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ON THE SPECIAL CASE OF THE BOUNDARY VALUE PROBLEM FOR THE CARLEMAN-BERS-VEKUA EQUATION

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Abstract. In this paper the special case of the Rieman-Hilbert boundary value problem (problem of linear conjugation) for the Carleman-Bers-Vekua equation is obtained, when the transition function $G(t)$, given on the boundary curve Γ has the zeros and poles on Γ. The necessary and sufficient condition of solvability is obtained and an explicit formula is given for the solution of this problem.

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In this paper we continue the investigation of special cases of Carleman-Bers-Vekua equation [1] and related boundary value problems.

Consider the *Carleman-Bers-Vekua* equation

$$w_{\overline{z}} + Aw + B\overline{w} = 0. \quad A, B \in L_p(\mathbb{C}), p > 2. \tag{1}$$

Let D be a domain in $\mathbb{C}$. Denote by $U_{p,2}(A, B, D)$ the space of regular solutions of (1) in D. This is a vector space over reals.

Let Γ be a closed curve in $\mathbb{C}$ with interior D^+ and exterior D^-. Suppose $G_1(t), g(t)$ are defined on Γ functions of class $C_\alpha(\Gamma)$, $0 < \alpha \leq 1$ and $G_1(t) \neq 0$ everywhere on Γ. Denote by $\alpha_1, \alpha_2, ..., \alpha_l$, $\beta_1, \beta_2, ..., \beta_n$ marked points on Γ and denote by $m_1, m_2, ..., m_l$, $p_1, p_2, ..., p_n$ the nonnegative integers.

Consider the following boundary value problem:

Find piecewise regular solutions of (1) which satisfy the following boundary value conditions:

$$W^+(t) = G_1(t)W^-(t) + g(t) \tag{2}$$

$$W(z) = O(z^N), \quad z \to \infty, \tag{3}$$

where N is a given integer and

$$G(t) = \frac{\Pi_{k=1}^{l}(t-\alpha_k)^{m_k}}{\Pi_{k=1}^{n}(t-\beta_k)^{p_k}} G_1(t), \quad t \neq \beta_k, \quad k = 1, ..., l, \quad t \in \Gamma.$$

The point $\alpha_k \in \Gamma$ is called zero of the function $G(t)$, order m_k with respect to $t - \alpha_k$. Similarly, β_k is called a pole of $G(t)$ of order p_k.

The spatial inhomogeneous problem of linear conjugation was studied in [2] for piecewise analytic functions.

Suppose $X(z)$ is a canonical solution in class of piecewise analytic functions of the following boundary value problem:

$$X^+(t) = G_1(t)X^-(t), \quad t \in \Gamma. \tag{4}$$

Consider the following Carlemann-Bers-Vekua equation

$$V_{\bar{z}} + AV + B_1\overline{V} = 0, \tag{5}$$

where

$$B_1(z) = B(z)\frac{\overline{X(z)}\Pi_{k=1}^{n}(z-\beta_k)^{p_k}}{X(z)\Pi_{k=1}^{n}(\bar{z}-\bar{\beta}_k)^{p_k}}, \quad z \in D^+,$$

$$B_1(z) = B(z)\frac{\overline{X(z)}\Pi_{k=1}^{l}(z-\alpha_k)^{m_k}}{X(z)\Pi_{k=1}^{l}(\bar{z}-\bar{\beta}_k)^{m_k}}, \quad z \in D^-.$$

It is clear, that $B_1 \in L_{p,2}(\mathbb{C})$.

Let $\Omega_1(z,t)$ and let $\Omega_2(z,t)$ be main kernels of class $U_{p,2}(A, B_1, \mathbb{C})$ and let

$$V_{2k} = R_{\infty}^{-A,-\overline{B}_1}(z^k), \quad V_{2k+1} = R_{\infty}^{-A,-\overline{B}_1}(iz^k), \quad k = 0,1,2,... \tag{6}$$

be generalized power functions [1] of class $U_{p,2}(A, -\overline{B}_1, \mathbb{C})$.

Consider conjugate to (5) equation

$$U_{\bar{z}} - AU - B_1\overline{U} = 0. \tag{7}$$

Suppose

$$\chi = \frac{1}{2\pi}\left[\arg G_1(t)\right]_{\Gamma}$$

and

$$\sum_{k=1}^{l} m_k = m.$$

Theorem 1. *Let $\chi + N + m \geq -1$. Then the general solution of problem (1), (2), (3) is*

$$W(z) = \frac{1}{\Pi_{k=1}^{n}(z-\beta_k)^{p_k}} \times \tag{8}$$

$$\left(\frac{X(z)}{2\pi i}\int_{\Gamma}\Omega_1(z,t)\frac{g(t)}{X^+(t)}dt - \Omega_2(z,t)\frac{\overline{g(t)}}{\overline{X^+(t)}}d\bar{t} + X(z)Q_{\chi+N+m}(z)\right), \quad z \in D^+,$$

$$W(z) = \frac{1}{\Pi_{k=1}^{l}(z-\alpha_k)^{p_k}} \times \tag{9}$$

$$\left(\frac{X(z)}{2\pi i}\int_{\Gamma}\Omega_1(z,t)\frac{g(t)}{X^+(t)}dt - \Omega_2(z,t)\frac{\overline{g(t)}}{\overline{X^+(t)}}d\bar{t} + X(z)Q_{\chi+N+m}(z)\right), \quad z \in D^-,$$

where $Q_{\chi+N+m}(z)$ is a generalized polynomial of class $U_{p,2}(A, B_1, \mathbb{C})$ of degree at most $\chi + N + m$, and $Q_{-1} = 0$ by definition.

Proof. From (4) we have

$$G_1(t) = \frac{X^+(t)}{X^-(t)}. \tag{10}$$

After the setting (10) in (2) we obtain

$$\Pi_{k=1}^{n}(t-\beta_k)^{p_k}W^+(t) \tag{11}$$

$$= \Pi_{k=1}^{l}(t-\alpha_k)^{m_k}\frac{X^+(t)}{X^-(t)}W^-(t)+g(t),\ \ t\in\Gamma.$$

From (11) it follows that

$$\Pi_{k=1}^{n}(t-\beta_k)^{p_k}\frac{W^+(t)}{X^+(t)} \tag{12}$$

$$= \Pi_{k=1}^{l}(t-\alpha_k)^{m_k}\frac{X^+(t)}{X^-(t)}W^-(t)+\frac{g(t)}{X^+(t)},\ \ t\in\Gamma.$$

Consider the following function

$$V(z)=\Pi_{k=1}^{n}(z-\beta_k)^{p_k}\frac{W(z)}{X(z)},\ \ z\in D^+, \tag{13}$$

$$V(z)=\Pi_{k=1}^{l}(z-\alpha_k)^{m_k}\frac{W(z)}{X(z)},\ \ z\in D^-. \tag{14}$$

From (1), (13), (14) it follows, that V satisfies equation (5) in domains D^+ and D^-, therefore

$$V\in U_{p,2}(A,B_1,D^+)\quad V\in U_{p,2}(A,B_1,D^-). \tag{15}$$

From (12), (13), (14) we obtain

$$V^+(t)=V^-(t)+\frac{g(t)}{X^+(t)},\quad t\in\Gamma. \tag{16}$$

Consider the generalized Cauchy type integral of class $U_{p,2}(A,B_1,\mathbb{C})$:

$$h(z)=\frac{1}{2\pi i}\int_\Gamma \Omega_1(z,\zeta)\frac{g(\zeta)}{X^+(\zeta)}d\zeta-\Omega_2(z,\zeta)\frac{\overline{g(\zeta)}}{\overline{X^+(\zeta)}}d\overline{\zeta},\ z\in D^+,\ \ z\in D^-.$$

Since $\frac{g}{X^+}\in C_\alpha(\Gamma)$, $0<\alpha\le 1$ then due to Sokhotski-Plemelj theorem formula we have

$$h^+(t)=h^-(t)+\frac{g(t)}{X^+(t)},\ \ t\in\Gamma. \tag{17}$$

Difference of expressions (16) and (17) gives

$$(V(t)-h(t))^+=(V(t)-h(t))^-,\ \ t\in\Gamma. \tag{18}$$

As it is known, h is the solution of equation (5) in the domains D^+ and D^-:

$$h\in U_{p,2}(A,B_1,D^+),\ \ h\in U_{p,2}(A,B_1,D^-). \tag{19}$$

From (15) and (19) it follows that $V-h$ is a solution of (5) in domains D^+ and D^- :

$$V-h\in U_{p,2}(A,B_1,D^+),\ \ V-h\in U_{p,2}(A,B_1,D^-). \tag{20}$$

From (18) follows, that $V-h$ is continuous on $\mathbb{C}$, therefore

$$V-h\in C(\mathbb{C}). \tag{21}$$

From the continuity properties of generalized analytic functions and from inclusions (20) and (21) it follows, that $V-h$ is a regular solution of equation (5) on $\mathbb{C}:$

$$V-h\in U_{p,2}(A,B_1,\mathbb{C}) \tag{22}$$

$$X(z)=O(z^{-\chi}),\ \ z\to\infty,\ \ \frac{1}{X(z)}=O(z^{\chi}),\ \ z\to\infty$$

and

$$\Pi_{k=1}^{l}(z-\alpha_k)^{m_k}=O(z^m),\quad z\to\infty. \tag{23}$$

From (3), (14), (23) it follows that

$$V(z)=O(z^{\chi+N+m}),\ \ z\to\infty. \tag{24}$$

It is clear that

$$h(z)=O(z^{-1}),\quad z\to\infty. \tag{25}$$

Since $\chi+N+m\geq -1$ from (25) it follows that

$$h(z)=O(z^{\chi+N+m}),\quad z\to\infty. \tag{26}$$

From (24) and (26) we obtain

$$V(z)-H(z)=O(z^{\chi+N+m}),\quad z\to\infty. \tag{27}$$

Due to Loiuville theorem for generalized analytic functions from (22) and (27) it follows that $V-h$ is a generalized polynomial $Q_{\chi+N+m}$ of degree at most $\chi+N+m$ of class $U_{p,2}(A,B_1,\mathbb{C})$. Therefore,

$$V(z)-h(z)=Q_{\chi+N+m}(z),\ \ z\in\mathbb{C}. \tag{28}$$

From (28) we obtain

$$V(z)=h(z)+Q_{\chi+N+m}(z),\ \ z\in D^+,\ \ z\in D^-. \tag{29}$$

Let $z\in D^+$, then from (13) and (29) it follows that

$$\Pi_{k=1}^{n}(z-\beta_k)^{p_k}\frac{W(z)}{X(z)}=h(z)+Q_{\chi+N+m}(z). \tag{30}$$

From (30) we obtain

$$W(z)=\frac{1}{\Pi_{k=1}^{n}(z-\beta_k)^{p_k}}(X(z)h(z)+X(z)Q_{\chi+N+m}(z)). \tag{31}$$

Therefore, we obtained identity (8) from the theorem. Let $z\in D^-$. Then from(14) and (29) it follows that

$$\Pi_{k=1}^{l}(z-\alpha_k)^{m_k}\frac{W(z)}{X(z)}=h(z)+Q_{\chi+N+m}(z). \tag{32}$$

From (32) we obtain

$$W(z)=\frac{1}{\Pi_{k=1}^{l}(z-\alpha_k)^{m_k}}(X(z)h(z)+X(z)Q_{\chi+N+m}(z)) \tag{33}$$

This is expression (9) from Theorem 1.

Theorem 2. *Let $\chi + N + m \neq -2$. The necessary and sufficient conditions for solvability of the problem (1), (2), (3) are the following identities*

$$\Im \int_\Gamma V_k \frac{g(t)}{X^+(t)} dt = 0, \quad k = 0, 1, 2, ..., 2(-\chi - N - m) - 3. \tag{34}$$

If conditions are satisfied, then the solution of problem (1), (2), (3) is

$$W(z) = \frac{1}{\Pi_{k=1}^{n} (z - \beta_k)^{p_k}} \tag{35}$$

$$\times \left(\frac{X(z)}{2\pi i} \int_\Gamma \Omega_1(z,t) \frac{g(t)}{X^+(t)} dt - \Omega_2(z,t) \frac{\overline{g(t)}}{\overline{X^+(t)}} d\bar{t} + X(z) Q_{\chi+N+m}(z) \right), \quad z \in D^+,$$

$$W(z) = \frac{1}{\Pi_{k=1}^{l} (z - \alpha_k)^{p_k}} \tag{36}$$

$$\times \left(\frac{X(z)}{2\pi i} \int_\Gamma \Omega_1(z,t) \frac{g(t)}{X^+(t)} dt - \Omega_2(z,t) \frac{\overline{g(t)}}{\overline{X^+(t)}} d\bar{t} + X(z) Q_{\chi+N+m}(z) \right), \quad z \in D^-.$$

Proof. Since $\chi + N + m \geq -2$ from (24) we have

$$V(z) = O(z^{-2}), \quad z \to \infty. \tag{37}$$

From (25), (36) it follows that

$$V(z) - h(z) = O(z^{-1}), \quad z \to \infty. \tag{38}$$

Due to Liouville theorem, (22) and (38) we have

$$V(z) - h(z) = 0, \quad z \in \mathbb{C}.$$

Therefore,

$$V(z) = h(z), \quad z \in D^+, \ z \in D^-. \tag{39}$$

From (13), (14), (39) follows (35) and (36). From (24) and (39) it follows that

$$h(z) = O(z^{\chi+N+m}), \quad z \to \infty. \tag{40}$$

The condition (40) is satisfied if and only if (34) is satisfied. This completes the proof.

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R E F E R E N C E S

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