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COMPUTATION OF EXTERIOR MODULI BY POWER SERIES

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Abstract. We study the problem of computing the exterior modulus of a bounded quadrilateral.

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1. Introduction

Let $U, V \subset \mathbb{C}$ be some regions in $\mathbb{C}$ and let $f : U \to V$ be C^1-homeomorphism. Suppose $u = x + iy$ and $w = u + iv$ are local coordinate systems at the points $z_0 \in U$ and $f(z_0) \in V$, respectively. Then $d_{z_0}f : T_{z_0}U \to T_{f(z_0)}V$ are linear map transforms (dx, dy) cotangent vector at the point z_0 to (du, dv) by the rule

$$du = u_x dx + u_y dy, \quad dv = v_x dx + v_y dy.$$

Let $du^2 + dv^2 = Edx^2 + 2Fdxdy + Gdy^2$, where

$$E = u_x^2 + v_x^2, \quad F = u_x v_x v_y, \quad G = u_y^2 + v_y^2.$$

The ration of the axes is

$$\sqrt{\frac{\lambda_1}{\lambda_2}} = \frac{E + G + \sqrt{(E-G)^2 + 4F^2}}{2\sqrt{EG - F^2}},$$

where λ_1, λ_2 are the solutions of the quadratic equation $(E - \lambda)(G - \lambda) - F^2 = 0$. Above in complex notation gives the Jacobian (see [1])

$$J = |f_z|^2 + |f_{\bar{z}}|^2 = u_x v_y - u_y v_x,$$

where

$$f_z = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y), \quad f_{\bar{z}} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y).$$

The Jacobian is positive for sense preserving and negative for sense reversing mappings. We consider only the sense preserving case. Then $|f_{\bar{z}}| < |f_z|$.

From the identity $dw = f_z dz + f_{\bar{z}} d\bar{z}$ it follows, that

$$(|f_z| - |f_{\bar{z}}||dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|)|d\bar{z}|.$$

By definition the *delatations* at the point z is the ration $D_f = \frac{|f_z|+|f_{\bar{z}}|}{|f_z|-|f_{\bar{z}}|} \geq 1$, or equivalence $d_f = \frac{|f_{\bar{z}}|}{|f_z|} < 1$, where

$$D_f = \frac{1 + d_f}{1 - d_f}, \quad d_f = \frac{D_f - 1}{D_f + 1}.$$

The maximum is attained when the ratio $\frac{f_{\bar{z}}d\bar{z}}{f_z dz}$ is positive, the minimum when it is negative. If we denote by μ_f the complex delatations then $\mu_f = \frac{f_{\bar{z}}d\bar{z}}{f_z dz}$ and the problem to find such function f that preserves given delatations $\mu(z)$ is the *Beltrami* equation [1]:

$$f_{\bar{z}} = \mu(z) f_z.$$

The mapping f is said quasiconformal if D_f is bounded. It is K-quasiconformal if $D_f \leq K$. The condition $D_f \leq K$ is equaivalent to $d_f \leq k$, where $k = \frac{K-1}{K+1}$. A 1-quasiconformal mapping is conformal. Therefore, quasiconformal mapping is measure of approximation conformality. On the other hand quasiconformal maps, which are solutions of the Beltrami equation, play important role for investigation of the space of complex structures on punctured Riemann sphere and developed the effective numerical methods for solution such type equations is an important problem [2], [3].

In this paper we consider the method of computation of exterior moduli of quadrilaterals. For the necessary notation we follow the paper [4].

A bounded Jordan curve in the complex plane divides the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ into two domains D_1 and D_2, whose common boundary is the Jordan curve. One of these domains, say D_1, is bounded and the other one is unbounded. The domain D_1 together with four distinct points z_1, z_2, z_3, z_4 in ∂D_1, which occur in this order when traversing the boundary in the positive direction, is called a quadrilateral and denoted by $(D_1; z_1, z_2, z_3, z_4)$. The domain D_1 can be mapped conformally onto a rectangle $f : D_1 \to (0, 1) \times (0, h)$ so that the four distinguished points are mapped onto the vertices of the rectangle $f(z_1) = 0, f(z_2) = 1, f(z_3) = 1 + ih, f(z_4) = ih$. The unique number h is called the *conformal modulus* of the quadrilateral $(D_1; z_1, z_2, z_3, z_4)$. Similarly, one can map D_2, the complementary domain, conformally $g : D_2 \to (0, 1) \times (0, k)$ such that the four boundary points are mapped onto the vertices of the rectangle $g(z_1) = 0, g(z_2) = 1, g(z_3) = 1 + ik, g(z_4) = ik$, reversing the orientation. The number k is unique and it is called the *exterior modulus* of $(D_1; z_1, z_2, z_3, z_4)$. An important example of a quadrilateral $(D_1; z_1, z_2, z_3, z_4)$ is the case when D_1 is a polygon with z_1, z_2, z_3, z_4 as the vertices. In particular, in [4] investigated cyclic configurations and moduli spaces of spherical quadrilaterals. For nondegenerate quadrilateral linkage established that cyclic configurations are critical points of the signed area function on moduli space and their number is determined by the topology of moduli space. The numerical computation given in below is an powerful tool for investigate such type geometric and variational problems. In the case of domains with polygonal boundary, numerical methods based on the Schwarz- Christoffel formula and have been extensively studied (see [3]).

2. Computation of moduli of the quadrilaterals

We use the notation $M(\Gamma)$ for the modulus for a family of curves Γ in the plane as in paper [1]. For instance, if Γ is the family of all curves joining the opposite b-sides within the rectangle $[0, a] \times [0, b]$, $a, b > 0$, then $M(\Gamma) = \frac{a}{b}$. If we consider the rectangle as a quadrilateral Q with distinguished points $a + ib$, ib, 0, a we also have $M(Q; a + ib, ib, 0, a) = \frac{b}{a}$.

For the exterior moduli for a family of curves Γ in the plane we have the following formula

$$M(\Gamma) = \frac{\mathcal{K}'(k)}{2\mathcal{K}(k)}, \quad \text{where} \;\; k = \psi^{-1}\left(\frac{a}{b}\right).$$

From the Düren-Pfalzgraff formula ψ is defined as

$$\psi(r) = \frac{2\left(\mathcal{E}(r) - (1-r)\mathcal{K}(r)\right)}{\mathcal{E}'(r) - r\mathcal{K}'(r)},$$

where $\psi : \;(0,1) \to (0,\infty)$ defines an increasing homeomorphism with limiting values 0, ∞ at 0, 1, respectively and its inverse is well-defined.

The function ψ is defined by hypergeometric function and complete elliptic integrals

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2 t}}, \quad \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2 t}\,dt$$

$$\mathcal{K}'(r) = \mathcal{K}(r'), \quad \mathcal{E}'(r) = \mathcal{E}(r'),$$

where $r \in (0,1)$ and $r' = \sqrt{1-r^2}$. This function can be written using the power series

$$\mathcal{K}(r) = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,r^2\right), \quad \mathcal{E}(r) = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},-\frac{1}{2},1,r^2\right).$$

Let's introduce the following notation

$$F(a,b;c;z) = \sum_{n=0}^{\infty} f_n(a,b,c)z^n, \quad f_n(a,b,c) = \frac{(a,n)(b,n)}{(c,n)}\frac{1}{n!}.$$

Then

$$\mathcal{K}(r) = \frac{\pi}{2}F(\frac{1}{2},\frac{1}{2};1;r^2) = \frac{\pi}{2}\sum_{n=0}^{\infty} f_n\left(\frac{1}{2},\frac{1}{2},1\right)r^{2n} = \sum_{n=0}^{\infty} k_n r^{2n},$$

where

$$k_n = \frac{\pi}{2}f_n\left(\frac{1}{2},\frac{1}{2},1\right)$$

and

$$\mathcal{E}(r) = \frac{\pi}{2}F(\frac{1}{2},-\frac{1}{2};1;r^2) = \frac{\pi}{2}\sum_{n=0}^{\infty} f_n\left(\frac{1}{2},-\frac{1}{2},1\right)r^{2n} = \sum_{n=0}^{\infty} e_n r^{2n},$$

where

$$e_n = \frac{\pi}{2}f_n\left(\frac{1}{2},-\frac{1}{2},1\right).$$

Therefore,

$$\mathcal{K}'(r) = \mathcal{K}(r') = \sum_{n=0}^{\infty} k_n(1-r^2)^n, \;\; \mathcal{E}'(r) = \mathcal{E}(r') = \sum_{n=0}^{\infty} e_n(1-r^2)^n.$$

In order to express the function ψ we consider the numerator and denominator separately. First we have

$$\mathcal{E}(r)-(1-r)\mathcal{K}(r)=\sum_{n=0}^{\infty}(e_n-k_n)r^{2n}+\sum_{n=0}^{\infty}k_n r^{2n+1}=\sum_{n=0}^{\infty}u_n r^n,$$

where

$$u_{2n}=e_n-k_n,\quad u_{2n+1}=k_n,\quad u_0=0$$

and for the denominator we need to do extra work

$$\mathcal{E}(r)-r'\mathcal{K}(r)=\sum_{j=0}^{\infty}e_j r^{2j}-\sqrt{1-r^2}\sum_{j=0}^{\infty}k_j r^{2j}$$

$$=\sum_{j=0}^{\infty}e_j r^{2j}-\sum_{i=0}^{\infty}\binom{1/2}{i}(-1)^i r^{2i}\sum_{j=0}^{\infty}k_j r^{2j}$$

$$=\sum_{j=0}^{\infty}e_j r^{2j}-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^i\binom{1/2}{i}k_j r^{2(i+j)}$$

$$=\sum_{n=0}^{\infty}e_n r^{2n}-\sum_{n=0}^{\infty}\left(\sum_{i+j=n}^{\infty}(-1)^i\binom{1/2}{i}k_j\right)r^{2n}$$

$$=\sum_{n=0}^{\infty}e_n r^{2n}-\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}(-1)^{n-j}\binom{1/2}{n-j}k_j\right)r^{2n}$$

$$=\sum_{n=0}^{\infty}\left(e_n-\sum_{j=0}^{n}(-1)^{n-j}\binom{1/2}{n-j}k_j\right)r^{2n}=\sum_{n=0}^{\infty}v_n r^{2n},$$

where

$$v_n=e_n-\sum_{j=0}^{n}(-1)^{n-j}\binom{1/2}{n-j}k_j,\quad v_0=v_1=0.$$

Now we can write

$$\frac{1}{\mathcal{E}(r)-r'\mathcal{K}(r)}=\frac{1}{v_2r^4+v_3r^6+v_4r^8+\cdots}$$

$$=\frac{1}{v_2r^4}\frac{1}{1+(v_3/v_2)(r^2)^1+(v_4/v_2)(r^2)^2+\cdots}$$

$$=\frac{1}{v_2r^4}\left(1+w_1(r^2)^1+w_2(r^2)^2+\cdots\right)\frac{1}{v_2r^4}\sum_{n=0}^{\infty}w_n r^{2n},$$

where

$$w_0=1,\quad w_n=\sum_{s_1+2s_2+\cdots+ns_n=n}(-1)^{s_1+\cdots+s_n}(s_1+\cdots+s_n)!\prod_{j=1}^{n}\frac{1}{s_j!}\left(\frac{v_{j+2}}{v_2}\right)^{s_j}.$$

Therefore, the function ψ has the following form

$$\psi(r) = \frac{2}{v_2(1-r^2)^2}\left(\sum_{n=0}^{\infty} u_n r^n\right)\left(\sum_{n=0}^{\infty} w_n (1-r^2)^n\right).$$

We can use ψ as a multiplication of two power series, or expand the multiplication

$$\frac{1}{(1-r^2)^2}\left(\sum_{n=0}^{\infty} u_n r^n\right) = \left(\sum_{i=0}^{\infty}(i+1)r^{2i}\right)\left(\sum_{j=0}^{\infty} u_{2j}r^{2j} + \sum_{j=0}^{\infty} u_{2j+1}r^{2j+1}\right)$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(i+1)u_{2j}r^{2(i+j)} + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(i+1)u_{2j+1}r^{2(i+j)+1}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{i+j=n}(i+1)u_{2j}\right) r^{2n} + \sum_{n=0}^{\infty}\left(\sum_{i+j=n}(i+1)u_{2j+1}\right) r^{2n+1}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}(n-j+1)u_{2j}\right) r^{2n} + \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}(n-j+1)u_{2j+1}\right) r^{2n+1}$$

$$= \sum_{n=0}^{\infty} h_n r^n,$$

$$h_{2n} = \sum_{j=0}^{n}(n-j+1)u_{2j}, \quad h_{2n+1} = \sum_{j=0}^{n}(n-j+1)u_{2j+1}.$$

So now we have

$$\psi(r) = \frac{2}{v_2}\left(\sum_{n=0}^{\infty} h_n r^n\right)\left(\sum_{n=0}^{\infty} w_n (1-r^2)^n\right).$$

Let's take a slice and expand it

$$\psi_N(r) = \frac{2}{v_2}\left(\sum_{n=0}^{2N+1} h_n r^n\right)\left(\sum_{n=0}^{N} w_n (1-r^2)^n\right)$$

$$= \frac{2}{v_2}\left(\sum_{i=0}^{2N+1} h_i r^i\right)\left(\sum_{n=0}^{N} w_n \sum_{j=0}^{N}\binom{n}{j}(-1)^j = r^{2j}\right)$$

$$= \frac{2}{v_2}\left(\sum_{i=0}^{2N+1} h_i r^i\right)\sum_{j=0}^{N}\left(\sum_{n=0}^{N}\binom{n}{j}w_n\right)(-1)^j r^{2j}$$

$$= \frac{2}{v_2}\left(\sum_{i=0}^{N} h_{2i} r^{2i} + \sum_{i=0}^{N} h_{2i+1} r^{2i+1}\right)\sum_{j=0}^{N}\left(\sum_{n=0}^{N}\binom{n}{j}w_n\right)(-1)^j r^{2j}$$

$$= \frac{2}{v_2}\sum_{i=0}^{N}\sum_{j=0}^{N}(-1)^j W_{N,j} h_{2i} r^{2(i+j)} + \frac{2}{v_2}\sum_{i=0}^{N}\sum_{j=0}^{N}(-1)^j W_{N,j} h_{2i+1} r^{2(i+j)+1}$$

$$= \sum_{s=0}^{2N} H_{N,s} r^s,$$

$$W_{N,j} = \sum_{n=0}^{N} \binom{n}{j} w_n, \quad H_{N,2s} = \frac{2}{v_2} \sum_{i=0}^{N} (-1)^{s-i} W_{N,s-i} h_{2i},$$

$$H_{N,2s+1} = \frac{2}{v_2} \sum_{i=0}^{N} (-1)^{s-j} W_{N,s-i} h_{2i+1},$$

since $H_{N,0} = 0$, we get

$$\psi_N(r) = \sum_{n=1}^{2N} H_{N,n} r^n = H_{N,1} r + H_{N,2} r^2 + \cdots + H_{N,2N} r^{2N}.$$

Let

$$\mathcal{K}(r) = \sum_{n=0}^{\infty} k_n r^{2n} \quad \mathcal{K}'(r) = \sum_{n=0}^{\infty} k_n (1-r^2)^n,$$

then

$$\frac{1}{\mathcal{K}(r)} = \frac{1}{k_0 + k_1 r^2 + k_2 \left(r^2\right)^2 + k_2 \left(r^2\right)^3 + \cdots}$$

$$= \frac{1}{k_0} \left(1 + \tilde{k}_1 r^2 + \tilde{k}_2 \left(r^2\right)^2 + \tilde{k} \left(r^2\right)^3 + \cdots \right),$$

where

$$\tilde{k}_0 = 1, \quad \tilde{k}_n = \sum_{s_1+2s_2+\cdots+ns_n=n} (-1)^{s_1+\cdots+s_n} (s_1+\cdots+s_n)! \prod_{j=1}^{n} \frac{1}{s_j!} \left(\frac{k_j}{k_0}\right)^{s_j}.$$

From this

$$\frac{\mathcal{K}'(r)}{2\mathcal{K}(r)} = \frac{1}{2k_0} \left(\sum_{n=0}^{\infty} k_n (1-r^2)^n \right) \left(\sum_{n=0}^{\infty} \tilde{k}_n r^{2n} \right).$$

The calculation gives the following expression

$$\begin{aligned}
\frac{\mathcal{K}'(r)}{2\mathcal{K}(r)} &\approx 0.0004r^{94} - 0.0049r^{92} + 0.0357r^{90} - 0.1872r^{88} \\
&+ 0.7492r^{86} - 2.3741r^{84} + 6.1111r^{82} - 13.0054r^{80} \\
&+ 23.1672r^{78} - 34.8425r^{76} + 44.497r^{74} - 48.4176r^{72} \\
&+ 44.9466r^{70} - 35.5739r^{68} + 23.9426r^{66} - 13.6384r^{64} \\
&+ 6.5286r^{62} - 2.6005r^{60} + 0.8507r^{58} - 0.225r^{56} \\
&+ 0.0474r^{54} - 0.0081r^{52} + 0.0012r^{50} + 0.0178r^{48} \\
&- 0.3621r^{46} + 3.7436r^{44} - 26.0982r^{42} + 133.7881r^{40} \\
&- 531.8842r^{38} + 1695.6436r^{36} - 4434.9631r^{34} + 9670.8796r^{32} \\
&- 17785.5001r^{30} + 27813.2318r^{28} - 37193.9707r^{26} + 42684.2543r^{24} \\
&- 42109.6719r^{22} + 35712.7426r^{20} - 25994.0702r^{18} + 16182.9191r^{16} \\
&- 8571.5114r^{14} + 3833.5577r^{12} - 1433.2443r^{10} + 442.1954r^{8} \\
&- 110.9771r^{6} + 22.4047r^{4} - 3.9393r^{2} + 1.0439.
\end{aligned}$$

R E F E R E N C E S

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