)$ such that $\rho'(0) = 0$.

Definition 1.2. A point 0 is called a point B-fold for f, if

$$\frac{d}{dt}(f(\rho(t)))\big|_{t=0} \neq 0$$

and B-cusp if

$$\frac{d}{dt}(f(\rho(t)))\big|_{t=0} = 0 \quad \text{and} \quad \frac{d^2}{dt^2}(f(\rho(t)))\big|_{t=0} \neq 0.$$

Now we'll find algebraic criterion of stability by means of Definition 1.2 and then show that by canonical form, given by splitting lemma, one may be easily distinguish folds and cusps. From this immediately follows that Definition 5.1 is equivalent to the Definition 1.2 and we come to criterion soon.

The first part we'll do for general case $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$.

Let $A = (A_1, \dots, A_n)$ be $(2 \times n)$ matrix. By the $J_i^1(A), \dots, J_i^{n-1}(A)$ denote (2×2) minors,where column A_i is absent, they are $n - 1$ pieces.

If A has rank 1 and $A_i \neq 0$, then it's obvious that system $J_i^1(A') = \cdots = J_i^{n-1}(A') = 0$, $A' \in \mathrm{Nom}(\mathbf{R}, \mathbf{R}^2)$ which defines neighbourhood of a submanifold $\sum^{n-1}$, constructed from the matrix of rank 1. It seems that they correspond to 1-jets from $\sum^{n-1}$.

Further, let f be a map of class C^∞. If for a point $p \in \mathbf{R}^n$, $J_i^k(Df(x)) = J_i^k(f(x))$, then it is obvious that $j^1 f$ in point p is equivalent to the map

$$J_i(f) = (J_i^1(f), \ldots, J_i^{n-1}(f)) : \mathbf{R}^n \to \mathbf{R}^{n-1}$$

is regular in point p.

Let's consider $n \times (n-1)$ matrix

$$M_i(f) = (\mathrm{grad}\, J_i^1(f), \ldots, \mathrm{grad}\, J_i^{n-1}(f)),$$

which in a neighbourhood of point p has rank $n-1$.

Denote by $M_i^k(f)$ a $(n-1) \times (n-1)$ minor which is obtained from $M_i(f)$, when the line with number k is deleted.

Let's consider also a vector-function defined in the following way:

$$T_i(f) = (-1)^n \begin{pmatrix} -M_i(f) \\ \vdots \\ (-1)^k M_i^k(f) \\ \vdots \\ (-1)^n M_i^n(f) \end{pmatrix}.$$

It is easy to check that $T_i(f)$ is a non-zero vector of tangent space $T_x(\sum(f))$ in the point $x \in \sum(f)$ in a neighbourhood of point p.

Indeed, $J_i^1(f) = \cdots = J_i^{n-1}(f)$ defines $\sum(f)$ in the neighbourhood of point p and then

$$\mathrm{grad}\, J_t^1(f)(x), \ldots, \mathrm{grad}\, J_i^{n-1}(f)(x)$$

draws on normal space of the space $T_x(\sum(f))$ in the mentioned neighbourhood.

Further, we can assume that $M_i(f)$ in point p has rank $n-1$. Knowing that $T_i(f)(x) \neq 0$ in neighbourhood of p, it remains to check that $T_i(f)(x)$ is orthogonal to each $J_i^k(f)(x)$, $1 \leq k \leq n-1$.

Notice that in the considered $n \times n$ matrix the first $n-1$ columns equal to $\mathrm{grad}\, J_i^1(f), \ldots, \mathrm{grad}\, J_i^{n-1}(f)$ and the last one to the one of $\mathrm{grad}\, J_j^k(f)$, for some $1 \leq k \leq n-1$. As the column $\mathrm{grad}\, J_i^k(f)$ is repeated, the determinant of matrix vanishes but decomposition by the last column we get that determinant equals to the scalar product of $T_i(f)$ and $\mathrm{grad}\, J_i^k(f)$. So the scalar product equals to zero for every k, and vector $T_i(f)(x)$ lies into the space $T_x(\sum(f))$.

Finally, we get that for $x \in \sum(f)$ and for each i, $T_i(f)(x)$ is either zero, either non-zero vector in the space $T_x(\sum(f))$.

But $T_i(f)(x) \neq 0$ as there are $(n-1) \times (n-1)$ minors of matrices $M_i(f)$, whose rank equals to $n-1$ by assumption.

Denote also by $K_i(f)(x) = Df(x)T_i(f)(x)$.

Then we get the result: f in point 0 is fold $\Longleftrightarrow$

$$\sum_{i=1}^{n} |J_i(f)(0)|^2 = 0 \quad \text{and} \quad \sum_{i=1}^{n} \|K_i(f)(0)\| \neq 0,$$

and f in point 0 is a cusp $\Longleftrightarrow$

$$\sum_{i=1}^{n}(|J_i(f)(0)|^2 + \|K_i(f)(0)\|^2) = 0$$

and

$$\sum_{i=1}^{n} \|DK_i(f)(0)T_i(f)(0)\|^2 \neq 0.$$

Indeed, let $\rho(t)$ be a parametrization of $\sum(f)$ such that $\rho(0) \neq 0$. We have:

$$\frac{d}{dt} f(\rho(t)) = Df(\rho(t))\rho'(t).$$

From aforesaid it follows that $T_1(f)(\rho(t))$ is a non-zero tangent vector to the $\sum(f)$ in neighbourhood of zero, so we can write $\vec{\rho}'(t) = \beta(t)\vec{T_1}(t)\rho(t)$ for some $\beta(t)$, $\beta(0) \neq 0$.

From here it follows that

$$\frac{d}{dt} f(\rho(t)) = \beta(t)\big(K_1(f)(\rho(t))\big).$$

But as $\beta(0) \neq 0$, then

$$\frac{d}{dt} f(\rho(t))\big|_{t=0} = 0$$

iff $K_1(f)(\rho(0)) \neq 0$ and so for each i i.e., fold $\Longleftrightarrow$

$$\sum_{i=1}^{n} \|K_i(f)(0)\|^2 \neq 0.$$

Further,

$$\frac{d^2}{dt} f(\rho(t)) = \beta'(t)(K_1(f)(\rho(t)) + \beta(t)D(K_1(f)(\rho(t))\rho'(t).$$

Let us assume that f is not a fold, then we get that the first summand is zero, so

$$\frac{d^2}{dt} f(\rho(t))\big|_{t=0} = \beta(0)D(K_1(f)(0)\rho'(0) = \beta(0)^2 D(K_1(f)(0)T_1(f)(0).$$

Again as $\beta(0) = 0$, we get that f is a cusp iff

$$\sum_{i=1}^{n}(|J_i(f)(0)|^2 + \|K_i(f)(0)\|^2) = 0$$

and

$$\sum_{i=1}^{n} \|DK_i(f)(0)T_i(f)(0)\|^2 \neq 0.$$

In case where $n = 2$, $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$, there exists only one 2×2 minor of Jacobian $J(f)$. Consequently,

$$J_1(f) = J_2(f) = J(f),$$

$$M_1 = M_2 = \begin{pmatrix} \frac{d}{dx_1} J(f) \\ \frac{d}{dx_2} J(f) \end{pmatrix}, \quad T(f) = T_1(f) = T_2(f) = \begin{pmatrix} \frac{d}{dx_2} J(f) \\ \frac{d}{dx_1} J(f) \end{pmatrix}$$

and

$$K(f)(x) = K_1(f)(x) = K_2(f)(x) = Df(x)T(f)(x)$$

i.e.,

$$M_1 = M_2 = \text{grad}\, J = (J_{x_1}, J_{x_2}) \perp \sum(f),$$

where $\sum(f) = \{J = 0\}$, and

$$T(f) = T_1(f) = T_2(f) = (-J_{x_1}(x), J_{x_2}(x)) \in T_x(\sum(f)),$$

We can write exactly the same formulae with a little simplification

$$\vec{\rho}'(t) = \beta(t)\vec{T}(f)(\rho(t)),$$

for some $\beta(t)$, $\beta(0) \neq 0$; and also

$$\frac{d}{dt} f(\rho(t)) = \beta(t)(K(f)(\rho(t)))$$

and

$$\frac{d^2}{dt^2} f(\rho(t)) = \beta'(t)K(f)(\rho(t)) + \beta(t)DK(f)(\rho(t))\rho'(t).$$

And finally: a) fold $\iff J(f)(x) = 0$, $K(f)(x) \neq 0$.

b) cusp $\iff J(f)(x) = 0$, $K(f)(x) = 0$.

$$DK(f)(x)T(f)(x) \neq 0.$$

As a result, we get that points of cups are given by roots of the following system:

$$\begin{cases} J(f)(x) = 0, \\ -f'_{1x_1} J_y + f'_{1x_2} J_x = 0, \\ -f'_{2x_1} J_y + f'_{2x_2} J_x = 0. \end{cases} \tag{2}$$

Remark that analogious equations are obtained in [7]

Remark 1.1. Inequality in (a) and in (b) is accomplished automatically, as other singularities of stable germs by Whitney theorem don't exist.

Generally, a common zero of three functions exists quite seldom and it is non-stable. But in our case there is a dependence between equations, as from the first one follows proportionality of lines:

$$f'_{1x_1} \cdot f'_{2x_2} = f'_{2x_1} \cdot f'_{1x_2} \Longrightarrow \frac{f'_{1x_1}}{f'_{2x_2}} = \frac{f'_{1x_2}}{f'_{1x_2}} = c.$$

As we've already mentioned, for $n = m = 2$, Splitting Lemma gives that $\varphi(x, y) = (x, bxy + cy^2 + g(x, y))$, and in stable case $g(0, y) \in \mathfrak{m}^3 \backslash \mathfrak{m}^4$.

Here

$$D\varphi = \begin{pmatrix} 1 & 0 \\ by + g'_x(x,y) & bx + 2cy + g'_y(x,y) \end{pmatrix}$$

and singular set

$$\sum(\varphi) = \{bx + 2cy + g'_y(x,y) = 0\},$$

If $c \neq 0$ singular set $\sum(\varphi)$ has the following parametrization

$$\rho(t) : \begin{cases} x(t), \\ y = -\frac{b}{2c}t + \alpha(t), \end{cases}$$

where $\alpha(t) = o(t^2)$, is obtained by theorem on the implicit function.

It is obvious that $\rho'(0) \neq 0$ and as $g'_{x_1}(\rho(t)) = m(t^2)$ then we can write

$$\frac{d}{dt}(\varphi(\rho(t)))\big|_{t=0} = \begin{pmatrix} 1 & 0 \\ b\Big(-\dfrac{b}{2c}t + 0(t^2)\Big) + 0(t^2) & b + 2c\Big(-\dfrac{b}{2c} + 0(t)\Big) + 0(t^2) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -\dfrac{b}{2c} + 0(t) \end{pmatrix}\Bigg|_{t=0} \neq 0,$$

it means that this is a fold.

If $c = 0$, then b must not be 0. Then parametrization must have the following form

$$\rho(t) : \begin{cases} x = \beta(t), \\ y = t, \end{cases}$$

where $\beta(t) = m(t^2)$.

Indeed, our equation has the following form

$$bx + g'_y(x,y) = 0.$$

Here again $\rho'(0)$ and

$$\frac{d}{dt}(\varphi(\rho(t)))\big|_{t=0} = \begin{pmatrix} 1 & 0 \\ bt + g'_x(\rho(t)) & b0(t^2) + g'_y(\rho(t)) \end{pmatrix} \cdot \begin{pmatrix} 0(t) \\ 1 \end{pmatrix}\Bigg|_{t=0} = 0,$$

because of

$$\varphi(\rho(t)) = (0(t^n), 0(t^2)bt + g^{(3)}(0(t^2), t)), g^{(3)} \in \mathfrak{m}^3.$$

$$\frac{d}{dt}(\varphi(\rho(t)))\big|_{t=0} = (\text{conct}, 0(t)b + g^{(1)}(0(t^2), t))\big|_{t=0} \neq 0, \quad g^{(1)} \in \mathfrak{m}$$

is the cusp.

Thus the equivalence of Definition 1.2 and Theorem 1.1 is proved.

2. On number of cusps of polynomial mapping $f : \mathbf{R}^n \to \mathbf{R}^2$

In this chapter we'll describe local normal forms of stable mapping $\mathbf{R}^n \to \mathbf{R}^2$, originally obtained by H. Levine [6]. We'll show another proof of Levine's theorem, based on the Splitting Lemma. Also, according to Mather theorem on equivalence of stability to infinitesimal stability it's enough to find the normal forms of stable germs $\mathbf{R}^n \to \mathbf{R}^2$.

Let $f : \mathbf{R}^n \to \mathbf{R}^2$ be a germ of polynomial mapping which has rank 1. (It's easily seen that there doesn't exist stable mapping of rank 0, and in case of rank 2, it's regular point with normal form $(x_1, x_2) \to (x_1, x_2)$. Then by Implicit Function Theorem, mapping f can be reduced to

$$(x_1, x_2, \ldots, x_n) \to (x, \widetilde{f}(x_1, x_2, \ldots, x_n)).$$

As linear members in function $\widetilde{f}$ there are no, it can be represented as the sum of a quadratic form, and function $g \in \mathfrak{m}^3$, i.e.,

$$(x_1, x_2, \ldots, x_n) \longmapsto \Big(x_1, \sum_{i,j=1}^{n} a_{ij}x_i x_j + g(x_2, \ldots, x_n)\Big).$$

By changing coordinate in the image we can always remove a term with x_1^2. In fact, if $a_{1l} \neq 0$, we use the following change

$$\widetilde{y}_1 = y_1, \quad \widetilde{y}_2 = y_2 - a_{11}y_1^2.$$

So, without loss of generality we can assume that $a_{11}x_1^2$ in $\sum_{i,j=1}^{n} a_{ij}x_i x_j$, is absent. Consider this expression as quadratic form of variables $x_2, \ldots, x_n$, assume that x_1 is a parameter and denote by $k-1$ the rank

$$k - 1 = rk\,(a_{ij})_{i,j=2}^{n} \leq n - 1.$$

Now, we use parametric form of the Morse lemma. It will enable us to reduce to the normal type, i.e., to sums of squares of $x_2, \ldots, x_k$, taking into consideration other variables as parameters.

The function g won't depend on variables $x_2, \ldots, x_k$ and with this in the quadratic form we get mixed products like $x_1 x_j$, $j = \overline{2, k}$. So, after reducing to the normal form we can meet only the following mixed products

$$x_1 x_{k+1}, x_1 x_{k+2}, \ldots, x_1 x_n.$$

Taking x_1 out of brackets, we'll have

$$(x_1, x_2, \ldots, x_n) \longmapsto \Big(x_1, \sum_{j=2}^{k} \pm x_1^2 + x_1(a_{1,k+1}x_{k+1} + \cdots +$$
$$+ a_{1n}x_n) + g(x_1, x_{k+1}, \ldots, x_n)\Big).$$

There may be the following possibilities.

Case a) quadratic form doesn't depend on x_1, which is the same as $a_j = 0$, $j = \overline{k+1, n}$.

Then we have

$$(x_1, x_2, \ldots, x_n) \longmapsto \Big(x_1, \sum_{j=2}^{k} \pm x_j^2 + g(x_1, x_{k+1}, \ldots, x_n)\Big),$$

where $g \in \mathfrak{m}^3_{\mathfrak{n}-\mathfrak{k}+1}$ (lower index is the number of variables)

Case b): there exists $j \in [k+1, n] \cap \mathbb{Z}$ such that $a_{1j} \neq 0$. We can assume that $j = k+1$.

Now let's use the following change of coordinates:

$$\begin{aligned} x'_1 &= x_1 \\ &\cdots\cdots\cdots \\ x'_k &= x_k \\ x'_{k+1} &= a_{1,k+1}x_{k+1} + \cdots + a_{1n}x_n \\ x'_{k+2} &= x_{k+2} \\ &\cdots\cdots\cdots \\ x'_n &= x_n \end{aligned}$$

and the map becomes

$$(x_1, x_2, \ldots, x_n) \longmapsto \Big(x'_1, x'_1 x'_{k+1} + \sum_{j=2}^{k} \pm {x'_j}^2 + g(x'_1, x'_{k+1}, \ldots, x'_n)\Big),$$

$g \in \mathfrak{m}^3_{\mathfrak{n}-\mathfrak{k}+1}$. Consequently, we get full splitting of mapping f and the following result.

Theorem 2.1. *Let $f : (R^n, 0) \to (R^2, 0)$ be a germ of polynomial mapping of rank at the origin.*

Then either

a) $f(x_1, x_2, \ldots, x_n) = \Big(x_1, \sum_{j=2}^{k} \pm x_j^2 + g(x_1, x_{k+1}, \ldots, x_n)\Big),$

either

b) $f(x_1, x_2, \ldots, x_n) = \Big(x_1, x_1 x_{k+1} + \sum_{j=2}^{k} \pm x_j^2 + g(x_1, x_{k+1}, \ldots, x_n)\Big),$

where $g \in \mathfrak{m}^3_{\mathfrak{n}-\mathfrak{k}+1}$.

We'll show that Definition 1.2 is equivalent to the usual definition of fold and cusp for the case $\mathbf{R}^n \to \mathbf{R}^2$, which was given by Levin [6].

Theorem 2.2. *According to [6] each stable smooth germ $(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^2, 0)$ is equivalent to one of the following .*

I) $(x_1, x_2, \ldots, x_n) \longmapsto (x_1, x_2)$ *is of maximal rank and the point is regular,*

II) $(x_1, x_2, \ldots, x_n) \longmapsto \Big(x_1, \sum_{j=2}^{k} \pm x_j^2\Big)$ *is a fold,*

III) $(x_1, x_2, \ldots, x_n) \longmapsto \Big(x_1, \sum_{j=2}^{k} \pm x_j^2 + x_1 x_n + x_n^3\Big)$ *is a cusp.*

Consider case a) of previous theorem. Here it's clearly seen that stability is possible only when $k = n$.

Indeed, otherwise the type of the germ changes by adding εx^2_{k+1} . So, for a stable germ, case a) of Theorem 2.1 will be following.

a′) $(x_1, x_2, \ldots, x_n) \longmapsto \Big(x_1, \sum_{j=2}^{k} \pm x_j^2 + g(x_1)\Big).$

It's obvious that with a coordinates change in image we can delete $g(x_1)$: $\ddot{y}_1 = y_1$, $\ddot{y}_2 = y_2 - g(y_1)$, so in this case we have exactly the Levine fold. Let us show that in this case we also have the B-fold by Definition 1.2. Let's do this for $n = 2$.

We have

$$(x_1, x_2, \ldots, x_n) \overset{f}{\longmapsto} \Big(x_1, \sum_{j=2}^{k} \pm x_1^2 + g(x_1)\Big).$$

So

$$Df = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 3x_1^2 + 0(x_1) & 2x_2 & 2x_+3 & \cdots & 2x_n \end{pmatrix},$$

and the singular set is

$$\sum(f) = \{2x_2 = 2x_3 = \cdots = 2x_n = 0\} \Longrightarrow \sum(f) = 0x_1\}$$

The following parametrization $\rho(t)$ can be used:

$$x_1 = t, \quad x_2 = \cdots = x_n = 0$$

Calculating the derivative

$$\frac{d}{dt} f(\rho(t))\Big|_{t=0} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 3t^2 + 0(t) & 0 & \cdots & 0 \end{pmatrix} \left.\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right|_{t=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0.$$

We see that it is a B-fold by Definition 1.2.

Case b): Here it's also obvious that stable germs can only be when $k = n - 1$ (otherwise we add εx_{k+2}^2). In this case our mapping splits in $\sum\limits_{i=2}^{n-1} \pm x_i^2$ and a mapping in plane (x_1, x_n). As it's known, this kind of sum is stable if both addends are stable, so we must check the stability of only for the second summand

$$(x_1, x_n) \longmapsto (x_1, x_1 x_n + g(x_1, x_n).$$

As for $n = 2$, we can make sure that

$$\frac{\partial^3}{\partial x_n^3} g(0, x_n)\Big|_{x_n=0} \neq 0,$$

Otherwise there is no stability (with adding εx_n^3 we get a small perturbation which isn't equivalent to $x_1 x_n + g(x_1, x_n)$).

Now, with coordinate change described in 5, the second function can be transformed into $x_1 x_n + x_n^3$.

At the end, having the stability, the normal form of mapping will be following

$$(x_1, x_2, \ldots, x_n) \longmapsto \Big(x_1, x_1 x_n + \sum_{j=2}^{n-1} \pm x_j^2 + x_n^3\Big)$$

and it is a cusp by Levine.

This shows that it is B-cusp by Definition 1.2.

When $k = n-1$ b) takes the form

$$b')\quad (x_1, x_2, \ldots, x_n) \xrightarrow{f} \Big(x_1, x_1x_n + \sum_{j=2}^{n-1} \pm x_i^2 + g(x_1, x_n)\Big)$$

then we'll do everything as in case a).

$$Df = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ x_n + 3x_1^2 + o(x_1) & 2x_2 & \cdots & x_1 + 3x_n^2 + o(x_n) \end{pmatrix},$$

$$\sum(f) = \{2x_2 = \cdots = x_1 + 2x_n^2 + 0(x_n) = 0\}.$$

Further, the following parametrization

$$\rho(t) : \begin{cases} x_1 = -3t^2 + o(t) \\ x_2 = \cdots = x_{n-1} = 0 \\ x_n = t \end{cases}$$

Then

$$\frac{d}{dt} f(\rho(t))\Big|_{t=0} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ t^2 + 9t^4 + o(t) & 0 & \cdots & o(t) \end{pmatrix} \cdot \begin{pmatrix} -6t + o(t) \\ 0 \\ \vdots \\ 1 \end{pmatrix}\Bigg|_{t=0} = 0.$$

Let's calculate the second derivative

$$f(\rho(t)) = \big(-3t^2 + o(t), -3t^3 + o(t^2) + g^{(3)}(-3t^2 + o(t), t)\big), \quad g^{(3)} \in \mathfrak{m}^3,$$

$$\frac{d^2}{dt^2} f(\rho(t))\Big|_{t=0} = \big(\text{conct}, -18t + o(t) + g^{(1)}(-3t^2 + o(t), t)\Big|_{t=0} \neq 0.$$

So by Definition 1.2 we get a B-cusp.

Thus, the Levine definition and Definition 1.2 are equivalent and we get this equations:

for folds: $J_1(f)(0) = J_2(f)(0) = \cdots = J_n(f)(0) = 0$,

for cusps: $\begin{cases} J_1(f)(0) = J_2(f)(0) = \cdots = J_n(f)(0) = 0, \\ K_1(f)(0) = K_2(f)(0) = \cdots = K_n(f)(0) = 0. \end{cases}$

As to the inequality we see that not all summands will be zeroes at the same time, i.e., $\exists i \in [1, n] \cap \mathbb{Z}$ is such that $DK_i(f)(0) \cdot T_i(f)(0) > 0$ either < 0.

And this can be expressed by condition of type $h_i > 0$ or < 0, or by any condition defined by combinatorics of "+" and "−". As we've already seen in 2 it can be reduced on the solution of system of linear equation for the number of roots emerging in crossing.

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