**Proceedings of I. Vekua Institute
of Applied Mathematics
Vol. 69, 2019**

THREE POINT CHARGES ON CONCENTRIC CIRCLES

Giorgadze G., Khimshiashvili G.

Abstract. Equilibrium configurations of three equal point charges with Coulomb interaction confined to a system of concentric coplanar circles are discussed. For arbitrary values of radii of the given circles, it is shown that the number of equilibria is finite. The main results are concerned with detailed investigation of aligned equilibrium configurations. In particular, it is shown that, for generic values of radii, all aligned configurations are non-degenerate critical points of Coulomb potential, and explicit formulas for their Morse indices are given. It is also proven that, for certain non-generic values of radii, a pitchfork bifurcation happens at one of the aligned equilibrium configurations, which enables us to determine the exact number of equilibria for arbitrary values of radii of the given circles. Some related results and an application to Bohr's 1913 model of lithium atom are also given.

Keywords and phrases: Bohr's atom model, point charge, Coulomb interaction, Morse index, pitchfork bifurcation.

AMS subject classification (2010): 52C35, 32S40.

1. Introduction

Geometric properties of equilibrium configurations of constrained point charges with Coulomb interaction have been actively studied for a long time, in particular, in connection with the famous Thomson problem (see, e.g.,[11]) and Bohr's 1913 molecular model [2], [3]. However, many problems arising in this context remain unsolved or lack rigorous mathematical investigation. Specifically, this refers to mathematical aspects of Bohr's 1913 molecular model which continues to attract attention of researchers from physical and mathematical points of view [3], [12]. Recently, certain mathematical aspects of Bohr's model have been discussed in big detail in [3]. The mathematical setting, accepted in [3], involved investigation of equilibrium configurations of electrons confined to coplanar concentric circles, playing the role of electronic orbits. Along the same lines, in the present paper we use methods of real algebraic geometry and singularity theory to shed more light on the geometry of equilibrium configurations of three equal charges confined to a system of concentric coplanar circles.

In particular, we obtain some results in the spirit of Morse theory for Coulomb potential. We give examples showing that the number of equilibria can be four or six. We also prove that the aligned equilibrium configurations are generically non-degenerate critical points of Coulomb potential and compute their Morse indices (Theorem). Using these results we show that for certain non-generic values of radii, a pitchfork bifurcation happens at one of the aligned equilibrium configurations (Theorem). As an application we present more detailed results in the case where two electrons belong to the same circle, which is related to Bohr's model of lithium atom [2]. We also present a few illustrative examples and open problems.

2. Definitions and auxiliary results

We begin by recalling necessary definitions and auxiliary results. Let $S = (P, Q)$ be a given pair, where $P = (p_1, p_2, ..., p_n)$ is a set of n points of three-dimensional Euclidean space $\mathbb{R}^3$ and $Q = (q_1, ..., q_n)$ is a set of n nonzero real numbers. Such a pair is interpreted as a configuration of point charges $\{q_j\}$ with Coulomb interaction, located at the points $\{p_j\}$, and this interpretation is expressed by the notation $S = Q@P$.

The Coulomb (electrostatic) energy $E(S)$ of such a system is defined as

$$E(S) = \sum_{i \neq j} \frac{q_i q_j}{d_{ij}}, \tag{1}$$

where d_{ij} is the distance between the points p_i and p_j. Thus, for fixed charges Q, the Coulomb energy $E(S) = E_Q(P)$ is a rational function of $3n$ variables representing the Cartesian coordinates of the points p_j. Assume that all the charges lie in a fixed compact subset $X \subset \mathbb{R}^3$. Then the system of point charges $S = (P_0, Q)$ is called an *equilibrium configuration* of point charges in X if the configuration $P_0 \in X^n$ is a critical point of the Coulomb energy of Q, regarded as a function $E_Q(P)$ of a configuration, belonging to the space X^n. The general problem with which we are concerned in the sequel can now be formulated as follows.

For a given system of n non-zero real numbers Q, find all n-tuples of points P in a given set X such that the system $Q@P$ is an equilibrium configuration of point charges in X. In this context X is referred to as *conductor* and any configuration where Coulomb energy, has global minimum is called a *ground state.* If the given conductor X is a compact differentiable submanifold of $\mathbb{R}^3$ this problem can be reformulated as the problem of finding and classifying the critical points of Coulomb energy considered as a function on X^n. Equilibrium configurations possessing certain stability properties are especially important from the physical point of view. Recall that an equilibrium configuration of a system of point charges is called *stable* if the Coulomb energy has an isolated local minimum at the considered configuration. If E appears to be a Morse function on X^n then stable configurations can be found by computing the Morse indices of all equilibrium configurations.

Having in mind configurations of electrons in the framework of Bohr's 1913 molecular model [2], we basically deal with configurations of equal charges, belonging to a system of coplanar concentric circles Γ and call them Γ-*orbital* configurations. Such configurations of point charges have also been studied in connection with the Coulomb control of swarms of small satellites [13], [6], [5].

In view of the rotational symmetry of the circle and rotational invariance of Coulomb energy, an orbital configuration cannot be an isolated minimum of the Coulomb energy. Therefore, for orbital configurations, it is reasonable to use a weaker definition of stability. Orbital configuration of charges is said to be *weakly stable* if any sufficiently small rotation of any of its points, with the other points being fixed, increases its electrostatic energy. In the sequel we basically deal with the case of three equal charges, confined to two or three coplanar concentric circles.

3. Three charges on concentric circles

We proceed by formulating a natural generalization of Bohr's 1913 model for atoms with three electrons. As usual, two point charges are called *like* or *repelling* if they are both of the same sign. We consider three fixed like charges q_1, q_2, q_3, confined to a system of concentric circles Γ playing the role of electronic orbits. Obviously, there are three combinatorially different settings of such type: all charges on the same circle, two charges on a circle and the third one on another concentric circle, and one charge on each of three concentric circles. The case where all three charges are confined to the same circle was studied in some detail in [4], [5] (see also [8], [9]). The second case is related to Bohr's model of lithium [2], [3]. The most general case where charges are not-necessarily equal is related to the Coulomb control scenario for swarms of small satellites moving on concentric orbits [13], [4].

We will first deal with the third case and then derive some consequences for the second one. Without loss of generality, it can be assumed that one of the charges is fixed in the outer circle, while the other two run over the remaining two circles. Then the Coulomb energy defines a smooth function on a reduced configuration space, which is diffeomorphic to the two-dimensional torus T^2. The following auxiliary statements will be used in the sequel. The first two of them hold true for for n mutually repelling charges on n concentric circles but for simplicity we formulate them only for $n = 3$. Let us say that an orbital configuration P is *separable* if, for any straight line through the origin, each of the two arising open half-planes contains at least one point of P.

Lemma 1. *Any equilibrium configuration of three like point charges on three concentric circles is separable.*

Notice that if all charges are in the same semi-circle it is geometrically obvious that one can move the charges in such a way that all pairwise distances will increase so the Coulomb energy will become smaller. In particular, this statement holds true for ground states. It follows in full generality by considering the force diagram. It is also easy to see that this holds true in all cases, i.e., two or all circles may coincide. As was already mentioned, an analogous statement holds true for n like charges on n concentric circles.

Lemma 2. *The ground state energy of three like point charges on three concentric circles is strictly bigger than*

$$\frac{q_1 q_2}{r_1 + r_2} + \frac{q_2 q_3}{r_2 + r_3} + \frac{q_1 q_3}{r_1 + r_3}.$$

This is evident since in the denominators stay the maximal possible pairwise distances and all pairwise distances cannot attain their maximal possible values simultaneously. Recall that if all charges are q and all radii are r then the ground state energy is $\frac{q^2}{r}\sqrt{3}$.

The next result refers to the case of three equal point charges.

Lemma 3. *The ground state energy of three unit charges on three concentric circles does not exceed*

$$(r_1^2 + r_2^2 + 2tr_1 r_2 r_3^{-1})^{-1/2} + (r_1^2 + r_3^2 + 2tr_1 r_2^{-1} r_3)^{-1/2} + (r_2^2 + r_3^2 + 2tr_1^{-1} r_2 r_3)^{-1/2},$$

where t is the unique real root of the cubic equation

$$2r_1r_2r_3t^3 + (r_1^2r_2^2 + r_2^2r_3^2 + r_1^2r_3^2)t^2 - r_1^2r_2^2r_3^2 = 0.$$

This upper estimate follows from the formula for the perimeter of the so-called maximal connecting circuit for Γ given in [7]. Obviously, the Coulomb energy of maximal connecting circuit cannot be less than the ground state energy by the very definition of the latter.

Remark 1. As will be shown below, this upper estimate can be explicated in the case where two radii coincide. Actually, the estimate given above is close to an optimal one since it is easy to verify that, for equal radii, it coincides with the actual value of ground state energy equal to $\frac{q^2}{r}\sqrt{3}$.

We can now present our first main result.

Theorem 1. *For any positive values of charges and radii, the Coulomb energy of a three-point Γ-orbital configuration of fixed charges has a finite set of critical points.*

Proof. In the sequel we assume that point charges q_1, q_2, q_3 are located on three circles, centered at the origin with the radii r_1, r_2, r_3 satisfying $r_1 < r_2 < r_3$. Then denoting by φ, ψ the angles between the first and third, and second and third point respectively, one has:

$$E(\varphi,\psi) = \frac{q_1q_3}{\sqrt{r_1^2 + r_3^2 - 2r_1r_3\cos\varphi}} + \frac{q_2q_3}{\sqrt{r_2^2 + r_3^2 - 2r_2r_3\cos\psi}} + \frac{q_1q_2}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi)}},$$

where $0 \le \varphi < 2\pi$, $0 \le \psi < 2\pi$, i.e., we have a function on the two-dimensional torus $E : \mathbb{T}^2 \to \mathbb{R}$.

The critical points of E are the solutions of the system of equations $\frac{\partial E}{\partial \varphi} = 0$ and $\frac{\partial E}{\partial \psi} = 0$:

$$-\frac{q_1q_3r_1r_3\sin\varphi}{(r_1^2 + r_3^2 - 2r_1r_3\cos\varphi)^{\frac{3}{2}}} + \frac{q_1q_2r_1r_2\sin(\psi-\varphi)}{(r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi))^{\frac{3}{2}}} = 0, \tag{2}$$

$$-\frac{q_2q_3r_2r_3\sin\psi}{(r_2^2 + r_3^2 - 2r_2r_3\cos\psi)^{\frac{3}{2}}} - \frac{q_1q_2r_1r_2\sin(\psi-\varphi)}{(r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi))^{\frac{3}{2}}} = 0, \tag{3}$$

Obviously, the above system of equations is satisfied if $\sin\varphi = 0, \sin\psi = 0$, i.e., if $\varphi = \psi = \pi k$, $k = 0, 1, ...$ This gives four aligned equilibrium configurations: $(\varphi = 0, \psi = 0)$, $(\varphi = 0, \psi = \pi)$, $(\varphi = \pi, \psi = 0)$, $(\varphi = \pi, \psi = \pi)$. These configurations will be called *parades* by a way of analogy with the term "parades of planets" used in celestial mechanics.

Notice now that all solutions of this system should satisfy the equation:

$$-\frac{q_1r_1\sin\varphi}{(r_1^2 + r_3^2 - 2r_1r_3\cos\varphi)^{\frac{3}{2}}} = -\frac{q_2r_2\sin\psi}{(r_2^2 + r_3^2 - 2r_2r_3\cos\psi)^{\frac{3}{2}}}. \tag{4}$$

Using the latter equation we show that system (2), (3) has only isolated solutions. To this end we square the both sides of (4) and rewrite them in terms of $\cos\varphi$ and $\cos\psi$. This gives a cubic relation between $\cos\varphi$ and $\cos\psi$ which enables one to express $\cos\varphi$ as an algebraic function of $\cos\psi$. Thus, for each branch we can write $\cos\varphi = \Phi_i(\cos\psi)$. Substituting this function into equation (2) one can rewrite this equation as an algebraic equation for $\tan\psi/2$ for each branch. Hence there are only finitely many solutions to system (2, 3) as was claimed. Moreover, it is clear that non-aligned solutions to (2, 3) appear in pairs which are mirror symmetric with respect to Ox axis.

Here are examples showing that the number of equilibrium configurations can be four or six depending on the radii.

Example 1. Let $q_1 = q_2 = q_3$, $r_1 = 1$, $r_2 = 2$, $r_3 = 3$. Solving the system (2), (3) numerically one finds out that there are exactly two non-aligned symmetric solutions: $(\varphi = 1.916, \psi = 4.242)$, $(\varphi = 4.242, \psi = 1.916)$. So in this case the total number of equilibrium configurations is six.

Example 2. Let $q_1 = q_2 = q_3$, $r_1 = 1$, $r_2 = 2$, $r_3 = 30$. Making the plots lot of the zero-sets of both equations one sees that they intersect only at four parades. So in this case there are no non-aligned solutions and the number of equilibrium configurations is four.

In accordance with general principles of singularity theory one can await that, for generic values of parameters q_i and r_j, Coulomb potential is a Morse function, i.e., all of its critical points are non-degenerate. The following result shows that this really holds true for parades.

Theorem 2. *For generic values of charges and radii, all parades are non-degenerate critical points of Coulomb potential of three equal charges on three concentric circles.*

Proof. To prove this we compute the Hessian matrix of E and evaluate it at aligned critical configurations (parades). We have:

$$\frac{\partial^2 E(\varphi,\psi)}{\partial\varphi^2} = \frac{3r_1^2 r_3^2 \sin^2(\varphi)}{(r_1^2 + r_3^2 - 2r_1r_3\cos(\varphi))^{\frac{5}{2}}} - \frac{r_1r_3\cos(\varphi)}{(r_1^2 + r_3^2 - 2r_1r_3\cos(\varphi))^{\frac{3}{2}}}$$

$$+\frac{3r_1^2 r_2^2 \sin^2(\psi-\varphi)}{(r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi))^{\frac{5}{2}}} - \frac{r_1r_2\cos(\psi-\varphi)}{(r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi))^{\frac{3}{2}}},$$

$$\frac{\partial^2 E(\varphi,\psi)}{\partial\varphi\partial\psi} = -\frac{3r_1^2 r_2^2 \sin^2(\psi-\varphi)}{(r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi))^{\frac{5}{2}}} + \frac{r_1r_2\cos(\psi-\varphi)}{(r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi))^{\frac{3}{2}}},$$

$$\frac{\partial^2 E(\varphi,\psi)}{\partial\psi^2} = \frac{3r_2^2 r_3^2 \sin^2(\psi)}{(r_2^2 + r_3^2 - 2r_2r_3\cos(\psi))^{\frac{5}{2}}} - \frac{r_2r_3\cos(\psi)}{(r_2^2 + r_3^2 - 2r_2r_3\cos(\psi))^{\frac{3}{2}}}$$

$$+\frac{3r_1^2 r_2^2 \sin^2(\psi-\varphi)}{(r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi))^{\frac{5}{2}}} - \frac{r_1r_2\cos(\psi-\varphi)}{(r_1^2 + r_2^2 - 2r_1r_2\cos(\psi-\varphi))^{\frac{3}{2}}},$$

It follows that

$$\frac{\partial^2 E(\varphi,\psi)}{\partial\varphi^2}$$

$$= \frac{3r_1^2 r_2^2 \sin^2\varphi}{(r_1^2 + r_2^2 - 2r_1r_2\cos\varphi)^{\frac{5}{2}}} - \frac{r_1r_2\cos\varphi}{(r_1^2 + r_2^2 - 2r_1r_2\cos\varphi)^{\frac{3}{2}}} - \frac{\partial^2 E(\varphi,\psi)}{\partial\varphi\partial\psi}$$

and

$$\frac{\partial^2 E(\varphi,\psi)}{\partial\psi^2}$$

$$= \frac{3r_1^2 r_3^2 \sin^2\varphi}{(r_1^2+r_3^2-2r_1r_3\cos\varphi)^{\frac{5}{2}}} - \frac{r_1 r_3 \cos\varphi}{(r_1^2+r_3^2-2r_1r_3\cos\varphi)^{\frac{3}{2}}} - \frac{\partial^2 E(\varphi,\psi)}{\partial\varphi\partial\psi}$$

Let us now evaluate the Hessian matrix H_E of E at parades, i.e. for $\varphi = \pi m, \psi = \pi k$, $k, m = 0, 1$, and find out when it is non-degenerate. Its determinant is denoted by h_E, i.e.,

$$h_E(\varphi,\psi)) = \frac{\partial^2 E(\varphi,\psi)}{\partial\varphi^2}\frac{\partial^2 E(\varphi,\psi)}{\partial\psi^2} - \left(\frac{\partial^2 E(\varphi,\psi)}{\partial\varphi\partial\psi}\right)^2.$$

We compute and investigate it at each parade separately.

After simplification we obtain that, for $\varphi = \psi = 0$, the hessian of $E(\varphi,\psi)$ has the form

$$h_E(0,0) = \frac{r_1^2 r_2 r_3}{(r_2-r_1)^3(r_3-r_1)^3} + \frac{r_1 r_2 r_3^2}{(r_3-r_2)^3(r_3-r_1)^3} + \frac{r_1 r_2^2 r_3}{(r_3-r_2)^3(r_2-r_1)^3}$$

$$= r_1 r_2 r_3 \left(\frac{r_1}{(r_2-r_1)^3(r_3-r_1)^3} + \frac{r_3}{(r_3-r_2)^3(r_3-r_1)^3} + \frac{r_2}{(r_3-r_2)^3(r_2-r_1)^3}\right)$$

and therefore always

$$h_E(0,0) > 0.$$

This means that this parade is an extremum of E and it is easy to check that it is an isolated maximum of E.

In a similar way we obtain:

$$h_E(0,\pi) \tag{5}$$

$$= r_1 r_2 r_3 \left(-\frac{r_1}{(r_2+r_1)^3(r_3-r_1)^3} + \frac{r_2}{(r_2+r_1)^3(r_2+r_3)^3} - \frac{r_3}{(r_2+r_3)^3(r_3-r_1)^3}\right);$$

$$h_e(\pi,0))$$

$$= r_1 r_2 r_3 \left(\frac{r_1}{(r_2+r_1)^3(r_3+r_1)^3} - \frac{r_2}{(r_2+r_1)^3(r_3-r_2)^3} - \frac{r_3}{(r_3-r_2)^3(r_1+r_3)^3}\right);$$

$$h_E(\pi,\pi)$$

$$= r_1 r_2 r_3 \left(-\frac{r_1}{(r_2-r_1)^3(r_3+r_1)^3} - \frac{r_2}{(r_2-r_1)^3(r_2+r_3)^3} + \frac{r_3}{(r_2+r_3)^3(r_3+r_1)^3}\right).$$

Let us find out when these expressions do not vanish and determine their signs. To find the sign of $h_E(0,\pi)$ we need to analyze the expression

$$-r_3(r_1+r_2)^3 - r_1(r_2+r_3)^3 + r_2(r_3-r_1)^3$$

$$= -6r_1r_2^2r_3 - 6r_1r_2r_3^2 - r_1^3r_3 - r_1^3r_2 - r_2^3r_3 - r_1r_2^3 - r_1r_3^3 + r_2r_3^3.$$

The equality

$$-6r_1r_2^2r_3 - 6r_1r_2r_3^2 - r_1^3r_3 - r_1^3r_2 - r_2^3r_3 - r_1r_2^3 - r_1r_3^3 + r_2r_3^3 = 0$$

considered as an equation for r_3 has a unique solution, satisfying the relation $0 < r_1 < r_2 < r_3$. More precisely, denoting this unique root by $R(r_1, r_2)$ we have:

$$R(r_1,r_2)=\frac{1}{2(r_2-r_1)}\left(r_2^2+5r_1r_2+\sqrt{r_2^4+14r_1r_2^3+21r_1^2r_2^2-4r_1^4+4r_1^3r_2}\right). \quad (6)$$

Thus this parade may become a degenerate critical point of E. If it is non-degenerate then it is easy to verify that it can be either a minimum or a saddle of E.

The sign of $h_E(\pi,\pi)$ depends on the expression $r_3(r_2-r_1)^3 - r_1(r_2+r_3)^3 - r_2(r_1+r_3)^3$. Simplifying this expression, we obtain

$$r_3(r_2-r_1)^3 - r_1(r_2+r_3)^3 - r_2(r_1+r_3)^3$$

$$= -6r_1r_2^2r_3 - 6r_1r_2r_3^2 - r_1^3r_3 - r_1^3r_2 - r_2^3r_3 - r_1r_2^3 - r_1r_3^3 + r_2r_3^3$$

Solving the equation

$$-6r_1r_2^2r_3 - 6r_1r_2r_3^2 - r_1^3r_3 - r_1^3r_2 - r_2^3r_3 - r_1r_2^3 - r_1r_3^3 + r_2r_3^3 = 0$$

with respect to r_3 we obtain: $r_3^{(1)} = -r_2$,

$$r_3^{(2)} = -\frac{1}{2(r_2+r_1)}\left(r_2^2-5r_1r_2+\sqrt{r_2^4-14r_1r_2^3+21r_1^2r_2^2-4r_1^4-4r_1^3r_2}\right) \quad (7)$$

and

$$r_3^{(3)} = \frac{1}{2(r_2+r_1)}\left(-r_2^2+5r_1r_2+\sqrt{r_2^4-14r_1r_2^3+21r_1^2r_2^2-4r_1^4-4r_1^3r_2}\right). \quad (8)$$

Analogously, to investigate the sign of $h_E(\pi,0)$ we consider the expression

$$-r_3(r_1+r_2)^3 + r_1(r_3-r_2)^3 - r_2(r_1+r_3)^3$$

$$= -6r_1^2r_2r_3 - 6r_1r_2r_3^2 - r_1^3r_3 - r_1^3r_2 - r_2^3r_3 - r_1r_2^3 + r_1r_3^3 - r_2r_3^3.$$

The solutions of the equation

$$-6r_1^2r_2r_3 - 6r_1r_2r_3^2 - r_1^3r_3 - r_1^3r_2 - r_2^3r_3 - r_1r_2^3 + r_1r_3^3 - r_2r_3^3 = 0$$

with respect to r_3 are $r_3^{(1)} = -r_1$,

$$r_3^{(2)} = -\frac{1}{2(r_2-r_1)}\left(r_1^2+5r_1r_2+\sqrt{r_1^4+14r_1^3r_2+21r_1^2r_2^2-4r_2^4+4r_1r_2^3}\right) \quad (9)$$

and

$$r_3^{(3)} = \frac{1}{2(r_2-r_1)}\left(-r_1^2-5r_1r_2+\sqrt{r_1^4+14r_1^3r_2+21r_1^2r_2^2-4r_2^4+4r_1r_2^3}\right). \quad (10)$$

The roots $r_3^{(1)}$ and $r_3^{(2)}, r_3^{(3)}$ of (7) - (10) are either complex, or do not satisfy the restriction $0 < r_1 < r_2 < r_3$. It follows that if $0 < r_1 < r_2 < r_3$ then both hessians $h_E(\pi,0)$ and $h_E(\pi,\pi)$ are negative. Thus, for generic values of radii, all parades are non-degenerate. Two of them are always saddles and can change their Morse type as the value of r_3 passes over the number $R(r_1, r_2)$ given by formula (6). The proof is complete.

Remark 2. Notice that as a by-product we have found the Morse indices of non-degenerate parades.

The formulas and analysis presented above enable us to describe an interesting bifurcation phenomenon for equilibrium configurations. This phenomenon, in particular, explains the change of the number of equilibria for varying radii of given circles.

Theorem 3. *For $r_3 = R(r_1, r_2)$, the Coulomb energy exhibits a cusp catastrophe at the $(0, \pi)$-parade.*

Proof. We prove this by analyzing the Taylor expansion of E up to fourth order at the $(0, \pi)$ parade for $r_3 = R(r_1, r_2)$ given by formula (6). For clarity, we do this first for concrete values of $r_1 = 1, r_2 = 3$ and $r_3 = 12.403$. Using computer we find that the Taylor expansion of E up to fourth order at the $(0, \pi)$ parade is:

$$0.4026171589+0.01925506182\varphi^2-0.04687500000(\psi-\pi)^2\varphi+0.02852841561(\psi-\pi)^2$$

$$+0.01990518624\varphi^4 - 0.05371093750(\psi - \pi)\varphi^3 + 0.08056640625(\psi - \pi)^2\varphi^2$$

$$-05371093750(\psi - \pi)^3\varphi + 0.01517344371(\psi - \pi)^4 + o(4).$$

Notice that the Hessian in this case vanishes on a line of the form $v = ku$, where we put $u = \varphi, \psi - \pi = v$. Restricting the above expansion to this line we obviously get an expression of the form $u^4 + au^2 + b$. Now one can treat this equation using standard results of catastrophe theory [10]. Namely, applying the so-called Siersma's trick (see, e.g., [10], Chapter 8, Section 12) one can transform this expression into the canonical form of the cusp catastrophe [10], which completes the proof in this concrete case. The proof in the general case is the same but involves complected expressions so we omit it.

4. Three point charges on two concentric circles

In this section we discuss a special case of our problem arising in connection with Bohr's model of lithium involving three electrons on two orbits [2], [3]. We use the same setting and notation as above and assume that $q_i = 1$, $r_1 = r_2 = r$, $r_3 = R > r$. We also fix $A = (R, 0)$, and assume that the points B, C belong to the smaller circle. Notice that this means that electrons are labeled so that they can be distinguished in any configuration. It turns out that no bifurcations may happen in this case so that the number of equilibrium configurations is constant.

Theorem 4. *For three equal charges on two circles as above, the number of equilibria equals four.*

Proof. First of all, for symmetry reasons equilibrium configurations should be symmetric with respect to Ox-axis. Indeed, it is obvious that the resultant at point A can be collinear with Ox-axis only if the points B and C are symmetric with respect to Ox-axis. In this case, there are two parades where one of the points coincides with $(r, 0)$ and the other one coincides with $(-r, 0)$. As we have seen, for a non-aligned equilibrium configuration $P = (A, B, C)$, the segment $[B, C]$ should be orthogonal to Ox-axis. Let D be the intersection point of the segment $[B, C]$ with Ox-axis. By Lemma 1 in an equilibrium configuration point

D should have a negative abscissa. So we put $D = (-x, 0)$ with positive number $x \in (0, 1)$. Using high-school geometry one easily computes the Coulomb energy of such a configuration:

$$E(P) = \frac{1}{2\sqrt{r^2 - x^2}} + \frac{2}{\sqrt{R^2 + 2Rx + r^2}}.$$

Computing its derivative and equating it to zero, we get the equation

$$\frac{x}{2(r^2 - x^2)^{3/2}} = -\frac{2R}{(R^2 + 2Rx + r^2)^{3/2}}.$$

Clearing the denominators and squaring both sides we obtain the polynomial form of equilibrium equation:

$$x^2(R^2 + 2Rx + r^2)^3 - 4R^2(r^2 - x^2)^3 = 0. \tag{11}$$

This is a sextic equation for x depending on two parameters r and R. Our aim is to estimate possible amounts of its real solutions in $(0, 1)$ for arbitrary r and R. Obviously, the number of solutions depends only on the ratio $\frac{r}{R}$. So without loss of generality from now on we put $r = 1$. By a general principle of singularity theory the number of real solutions to (11) may change only when the value of parameter crosses a real root of its discriminant δ considered as a function of R. So we compute the discriminant of equation (11) with respect to x and investigate the distribution of its real zeros. Using computer we get

$$\delta = 151552R^{44} - 995328R^{42} + 10174464R^{40} - 147996672R^{38} + 1235091456R^{36}$$
$$-6268796928R^{34} + 21343322112R^{32} - 51658407936R^{30}$$
$$+91815026688R^{28} - 121776545792R^{26} + 120945623040R^{24} - 89130516480R^{22}$$
$$+47492136960R^{20} - 17294379960R^{18} + 374995356R^{16} - 255934464R^{14}$$
$$-70447104R^{12} + 12275712R^{10} + 262144R^{8}.$$

It is obvious that 0 is a root of multiplicity 8. It is also easy to see that dividing δ by R^8 one gets a polynomial for which number 1 is a root of multiplicity 12. Using Sturm's algorithm one can verify that δ is positive in the segments $(0, 1)$ and $(0, \infty)$. This means that the number of real roots of (11) belonging to $(0, 1)$ is the same for all $R > 1$. Putting $R = 2$ we find that it has exactly one root in $(0, 1)$ equal approximately to 0.248. So the number of solutions is always equal to four: two parades and two symmetric configurations with non-zero y-coordinate. The theorem is proven.

Remark 3. Notice that the Morse equality for Euler characteristic is fulfilled if one takes into account the two poles corresponding to coincidence of charges at each of the two end-points of the horizontal diameter. It is also easy to check that the segment $[B, C]$ corresponding to a non-aligned equilibrium tends to Oy-axis as $R \to \infty$. Finally, it is well known that, for $r = R$ there are only two stable equilibria having the form of regular triangle. Solving the equilibrium equation for $R = r$ we get that $x = \frac{r}{2}$ as is expected.

In this case one can get more strict estimates for the ground energy using representation of the real root of (11) as a power series in R.

5. Concluding remarks

In conclusion we wish to mention a few feasible research perspectives suggested by our results. More or less complete description of open problems and possible generalizations would require a separate publication.

First of all, it is natural to investigate in more depth similar problems for three non-equal charges. In all examples with non-equal charges we have computed, the number of equilibria did not exceed six. So there is good evidence for the following conjecture.

Conjecture 1. For arbitrary values of three charges and radii, there are four or six equilibrium configurations in the setting considered above.

It is also plausible that the statement of Theorem 2 holds true in more strong formulation.

Conjecture 2. For any like charges and generic values of radii, all orbital equilibrium configurations are non-degenerate critical points of electrostatic energy.

A much more far-going generalization arises if one considers Coulomb equilibria of $n > 3$ point charges assuming that each charge is confined to one of n nested circles. Notice that there is no reason to require that all charges are like. In particular, analogs of Theorems - hold true for three charges of arbitrary signs. However it is unclear if such results are related to any physical problem or real life situation. Finally, analogs of our results can be obtained for the logarithmic potential and other central forces.

R E F E R E N C E S

1. Arnol'd V., Varchenko A., Gusein-Zade S. Singularities of Diferentiable Mappings (Russian). *Moscow*, 2005.

2. Bohr N. On the constitution of atoms and molecules. Part I. *Phil. Magz.*, **26** (1913), 1-25.

3. Chen G., Ding Z., Hsu S.B., Kim M., Zhou J. Mathematical analysis of a Bohr atom model. *J. Math. Phys.*, **47** (2006), 022107.

4. Giorgadze G.K., Khimshiashvili G.N. Equilibria of point charges on nested circles. *Bull. Georgian Natl. Acad. Sci.*, **9**, 3 (2015), 43-49.

5. Giorgadze G.K., Khimshiashvili G.N. On equilibrium concyclic configurations. *Doklady Mathematics*, **98**, 1 (2018), 1-5.

6. Khimshiashvili G. Equilibria of constrained point charges. *Bulletin of the Georgian National Academy of Sciences*, **7**, 2 (2013), 15-20.

7. Khimshiashvili G., Siersma D. Connecting cycles for concentric circles. *Bull. Georgian Natl. Acad. Sci.*, **13**, 1 (2019), 13-21.

8. Khimshiashvili G., Panina G., Siersma D. Equilibria of point charges on convex curves. *J. Geom. Phys.*, **98**, 1 (2015), 110-117.

9. Khimshiashvili G., Panina G., Siersma D. Equilibria of three contrained point charges. *J. Geom. Phys.*, **106**, 1 (2016), 42-50.

10. Poston T. Stewart I. Catastrophe Theory and its Application. *Pitman*, 1978.

11. Smale S. Mathematical problems for the next century. *Mathematical Intelligencer*, **20**, 2 (1998), 7-15.

12. Svidzinsky A., Scully M., Herschbach D. Bohr's 1913 molecular model revisited. *Proc. Natl. Acad. Sci.*, **102** (2005), 11985-11988.

13. Wang T., Xia G., Zhao N. Reconfiguration and control of non-equal mass three-craft Coulomb formation. *The Journal of the Astronautical Science*, **63**, 1 (2016), 47-59.

Received 15.11.2019; accepted 26.11.2019.

Authors' addresses:

G. Giorgadze
I. Vekua Institute of Applied Mathematics
of I. Javakhishvili Tbilisi State University
2, University str., Tbilisi 0186
Georgia
E-mail: gia.giorgadze@tsu.ge

G. Khimshiashvili
School of Natural Sciences and Engineering
Ilia State University
3/5, Kakutsa Cholokashvili ave., Tbilisi 0162
Georgia
E-mail: giorgi.khimshiashvili@iliauni.edu.ge