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ON ONE WAY OF SOLUTION OF THE BASIC MIXED PLANE BOUNDARY VALUE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE

Svanadze K.

Abstract. Applying the representation of the stress vector by the so-called mutually associated vector-functions we solve the basic mixed plane boundary value problem of statics of the linear theory of elastics mixture for a simply connected finite domain in the case, when the displacement vector is given on one part of the boundary and the stress vector on the remaining part.

On the basis of the generalized Kolosov-Muskhelishvilis formulas the displacement vector and so-called its adjoint vector are represented by means of potentials with complex densities. As a result, our problem is reduced to a system of singular integral equations with discontinuous coefficients of special kind. The solvability of the system in a certain class is proved, which implies that the basic plane boundary value problem has a unique solution and the solution of the problem is given in the form of a generalized double layer potential.

Keywords and phrases: Basic mixed boundary value problem, elastic mixture theory, equation of statics analogues of the general Kolosov-Muskhelishvilis representations, equations singular integral equations with discontinuous ciefficients, discontiunty point mutually adjoint vector-feunctions.

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1. Introduction

The construction and the intensive investigation of the mathematical models of elastic mixtures arise by the wide use composites into practice. The diffusion and shift models of the linear theory of elastic mixtures are presented by several authors.

[1, 3] for a simply connected finite and infinite domain the basic plane boundary value problems of statics of the elastic mixture are considered when on the boundary a displacement vector (the first problem) a stress vector (the second problem) differences of partial displacements and the sum of strees vector components are given.

In [3] using potentials with complex densities the solutions of basic plane boundary value problems are reduced to solution of Fredholm linear integral equation of the second kind.

In [4] by the method of N. Muskhelishvili developed in [6] the basic mixed plane boundary value problem of statics of the theory of elastic mixture for a simply connected finite domain is solved, when the displacement vector is given on one partl of the boundary and the stress vector on the remeining part.

In the present work in a somewhat different way we study the problem which is considered in [4]. For the solution of the problem we use the generalized formulas due to Kolosov-Muskhelishvili [3] and the method described in the works of M. Basheleishvili [2]. Further, note that to solve the problem we also use some result, obtained in [4].

2. Some auxiliary formulas and operators

The homogeneous equation of statics of the linear theory of elastic mixtures in a complex form has the form [3]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \tag{2.1}$$

where $U = (u_1 + iu_2, u_3 + iu_4)^T$, $u' = (u_1, u_2)^T$, and $u'' = (u_3, u_4)^T$, are partial displacements,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right),$$

$$K = -\frac{1}{2} l m^{-1}, \quad l = \begin{bmatrix} l_4 & l_5 \\ l_5 & l_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix}, \quad \Delta_0 = m_1 m_3 - m_2^2,$$

$$m_k = l_k + \frac{1}{2} l_{3+k}, \quad k = 1,2,3, \quad l_1 = \frac{a_2}{d_2}, \quad l_2 = -\frac{c}{d_2}, \quad l_3 = \frac{a_1}{d_2},$$

$$a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5, \quad d_2 = a_1 a_2 - c^2, \quad l_1 + l_4 = \frac{b}{d_1},$$

$$l_2 + l_5 = -\frac{c_0}{d_1}, \quad l_3 + l_6 = \frac{a}{d_1}, \quad a = a_1 + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d, \quad d_1 = ab - c_0^2,$$

$$b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2 \frac{\rho_2}{\rho}, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2 \frac{\rho_1}{\rho}, \quad \rho = \rho_1 + \rho_2,$$

$$\alpha_2 = \lambda_3 - \lambda_4, \quad d = \mu_3 + \lambda_3 - \lambda_5 - \alpha_2 \frac{\rho_1}{\rho} \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2 \frac{\rho_2}{\rho}.$$

Here μ_1, μ_2, μ_3 *and* λ_p, $p = \overline{1,5}$ are elastic modules, characterizing mechanical properties of the mixture, ρ_1 and ρ_2 are particular densities. The elastic constants $\mu_1, \mu_2, \mu_3, \lambda_p, p = \overline{1,5}$ and particular densities ρ_1 and ρ_2 will be assumed to satisfy the conditions of inequality [1].

In [3] M. Basheleishvili obtained the following representations (Kolosov- Muskhelishvili type formulas)

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} = m\varphi(z) + \frac{1}{2}\, lz\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2.2}$$

$$TU = \begin{pmatrix} (Tu)_2 - i(Tu)_1 \\ (Tu)_4 - i(Tu)_3 \end{pmatrix} = \frac{\partial}{\partial S(x)}\left[-2\varphi(z) + 2\mu U(x)\right]$$

$$= \frac{\partial}{\partial S(x)}[(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}], \tag{2.3}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions, $\frac{\partial}{\partial S(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}$ $\quad n = (n_1, n_2)^T$ is an arbitrary unit vector, E is the unit matrix,

$$A=\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}=2\mu m,\ B=\begin{bmatrix}B_1 & B_2\\ B_3 & B_4\end{bmatrix}=\mu e,\ \mu=\begin{bmatrix}\mu_1 & \mu_3\\ \mu_3 & \mu_2\end{bmatrix},\ m=\begin{bmatrix}m_1 & m_3\\ m_3 & m_2\end{bmatrix},$$

$$\Delta_0=detm>0,\qquad \Delta_1=det\mu>0,\quad \det A=4\Delta_0\Delta_1>0;\quad \Delta_2=\det(A-2E)>0,$$

$$(Tu)_p,\quad p=\overline{1,4}\quad are\quad the\quad components\quad of\quad stresses[1].$$

$$(Tu)_1=(a\theta'+c_0\theta'')n_1-(a_1\omega'+c\omega'')n_2-2\frac{\partial}{\partial S(x)}(\mu_1u_2+\mu_3u_4,)$$

$$(Tu)_2=(a\theta'+c_0\theta'')n_2+(a_1\omega'+c\omega'')n_1+2\frac{\partial}{\partial S(x)}(\mu_1u_1+\mu_3u_3),$$

$$(Tu)_3=(c_0\theta'+b\theta'')n_1-(c\omega'+a_2\omega'')n_2-2\frac{\partial}{\partial S(x)}(\mu_3u_2+\mu_2u_4),$$

$$(Tu)_4=(c_0\theta'+b\theta'')n_2+(c\omega'+a_2\omega'')n_1+2\frac{\partial}{\partial S(x)}(\mu_3u_1+\mu_2u_3),$$

$$\theta'=divu^{'},\quad \theta''=divu^{''}\quad \omega'=rotu^{'},\quad \omega''=rotu^{''}.$$

To investigate the basic mixed boundary value problem use will made of the following vectors [3]

$$V=\begin{pmatrix}v_1+iv_2\\ v_3+iv_4\end{pmatrix}=i[-m\varphi(z)+\frac{1}{2}lz\overline{\varphi'(z)}+\overline{\psi(z)}],\tag{2.4}$$

$$NU=\begin{bmatrix}(Nu)_2-i(Nu)_1\\ (Nu)_4-i(Nu)_3\end{bmatrix}=\frac{\partial}{\partial S(x)}[-2\varphi(z)+m^{-1}U(x)],$$

$$NV=\begin{bmatrix}(Nv)_2-i(Nv)_1\\ (Nv)_4-i(Nv)_3\end{bmatrix}=i\frac{\partial}{\partial S(x)}[2\varphi(z)-im^{-1}V(x)],$$

where N is the pseudostress operator.

It is not difficult to prove that (2.4) satisfies (2.1); moreover,

$$U(x)+iV(x)=2m\varphi(z),$$

$$NU=-im^{-1}\frac{\partial V}{\partial S(x)},\qquad NV=im^{-1}\frac{\partial U}{\partial S(x)},\tag{2.5}$$

$$TU=(2\mu-m^{-1})\frac{\partial U}{\partial S(x)}+NU=(2\mu-m^{-1})\frac{\partial U}{\partial S(x)}-im^{-1}\frac{\partial V}{\partial S(x)}.\tag{2.6}$$

Here we introduce the following

Definition 2.1. If U and V satisfy the relations (2.5) then they are mutually adjointed (associated).

Let $D_0^+(D_0^-)$ be a finite (infinite) two-dimensional domain, bounded by the contours $S_0\in C^{2,\beta}$, $\quad 0<\beta<1$, $\quad \overline{D_0^+}=D_0^+\bigcup S_0$, $\quad D_0^-=E_2\setminus\overline{D_0^+}\quad \overline{D_0^-}=D_0^-\bigcup S_0$, $\quad$ where E_2 is a two-dimensional Euclidean space

A vector $U = (u_1 + iu_2, u_3 + iu_4)$ defined in the domain $D_0^\pm$ is called regular if $U \in C^2(D_0^\pm) \bigcap C^1(\overline{D^\pm}_0)$ also in the case of the domain D_0^- we assume in addition the following conditions at infinity

$$U = O(1), \quad |x|^2 \frac{\partial U}{\partial x_j} = O(1), \quad j = 1, 2,$$

to be fulfilled with $|x|^2 = x_1^2 + x_2^2$.

Note that if $U = (u_1 + iu_2, u_3 + iu_4)^T$ is a regular solution of equation (2.1) in the domain $D_0^+(D_0^-)$ then (see [3])

$$\int_{D_0^\pm} T(u,u)dy = \pm Im \int_{S_0'} U^\pm (\overline{TU})^\pm ds, \qquad (2.7)^\pm$$

$$\int_{D_0^\pm} N(u,u)dy = \pm Im \int_{S_0'} U^\pm (\overline{NU})^\pm ds = \pm Im \int_{S_0'} V^\pm (\overline{NV})^\pm ds, \qquad (2.8)^\pm$$

where $T(u,u)$ and $N(u,u)$ defined in [1] (see pp 5-6) are positive definite quadratic forms. The formulas $(2.7)^\pm$ and $(2.8)^\pm$ will be called generalized Greens formulas in the theory of elastic mixtures for the equation of statics (2.1).

We have the following

Lemma 2.1. *The solutions of the equations $T(u,u) = 0$ and $N(u,u) = 0$ have, respectively, the form*

$$U = a^0 + i\varepsilon \begin{pmatrix} 1 \\ 1 \end{pmatrix} z, \qquad a^0 = (a_1^0, a_2^0)^T, \qquad (2.9)$$

$$U = C^0; \qquad C^0 = (C_1^0, C_2^0)^T, \qquad (2.10)$$

where a^0 and C^0 are an arbitrary constant complex vectors, and ε is an arbitrary real constant.

3. Formulation of the problem

Let us assume that an elastic mixture occupies the finite simply connected domain D^+, bounded by the closed curve S of Holder curvature.

Let the boundary S be divided into $2p$ arcs S_j, $\quad j = \overline{1, 2p}$, with edges C_j and C_{j+1} $\quad (C_{2p+1} \equiv C_1)$ whose pocitive directions coinside woth the positive direcction on S learing the domain D^+ on the left. We introduce the notation

$$S' = \bigcup_{j=1}^{p} S_{2j-1}, \quad S'' = \bigcup_{j=1}^{p} S_{2j}, \quad D^- = E_2 \setminus (D^+ \cup S),$$

where E_2 is a two -dimensional Eclidean space.

Definition 3.1. The vector $U = (u_1 + iu_2, u_3 + iu_4)^T$ is called regular if

$$(i) \qquad U \in C^2(D^+) \cap C^1(\overline{D^+})$$

$$(ii) \qquad TU = [(Tu)_2 - i(Tu)_1, (Tu)_4 - i(Tu)_3]^T$$

is continously extendable at every point of S from D^+ except perhaps the points C_j, $\quad j = \overline{1, 2p}$,

(iii) near the points C_j, $\quad j = \overline{1, 2p}$,

$$|(Tu)_q| < const|x - c_j|^{-\beta}, \quad 0 \le \beta < 1, \quad q = \overline{1,4}, \quad x = (x_1, x_2) \in D^+.$$

We consider the following problem. Find in D^+ a regular solution $U = (u_1 + iu_2, u_3 + iu_4)^T$ of equation (2.1) which satisfies the conditions

$$U^+(t) = f(t), \quad t \in S',$$
$$(TU(t))^+ = F(t), \quad t \in S'', \tag{3.1}$$

where $f = (f_1, f_2)^T$ and $F = (F_1, F_2)^T$ are given vector-functions, satisfying the conditions

$$f \in H, \quad \frac{\partial f}{\partial S} \in H^*, \quad \frac{\partial F}{\partial S} \in H^*. \tag{3.2}$$

The definition of the classes H and H^* can be found in [6]

Using the Green formula $(2.7)^+$ it is easy to prove (see Lemma 2.1.)

Theorem 3.1. *In the class of regular vectors, the homogeneous mixed plane boundary value problem* $(f = F = 0)$ *admits a trivial solution only.*

4. Reducing the basic mixed plane. Boundary value problem to integral equations

In view of (2.6) the boundary conditions of the mixed problem can be written as

$$2\mu U^+(t) = \phi(t), \quad t \in S', \quad (2\mu - m^{-1})U^+(t) - im^{-1}V(t) = \phi(t) + \Gamma(t), \quad t \in S'', \tag{4.1}$$

$$\phi(t) = \begin{cases} 2\mu f(t), & t \in S', \\ F^0(t), & t \in S'', \end{cases} \tag{4.2}$$

$$F_0(t) = \int_{C_{2j}}^{S(t)} F dS \qquad t \in S_{2j}, \quad j = \overline{1,p}, \tag{4.3}$$

the integrals are taken along the arcs S_{2j}, $\Gamma(t)$ is a piecewise constant vector on S'' i.e.

$$\Gamma(t) = \Gamma^{(j)} = (\Gamma_1^{(j)}, \Gamma_2^{(j)})^\Gamma, \quad for \quad t \in S_{2j}, j = \overline{1,p},$$

$\Gamma_1^{(j)}$ and $\Gamma_2^{(j)}$ are arbitrary complex constants. The $\Gamma_1^{(j)}$ and $\Gamma_2^{(j)}$ are not given in advance and defined while solving the problem.

The vector-function $\phi(z)$ will be assumed to belong to the class H_0, and $\frac{\partial \phi(t)}{\partial S(t)}$ to the class H^* with the nods C_j, $j = \overline{1,2p}$. Definition of the classes H_0 and H^* can be found in Muskhelishvili's monograph [6].

Now first we are to write a system of singular integral equations for the mixed boundary value problem. Using formulas (2.2), (2.4) and choosing $\phi(z)$ and $\psi(z)$ in the from

$$\varphi(z) = \frac{1}{2\pi i}\int_S \frac{\partial ln(z-\varsigma)}{\partial S(y)} g(y)d_y S,$$

$$\psi(z) = -\frac{m}{2\pi i}\int_S \frac{\partial ln(\overline{z}-\overline{\varsigma})}{\partial S(y)} g(y)d_y S - \frac{l}{4\pi i}\int_S \frac{\partial}{\partial S(y)} \frac{\zeta}{\overline{z}-\overline{\zeta}}\overline{g(y)}dgS$$

where $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$, $y = (y_1, y_2) \in S$ and $g = (g_1, g_2)^T$ are unknown complex vector-functions.

After a simple transformation we get

$$U(x,g) = \frac{m}{\pi}\int_S \frac{\partial ln|z-\varsigma|}{\partial S(y)} g(y)d_y S - \frac{l}{4\pi i}\int_S \frac{\partial}{\partial S(y)} \frac{z-\varsigma}{\overline{z}-\overline{\varsigma}}\overline{g(y)}d_y S, \tag{4.4}$$

$$V(x,g) = -\frac{m}{\pi}\int_S \frac{\partial ln|z-\varsigma|}{\partial S(y)} g(y)d_y S - \frac{l}{4\pi}\int_S \frac{\partial}{\partial S(y)} \frac{z-\varsigma}{\overline{z}-\overline{\varsigma}}\overline{g(y)}d_y S. \tag{4.5}$$

Note that (4.4) and (4.5) are the mutually associated potentials

Taking into account boundary properties of the potentials $U(x, g)$ and $V(x, g)$ as well as the boundary contitions (4.2) after simple transformations for determining $g = (g_1, g_2)^T$ we obtain the system of singular integral equations with discontinuous coefficients

$$Kg = a(t)g(t) + \frac{b(t)}{\pi i}\int_S \frac{g(\tau)d\tau}{\tau - t} + \int_S K_1(t,\zeta)g(y)d_y S +$$
$$+\int_S K_2(t,\zeta)\overline{g(y)}d_y S = \phi(t) + \Gamma(t), t \in S, \tag{4.6}$$

where $t = t_1 + it_2$, $\tau = \tau_1 + i\tau_2$, $\phi(t) = [\phi_1(t), \phi_2(t)]^T$ is defined by the formulas (4.2) and (4.3),

$$a(t) = \begin{cases} A, & t \in S', \\ A - E, & t \in S'', \end{cases} \qquad b(t) = \begin{cases} 0, & t \in S', \\ -E, & t \in S'', \end{cases} \tag{4.7}$$

$$K_1(t,\zeta) = \frac{1}{\pi}A\frac{\partial ln|t-\zeta|}{\partial n(y)},$$

$$K_2(t,\zeta) = -\frac{1}{2\pi i}B\frac{\partial}{\partial S(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}},$$

$$\Gamma(t) = \begin{cases} 0, & t \in S', \\ (\Gamma_1^{(j)}, \Gamma_2^{(j)})^T, & t \in S_{2j}, \quad j = \overline{1,p}, \end{cases}$$

as has already been said, the constants $\Gamma_1^{(j)}$ and $\Gamma_2^{(i)}$ are not given in advance and deffined while solving the problem. Meanwhile we assume the vector $\Gamma(t) \neq 0$ be given. It is obvious that $\phi(t) \in H_0$ and $\frac{\partial \phi(t)}{\partial S(t)} \in H^*$ at the nodes C_j, $j = \overline{1, 2p}$.

5. Investigation of the system of singular integral equations

1^0. The system of singular integral equations with discontinuous coefficients (4.6) is of special kind. It contains g and $\overline{g}$, $\overline{g}$ being only outside the characteristic part.

The general theory of such systems was developed on [6]. By this theory system (4.6) is of normal kind since (see [4])

$$\det[a(t)+b(t)] = \begin{cases} \det A > 0, \quad t \in S', \\ \det(A-2E) = \Delta_2 > 0, \quad t \in S'', \end{cases}$$

$$\det[a(t)-b(t)] = \det A > 0, \quad t \in S.$$

The characteristic homogeneous equation, corresponding to (4.6), is

$$K^0 g \equiv a(t)g(t) + \frac{b(t)}{\pi i}\int_S \frac{g(\tau)d\tau}{\tau - t} = 0$$

and the corresponding to it homogeneos problem of conjugation is of the form

$$\Psi^+(t) = G(t)\Psi^-(t), \qquad \Psi = (\Psi_1, \Psi_2)^T,$$

where (see (4.7))

$$G(t) = [a(t)+b(t)]^{-1}[a(t)-b(t)] = \begin{cases} E, \quad t \in S', \\ (A-2E)^{-1}A, \quad t \in S'' =, \end{cases}$$

$$= \begin{cases} E, \quad t \in S', \\ \frac{1}{\Delta_2}(4\Delta_0\Delta_1 E - 2A), \quad t \in S'', \end{cases} \tag{5.1}$$

Let us now determine which of discontinuity points $C_j, \quad j = \overline{1, 2p}$ of the matrices $a(t)$ and $b(t)$ are singular and which are nonsingular. To this end define the roots of the equation

$$\det[G^{-1}(\eta+0)G(\eta-0) - \sigma E] = 0,$$

at the discontinuity points $C_j, \quad j = \overline{1, 2p}$.

After simple calculations, from (5.1) we obtain (see [4])

$$\det[G^{-1}(C_{2j-1}+0)G(C_{2j-1}-0) - \sigma E)] =$$

$$\sigma^2 - \frac{2}{\Delta_2}(4\Delta_0\Delta_1 - A_1 - A_4)\sigma + \frac{4\Delta_0\Delta_1}{\Delta_2} = 0, \quad j = \overline{1, p}, \tag{5.2}$$

$$\det[G^{-1}(C_{2j}+0)G(C_{2j}-0) - \sigma E] =$$

$$\sigma^2 - \frac{2}{4\Delta_0\Delta_1}(4\Delta_0\Delta_1 - A_1 - A_4)\sigma + \frac{\Delta_2}{4\Delta_0\Delta_1} = 0, \quad j = \overline{1, p}. \tag{5.3}$$

Note that the roots of the characteristic equations (5.2) and (5.3) are resiprocal and negative [4]

Denote the roots of (5.2) by σ_1 and σ_2. Then the roots of (5.3) are $\frac{1}{\sigma_1}$ and $\frac{1}{\sigma_2}$ respectively

Let the numbers

$$\nu_1' = \frac{1}{2\pi i} ln\sigma_1 = \frac{1}{2} + \frac{1}{2\pi i} ln|\sigma_1| = \frac{1}{2} - i\beta, \quad \nu_2' = \frac{1}{2\pi i} ln\sigma_2 = \frac{1}{2} - i\beta''$$

correspond to all the points $C_{2j}, \quad j=\overline{1,p}$ and the numbers

$$\nu_1''=\frac{1}{2\pi i}ln\frac{1}{\sigma_1}=\frac{1}{2}+i\beta', \quad \nu_2''=\frac{1}{2\pi i}ln\frac{1}{\sigma_2}=\frac{1}{2}+i\beta''$$

correspond to all the points $C_{2j-1}, \quad j=\overline{1,p}$.

Since $Re\nu_1'=Re\nu_2'=Re\nu_1''=Re\nu_2''=\frac{1}{2}$, therefore all discontinuity points $C_j, \quad j=\overline{1,2p}$ of the coefficitns of the system (4.6) are nonsingular. [4].

The total index of the class $h_{2p}=(h_1,h_2,h_3,...h_p)$ of system (4.6) is equal to $(-2p)$

The solution $g(t)$ of the equation (4.6) will be sought in the class h_{2p}. Therefore if we take into account the restrictions, imposed both on the contour S and on the boundary vector-function ϕ, then repeating word by word the reasoning given in [5], we can show that $g(t)$ belongs to the Holder class on the entire contour S and $\frac{\partial g}{\partial S}$ belongs to the class H^*.

Since the equation (4.6) containing $g=(g_1,g_2)^T$ and $\overline{g}=(\overline{g_1},\overline{g_2})^T$ (where $\overline{g}$ lies only outside the characteristic part) therefore we have [6]

$$\nu-\nu'=-4p \tag{5.4}$$

where ν is the number of independent solutions of the class h_{2p} of the homogeneous equation $Kg=0$, and ν' is the number of independent solutions of the class $h_0=h_{2p}'$ of the associated homogeneous equation $K'\omega=0$.

2^0. Let us prove that $\nu=0$ and hence $\nu'=4p$. Indeed let $g_0(t)$ be any solution of the class h_{2p} of the associated homogeneous equation $Kg=0$. By g_0 we construct the mutually associated potentials $U_0(x,g_0)$ and $V_0(x,g_0)$ (see (4.4) and (4.5.)). Then $[U_0(t,g_0)]_+=0$ for $t\in S'$ and $[TU_0(t_1g_0)]^+=0$ for $t\in S''$, except maybe the points $C_j, \quad j=\overline{1,2p}$ (see (3.1)) at which the stress vector may have only integrable singularities.

Using the uniqueness theorem for the mixed boundary value problem (see Theorem 3.1.) we conclude that $U_0(x,g_0)=0, \quad x\in D^+$ In the case (see (2.5))

$$NU_0(x,g_0)=-im^{-1}\frac{\partial V_0(x,g_0)}{\partial S(x)}=0 \quad x\in D^+$$

and applying the equality (4.5) we have

$$[NU_0(t,g_0)]^+=[NU_0(t,g_0)]^-, \quad t\in S,$$

which takes place at all points of the contour S except possibly for the points $C_j, \quad j=\overline{1,2p}$, we obtain

$$[NU_0(t,g_0)]^-=0, \quad t\in S, \tag{5.5}$$

Since $U_0(x,g_0)$ satisfies the decrease conditions at infinity as well as the boundary condition (5.5.) using formula $(2.8)^-$ and Lemma 2.1. (see (2.10)) we conclude that $U_0(x,g_0)=0, \quad x\in D^-$.

The formula $U_0^+(x,g_0)-U_0^-(x,g_0)=2mg_0(t), \quad t\in S$, implies $g_0(t)=0$ which contradicts our assumption.

Thus the homogeneous equation $Kg=0$ has in the class h_{2p} only the trivial solution. Hence as above $\nu'=4p$ and the general theory (see [§6, 112)] the conditions for the solvability of (4.6.) have in the class h_{2p} the form

$$Re\int_S[\phi(t)+\Gamma(t)]\omega_j(t)dt=0,\quad j=\overline{1,4p},\tag{5.6}$$

where $\omega_j(t),\quad j=\overline{1,4p}$ is a complete system of linearly independent solutions of the class $h_0=h'_{2p}$ of the associated homogeneous system $K'\omega(t)=0$.

Assuming that in (5.6)

$$\Gamma(t)=\Gamma^{(j)}=(\Gamma_1^{(j)},\Gamma_2^{(j)})^T=(\gamma_j+i\gamma_{p+j},\gamma_{2p+j}+i\gamma_{3p+j})^T,t\in S_{2j}\quad j=\overline{1,p},$$

where $\gamma_j,\quad j=\overline{1,4p}$ are real constants to define the constants $\gamma_j,\quad j=\overline{1,4p}$ we obtain the following real system of linear algebraic equations:

$$\sum_{k=1}^{4p}M_{jk}\gamma_k=Q_j,\qquad j=\overline{1,4p},\tag{5.7}$$

where M_{jk} are the defined constants, not depending on $\phi(t)$ while Q_j are also constants but depending on $\phi(t)$

$$Q_j=-Re\int_S\phi(t)\omega_jdt,\quad j=\overline{1,4p}.$$

3^0. Let us prove that the determinant of the system (5.7) is different from zero. Indeed, if we assume that $\phi(t)=0$ then in system (5.7) all $Q_j=0$ and it becomes a homogeneous system. If the determinant of this system is equal t_0 zero, then it has a solution different from zero. Denote such a solution by $\gamma_j^0,\quad j=\overline{1,4p}$. Then system (4.6) is silvable in the class h_{2p} for $\phi(t)=0$ and

$$\Gamma^{(j)}=\Gamma_0^{(j)}=(\gamma_j^0+i\gamma_{p+j}^0,\gamma_{2p+j}^0+i\gamma_{3p+j}^0)^T.$$

Let $g^0(t)$ be its solution

$$a(t)g^0(t)+\frac{b(t)}{\pi i}\int_S\frac{g^0(\tau)d\tau}{\tau-t}+\int_S K_1(t,\zeta)g^0(y)d_yS$$

$$-\int_S K_2(t,\zeta)\overline{g^0(y)}d_yS=\Gamma_0^{(j)},\quad t\in S,\quad \Gamma_0^{(j)}(t)=0,\quad t\in S'.\tag{5.8}$$

Composing the potential (see (4.4)

$$U(x,g^0)=\frac{m}{\pi}\int_S\frac{\partial ln|z-\zeta|}{\partial m(y)}g^0(y)d_yS-\frac{l}{4\pi i}\int_S\frac{\partial}{\partial S(y)}\frac{z-\zeta}{\overline{z}-\overline{\zeta}}\overline{g^0(y)}d_yS,\quad x\in D^+$$

Obviously $U^0(x,g^0)$ on S satisfying the following homogeneous conditions $[U^0(t,g^0)]^+=0$, for $t\in S'$ and

$$[TU^0(t,g^0)]^+=0,\quad t\in S'',$$

except maybe $C_j,\quad j=\overline{1,2p}$ (see (3.1)).

Now repeating word by word the reasoning we have used above for $U_0(x,g_0)$ (see §5. 2^0) we find that $g^0=0$.

Owing to $g^0 = 0$, from (5.8) we have $\Gamma_0^{(j)} = [\Gamma_{01}^{(j)}, \Gamma_{02}^{(j)}]^T = 0 \quad j = \overline{1,p}$, which contradicts our assumprion.

Thus the determinant of system (5.7) is different from zero. Therefore the constant vectors $\Gamma^{(j)}; \quad j = \overline{1,p}$, are defined uniquely so that with these constant values the system of integral equations (4.6) is solvable uniquely in the class h_{2p}.

Substituting the solution $g = (g_1, g_2)^T$ in (4.4) we get a solution of the basic mixed boundary value problem (2.1), (3.1).

REFERENCES

1. Basheleishvili M.O. Two-dimensional boundary value problem of statics of the theory of elastic mixtures. *Mem. Differential Equations Math. Phys.*, **6** (1995), 59-106.
2. Basheleishvili M.O. Two-dimensional promlems of elasticity of anisotropic bedies. *Mem. Differential Equations Math. Phys.*, **16** (1999), 9-140.
3. Basheleishvili M.O. Svanadze K.N. A new method of solving the basic plane boundary volue problems of statics of the elastic mixture theory, *Georgian Math. J.*, **8**, 3 (2001), 427-446.
4. Basheleishvili M.O. Zazashvili Sh. The basic mixed plane boundary value problem of statics in the elastic mixture theory. *Georgian Math. J.*, **7**, 3 (2000), 427- 440.
5. Manjavidze G. On a singular integral equation with discontinuous coefficients and its application in the theory of elasticity (Russian). *Prikl. Math. Mekh.*, **15**, 3 (1951), 279-296.
6. Muskhelishvili N. Singular Integral Equations (Russian). Nauka, Moskow, 1968.

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Author's address:

K. Svanadze
A. Tsereteli Kutaisi State University
59, Tamar Mepe str., Kutaisi 4600
Georgia
E-mail. kostasvanadze@yahoo.com