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THE IMPACT OF BOJARSKI'S WORKS ON THE THEORY OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS ON THE PLANE

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In memory of **Bogdan Bojarski**

Abstract. The period of the formation of the theory of generalized analytic functions, nowadays known as the Bers-Vekua theory, meets the beginning of scientific activity of Bogdan Bojarski together with his supervisor Ilia Vekua in Moscow. Bojarski's fundamental new approach to the solution of the system of elliptic partial differential equations on a plane, opened up a direct pathway to many important issues in the geometric theory of analytic functions and related boundary value problems. In this paper we give a short overview of important results of Bogdan Bojarski in the theory of generalized analytic functions and his point of view on the theory of boundary value problems.

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Mathematics and love of Georgia as profession

In the era of globalization and scientific breakthroughs in which we've come to live and work, the Georgian mathematical school is noticeable by the monographs: "Singular Integral Equations" of N. Muskhelishvili and "Generalized Analytic Functions" of I. Vekua. In each of these monographs there are unique paragraphs that belong to Bogdan Bojarski, which he had written especially for the above books at the request of the authors.

Bogdan Bojarski spent his childhood in Poland occupied by fascist Germany. He prematurely graduated from the Lodz University and as it was the tradition for the outstanding youth of East Europe was sent to the Moscow State University for further studies. In Moscow his extraordinary scientific career took off under the guidance of Academician Ilia Vekua. It was the time when the generalized analytic functions theory was being created. In 1955-1965 B. Bojarski contributed with high-quality works in the theory [1-7] and at the age of 28 he was granted the doctor's degree by the prestigious Steklov Mathematical Institute.

He returned to Poland, occupying the position of a professor in the Warsaw University and mentored a whole league of mathematicians, who are actively engaged in the leading universities of Europe and US today. In 1970-1981he was the director of the Mathematical Institute of the Warsaw University, in 1986-2002 he leaded the Mathematical Institute of the Polish Academy of Sciences and the Banach International Center of Mathematics. He founded an international research and conference center in Bedlevo. B. Bojarski was a distinguished scientist and organizer of science in Europe. He had received several supreme state awards.

He was a ordinary member of the Polish Academy of Sciences. Was elected as an honorary doctor at the Tbilisi State University.

In the 60ies he arrived in Tbilisi and fell in everlasting love with Georgia. Over half a century he made friends with various generations of Georgian mathematicians. The recent achievements of the Department of Complex Analysis of the I. Vekua Institute of Applied Mathematics were greatly conditioned by his support and collaboration with the above department.

Due to this legacy, but mostly to his vast human traits, we dedicate this narrative to our great friend, colleague, co-author and teacher.

1. Generalized analytic functions and quasi-conformal mapping

The systematic study of elliptic systems of differential equations using the methods of singular integral equations was initiated by I. Vekua in the middle of the twentieth century.

Consider the general system of partial differential equation of two variables of the form

$$v_y = \alpha u_x + \beta u_y + au + bv + e, \tag{1}$$

$$-v_x = \gamma u_x + \delta u_y + cu + dv + f. \tag{2}$$

The elipticity condition takes the form

$$4\alpha\beta - (\beta + \alpha)^2 > 0.$$

System (1),(2) is uniformy elliptic if

$$4\alpha\beta - (\beta + \alpha)^2 \geq K_0 > 0$$

and the coefficients $\alpha, \beta, \gamma, \delta$ in (1),(2) are uniformly bounded.

The real system (1),(2) can be written in the complex form

$$w_{\overline{z}} - \mu w_z - \nu \overline{w}_z = Aw + B\overline{w} + C \tag{3}$$

which is a linear differential equations over the field of reals. Here we used the notations:

$$w = u + iv, \quad w_z = \frac{1}{2}(w_x - iw_y), \quad w_{\overline{z}} = \frac{1}{2}(w_x + iw_y).$$

The particular cases of equation (3) are the following important equations:

Beltrami equation

$$w_{\overline{z}} - \mu(z) w_z = 0, \tag{4}$$

general Beltrami equation

$$w_{\overline{z}} - \mu w_z - \nu(z)\overline{w}_z = 0 \tag{5}$$

and *Carleman-Bers-Vekua* equation

$$w_{\overline{z}} = Aw + B\overline{w}. \tag{6}$$

Equations (4), (5) at the point z are elliptic if

$$|\mu(z)| < 1 \quad \text{or} \quad |\mu(z)| + |\nu(z)| < 1.$$

The ellipticity is uniform in a subdomain U of the complex plane $\mathbb{C}$ if for some constant k

$$|\mu(z)| + |\nu(z)| \leq k < 1$$

for all $z \in U$.

The functions μ and ν are assumed measurable complex valued functions of z, and $A, B, C \in L_p(U)$, where $p > 2$ is a given number.

The generalized regular solution $w(z)$ of (3) is a function which is continuous in U and possesses generalized L_p derivatives w_z and $w_{\bar{z}}$ in U, i.e., $w(z) \in W_p(U)$. The functions $w_z, w_{\bar{z}}$ and w almost everywhere in U satisfy (3). As a sample of the results we quote: to each entire function $f(z)$ there corresponds a unique generalized regular solution $w(z)$ such that $w(z)$ is continuous in the entire plane $\mathbb{C}$ analytic in the complement of U, $w(z) \in W_p(\mathbb{C})$, and $w(z)\ f(z)$ for $z \to \infty$.

To solve this problem, the following method was applied. Solution is sought in the form

$$w(z) = f(z) - \frac{1}{\pi} \int\int \frac{\omega(t)}{t-z} dU = f(z) + T(\omega). \tag{7}$$

It is readily shown that ω must satisfy the equation of the form

$$\omega - q_1 S(\omega) - q_2 \overline{S(\omega)} = AT(\omega) + B\overline{T(\omega)} + C_0,$$

where C_0 is known and is 0 outside U, and

$$S(\omega) = -\frac{1}{\pi} \int\int \frac{\omega(t)}{(t-z)^2} d\sigma_t.$$

S can be estimated by the Calderron-Zygmund inequality and the problem is ultimately reduced to the fixed point theorem.

The Beltrami equation, which expresses the conformality of a mapping with respect to the Riemann metric, may be written in the complex form (4) for the case of a Hölder continuous μ. This has been done by Vekua. Bojarski extended Vekua's note for a bounded measurable μ [2],[4].

The solution of the form

$$w(z) = z - \frac{1}{\pi} \int\int \frac{\omega(t)}{t-z} dU$$

is called *principal homeomorphism* by terminology of Vekua and Bojarski. Here is one of possible formulations of main results on quasi-conformal mapping, which is known as *Ahlfors-Bers-Bojarski* theorem.

Theorem 1. *[3] Let D be the unit disc and assume that μ satisfies a Hölder condition at the origin. Let $f(\chi)$ be an analytic function defined in the unit disc. Then there exists a homeomorphism $\chi(z)$ of the unit disc onto itself which keeps the origin fixed and is such that the function (4) is a solution of (7).*

The proof is based on the solution of the nonlinear differential equation

$$\chi_{\bar{z}} = q(z) \frac{\overline{f'(\chi)}}{f'(\chi)} \overline{\chi}_{\bar{z}} \tag{8}$$

which must be satisfied by the function χ. A solution is sought in the form $\chi(z) = e^{\psi}$, where f is required to be imaginary on the unit circle. Transforming(8) into the integral equation for the function $\psi_{\overline{z}}$ we can use the methods of integral equations. In particular, the obtained integral equation is solved by means of the fixed-point theorem, the necessary a priori estimate being a consequence of the Calderron-Zygmund inequality.

Furthermore, the formal rules of computing derivatives in new variables are valid. The inverse mapping has the same properties. This implies, in particular, that every solution of the Beltrami system is an analytic function of the solution considered, and that the Jacobian of the constructed mapping is different from zero almost everywhere. The construction also implies that the derivatives of the solution w are absolutely integrable in some power $p > 2$.

For the behaviour of the complex dilatation $\mu_f = \frac{w_{\bar{z}}}{w_z}$ under composition of the quasi-conformal mappings $f = w \circ v^{-1}$ expressed by simple but important formula

$$\mu_f = \left[\frac{\mu_w - q_v}{1 - \bar{\mu}_v \mu_w} \frac{v_z}{\overline{v_z}} \right] \circ v^{-1}.$$

B. Bojarski also showed that the solution w takes measurable sets into measurable sets, the sets of zero measure into the sets of zero measure and continuous functions having generalized L_2 derivatives into functions with the same properties. The constructed solution gives a homeomorphic mapping of the whole plane onto itself.

Theorem 2. *[3] Every solution of (4) may be written in the form*

$$w(z) = f[\chi(z)] \tag{9}$$

where f is analytic and χ is a homeomorphism of the whole plane onto itself satisfying together with its inverse a uniform Hölder condition.

Deep results, given above, originated the investigations of analytical, algebraic and topological structures of Beltrami equation itself and for its solutions space as main object for many applications in modern mathematical physics. For example, as mentioned above Beltrami equation (4) has a solution if $|\mu(z)| < 1$ and it is known that under various assumptions, there exist the solutions of (4) when $|\mu|_\infty = 1$. This imply the set $\{z : |\mu(z)| > 1 - \epsilon\}$ is small when ϵ is small.

These properties can be extended for the Beltrami equation with two characteristics (5), where μ and ν are measurable functions with $|\mu| + |\nu| < 1$. Let

$$K(z) = \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|}.$$

f is called a regular solution of the equation (5) if $f \in W^{1,1}$, $K \in L^1_{loc}$ and $J_f \neq 0$ almost everywhere.

The regular solution of (5) exists on a domain U if for every $z \in U$ and some $\epsilon < \mathtt{dist}(z, \partial U)$, there is a one-parameter family of the functions $\psi_{z,\delta}$ defined on the positive reals so that

$$\int K(u) \psi^2_{z,\delta}(w - z) dx dy = o\left(\int_\delta^\epsilon \psi_{z,\delta}(t) d(t) \right),$$

where the left-hand integral is over the annulus $d < |w - z| < \epsilon$. This technical-looking result implies a number of "easy-looking" consequences. For example, the Beltrami equation (5) has a regular solution if

$$\limsup_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{B(z,\epsilon)} K(u) dx dy < \infty$$

for every $z \in U$, or more generally, if $K \leq Q$ for some Q in the space FMO (finite mean oscillation, a generalization of the well-known space BMO)[32].

Bojarski referred in [17],[19] on the priority of I. Vekua school in the development of a new approach for the Beltrami equation and in the solution of the key problems of this equation (see also a short version of this work [18]). Bojarski's research gave great impetus to the understanding of the quasi-conformal mappings and the geometric structure of the solutions space of the partial differential equations (see *K. Astala, T. Iwaniec, G. Martin. Elliptic Partial Differential Equations and Quasi-conformal Mappings in the Plane. Princeton Uni.Press. 2009*)

On the contribution of Vekua and Bojarski in the theory of the quasi-conformal mappings (Theorem 1 and Theorem 2) in 1978 Lars V. Ahlfors' writed (*Fields Medallists' Lectures, Ed. M.Atiyah ans D.Iagolnitzer, World Scientific, 1997*): "It must be clear that I am condensing years of research into minutes. The fact is that the post-Teichmüller era of quasiconformal mappings did not start seriously until 1954. In 1957 I. N. Vekua in the Soviet Union proved the existence and uniqueness theorem for the Beltrami equation, and in the same year L. Bers discovered that the theorem had been proved already in 1938 by C. Morrey. The great difference in language and emphasis had obscured the relevance of Morreys paper for the theory of q.c. mappings. The simplest version of the proof is due to B. V. Boyarski who made it a fairly straightforward application of the Calderon-Zygmund theory of singular integral transforms."

2. The Riemann-Hilbert boundary value problem (RHbvp) and partial indices of matrix functions

Let Γ be a smooth closed positively oriented loop in $\mathbb{CP}^1$ which separates $\mathbb{CP}^1$ into two connected domains U_+ and U_-. Suppose $0 \in U_+$ and $\infty \in U_-$. Let us denote by Ω the space of all Hölder-continuous matrix functions $f : \Gamma \to GL_n(\mathbb{C})$ with the natural topology.

RHbvp. *Find a piecewise holomorphic vector function $\Phi(t)$ in $U_+ \cup U_-$, which admits continuous boundary values on Γ and satisfies on Γ the boundary condition*

$$\Phi^+(t) = f(t)\Phi^-(t), t \in \Gamma$$

and has finite order at ∞.

Let the matrix function $X(z)$ be a solution of the RHbvp. It is called *canonical* if it has the form

$$\chi(z) = \chi_0(z) \text{ on } z \in U_+, \chi(z) = \chi_0(z) D^{-1}(z), \text{ on } z \in U^-,$$

where $\chi_0(z)$ is a holomorphic matrix function in $U^+ \cup U^-$, admitting a continuous inverse $\chi_0^{-1}(z)$ in $\overline{U}^+$ and $\overline{U}^-$, respectively, including the point $z = \infty$ and

$\mathsf{det}\chi_0(\infty) = 1$. The matrix function $D(z)$ is diagonal $D(z) = \mathsf{diag}(z^{k_1}, z^{k_2}, ..., z^{k_n})$ and the integers $k_1, k_2, ..., k_n$ satisfy the inequalities

$$k_1 \geq k_2 \geq ... \geq k_n.$$

It is known that, for every $f(t) \in \Omega$ the canonical solution always exists. The integer valued vector $K = (k_1, ..., k_n)$ does not depend on the considered canonical solution.

The integers $k_1, k_2, ..., k_n$ are called the *partial indices* of the boundary problem or of the matrix function $f(t)$. The *global index* k of the above boundary value problem may be calculated by the formula:

$$k = k_1 + k_2 + ... + k_n \text{ with } k = \frac{1}{2\pi}\Delta_\Gamma \mathsf{argdet} G(t)$$

Let

$\Omega^+ = \{f \in \Omega : f$ is the boundary value of the matrix function holomorphic in $U^+\}$.

$\Omega^- = \{f \in \Omega : f$ is the boundary value of the matrix function holomorphic in U^- and is regular at infinity $f(\infty) = \mathbf{1}\}$.

Any matrix function $f \in \Omega$ can be represented as

$$f(t) = f^-(t) d_K f^+(t), \tag{10}$$

where $f^\pm \in \Omega^\pm$ and d_K is a diagonal matrix $d_K = diag(t^{k_1}, ..., t^{k_n})$ satisfying the condition $k_1 \geq ... \geq k_n$.

The diagonal matrix d_K will be called the *characteristic loop* of the corresponding matrix function, $K = (k_1, k_2, ..., k_n)$ is called the *characteristic multi-index* or *partial indices* of f. Two matrix functions $f, g \in \Omega$ are called equivalent, if f and g have identical characteristic multi-indices.

For $K = (k_1, k_2, ..., k_n)$, denote by Ω_K the set of equivalence classes of loops Ω. The representation (10) is not unique, but if one fixes f^+ (or f^-) then f^- (respectively f^+) will be uniquely defined.

The partial indices of a matrix function are called *stable*, if in its sufficiently small neighborhood all matrix functions have the same partial indices.

The topological space Ω decomposes into a countable number of open components

$$\Omega^k = \{f \in \Omega, \Delta_\Gamma argdet f(t) = 2\pi k\}, \Omega = \cup_k \Omega^k, k \in \mathbb{Z}.$$

One has $\Omega^k = \cup_K \Omega_K$ and Ω^k is connected.

Theorem 3. *[5] The matrix functions $f_1(t)$ and $f_2(t)$ belong to the same Ω^k iff $f_1(t)$ and $f_2(t)$ are homotopic.*

Theorem 4. *[5] The set of partial indices is stable iff $|k_i - k_j| \leq 1$, $i, j = 1, 2, ..., n$.*

From the above theorems it follows that in Ω^k for every k there exists a diagonal matrix with stable partial indices $(p+1, p+1, ..., p+1, p, p, ..., p)$, where $k = np + r$, $0 \leq r < n$ and every matrix function can be transformed into such a stable diagonal matrix by elementary operations. Besides, the multi-index K as a function of $f \in \Omega^k$ has discontinuities only on the strata Ω_K.

The partial indices as the splitting type of holomorphic vector bundles on the Riemann sphere and the structures of spaces Ω_K, Ω^k for stable partial indices, played an important role in the final solution of Hilbert 21th problem [32].

Theorem 5. *[3],[5] Let $0 < k < n$, then in Ω^k, among the strata Ω_K the only ones which are open and dense subspaces are the ones with $K = (1, ..., 1, 0, ..., 0)$, i. e. for such K, $\Omega^k \setminus \Omega_K$ does not contain interior points.*

In the particular case, when $k = np$ and $k = 0$ we have

1) If $k = np$, from the stability of the partial indices it follows that $K = (p, p, ..., p)$.

2) If $k = 0$ and $K = (k_1, ..., k_n)$ is stable then $K = (0, 0, ..., 0)$.

Consider other setting of the boundary value problem.

Let $U^+ = \cup_{k=1}^{n} U_k$ be a multiply connected domain, where D_k are disjoint domains in the complex plane $\mathbb{C}$ bounded by simple smooth curves L_k. All curves are oriented counterclockwise. Let U be the complement of $\overline{U^+}$ in the extended complex plane $\overline{\mathbb{C}}$, and $L = \cup_{k=1}^{n} L_k$. Let $a(t), b(t)$ and $c(t)$ be Hölder-continuous functions on L, and let $a(t) \neq 0$.

Find a function $\varphi(z)$ that is analytic in U, U^+, continuous on $\overline{U}$, $\overline{U^+}$ and such that

$$\varphi^+(t) = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + c(t), \quad t \in L.$$

Here $\varphi^{\pm(t)}$ denote the limit values of $\varphi(z)$ as $z \to t$ from U^+ and from U, respectively. From this, in particular, it follows that $\varphi(\infty) = 0$.

If $|b(t)| < |a(t)|$, then for a simple connected domain in [8] the solvability condition of the problem is obtained. Generalization of the main theorem from [8] to the case of the multiply connected domain U^+ is considered in [33] and solvability condition from [33] is applied to the problem with piecewise constant coefficients $a(t), b(t)$. In this case, the convergence of the method of successive approximation for solving integral equations associated with the boundary value problem is proved [33].

The study of the classical Riemann-Hilbert problem was crucial to understand one-dimensional singular integral equations and the corresponding index problem. It has contributed essentially to the creation of the general abstract theory of Fredholm operators [14]. B. Bojarski with his followers [28],[29] investigated the boundary value problem from the point of view of homological algebra and success applied to the modern language of bordism theory, index theory of elliptic operators on manifolds and methods of C^*-algebra. The notion of a Fredholm pair of closed subspaces of a Hilbert (or Banach) space and the closely related notion of a Fredholm correspondence can be considered as a convenient generalization of the notion of a Fredholm operator. He investigated some topological and differential properties of the restricted Grassmannians of a polarized Hilbert space and proved the existence of a natural Fredholm manifold structure modeled by the space of compact operators in the corresponding Schatten ideal. This and many other results obtained in [28],[29], are key subjects for understanding the modern theory of elliptic equations in general.

One of the preconditions of I.Gelfand's conjecture that the index of elliptic boundary value problem depends only on the homotopy class of the boundary curve (known today as Atiyah-Singer index theorem) was Bojarski's work [1]. After the publication of Atiyah-Singer theorem, Bojarski's papers [10], [11] dedicated to the topological nature of the index of boundary value problem, are important not only from the viewpoint of the history. They may be considered as the maximal possible expansion of the methods of classical approach to the partial differential equations and the boundary value problems as well.

3. The theory of generalized analytic vectors and related topics

B Bojarski showed that the methods of generalized analytic functions have far-reaching generalizations [6],[13]. To be more precise, a vector $w(z) = (w_1, ..., w_n)$ is called a *generalized analytic vector* [6], [7], [13] in the domain U if it is a solution of an elliptic system of the form

$$\partial_{\bar{z}} w - Q(z)\partial_z w + A(z)w + B(z)\bar{w} = 0 \tag{11}$$

where $A(z), B(z)$ are given quadratic matrices of order n of the class $L_{p_0}(U), p_0 > 2$, and $Q(z)$ is a matrix of the following special form: it is quasi-diagonal and every block $Q^r = (q^r_{ik})$ is a lower (upper) triangular matrix satisfying the conditions

$$q^r_{11} = ... = q^r_{m_r,m_s} = q^r, \quad |q^r| \le q_0 < 1,$$

$$q^r_{ik} = q^r_{i+s,k+s} \ \ (i+s \le n, k+s \le n).$$

Moreover, we suppose $Q(z) \in W^1_p(\mathbb{C}), p > 2$, and $Q(z) = 0$ outside of some circle.

The equation

$$\partial_{\bar{z}} w - \partial_z (Q^T w) - A^T(z)w - \overline{B^T(z)w} = 0 \tag{12}$$

is called *conjugate* to the equation (11), and T denotes a transposition of a matrix.

If $A(z) \equiv B(z) \equiv 0$, equation (11) and (12) passes into

$$\partial_{\bar{z}} w - Q(z)\partial_z w = 0 \tag{13}$$

and

$$\partial_{\bar{z}} w - \partial_z (Q^T w) = 0. \tag{14}$$

The solutions of the equation (13) are called $Q-$ *holomorphic vectors.*

Theorem 6. *[6], [13] The equation* (13) *has a solution of the form*

$$\zeta(z) = zI + T\omega, \tag{15}$$

where I is the unit matrix and $\omega(z)$ is a solution of the equation

$$\omega(z) + Q(z)\Pi\omega = Q(z)$$

belonging to $L_p(\mathbb{C}), p > 2$.

The solution (15) of the equation (13) is analogous to the *fundamental (principal) homeomorphism* of the Beltrami equation.

Consider the following boundary value problem for (11) type nonhomogeneous elliptic system

$$w_{\bar{z}} - Qw_z = Aw + Bw + F.$$

Unknown w is a vector of n complex valued functions. Q is a triangular matrix vanishing outside of some large circle and admitting generalized derivatives Q_z, $Q_{\bar{z}}$ in $L_p, p > 2$. The elements of the matrices A and B are functions in L_p. Let U be a multiply connected domain in the complex plane with the boundary of $m+1$ Liapunov curves. In U the above system is assumed strongly elliptic.

In the *problem of Riemann* a solution should be Hölder continuous in $U+L$ and satisfy

$$\Re[G(t)w(t)] = f(t)$$

on L. The solutions $w_+(t)$ are sought in $U+L$ and $w_-(t)$ in $\mathbb{C} - U$ so that

$$w_+(t) = G(t)w_-(t) + h(t)$$

on L.

These problems are reduced to of certain singular integral equations and solution is given by the following theorem.

Theorem 7. *[6] Necessary and sufficient solvability conditions for inhomogeneous Hilbert problem is*

$$\Im \int_L (h(t), (Edt + Q'dt)f) = 0$$

for all solutions f of the adjoint homogeneous Hilbert problem.

A relation between the numbers l and l' of the solutions of the problem and its adjoint is:

$$l - l' = 2\kappa - n(m-1),$$

where $\kappa = \frac{1}{2\Pi}\Delta_L \arg detG(t)$.

By analogy with the one dimensional case,consider regular solutions of systems of $2n$ elliptic partial differential equations presented in a complex form

$$\partial_{\bar{z}} f(z) = A(z)f(z) + B(z)\overline{f(z)}, \tag{16}$$

where $A(z)$, $B(z)$ are bounded matrix functions on the domain $U \subset \mathbb{C}$ and $f(z) = (f^1(z), ..., f^n(z))$ is an unknown vector function and matrix elliptic system of the form:

$$\partial_{\bar{z}} \Phi(z) = A(z)\Phi(z). \tag{17}$$

For the system (17) an analogue of the similarity principle is given by the following theorem

Theorem 8. *[13] Each solution of the matrix equation (17) in U can be represented as*

$$\Phi(z) = F(z)V(z), \tag{18}$$

where $F(z)$ is an invertible holomorphic matrix function in U, and $V(z)$ is a single-valued matrix function invertible outside $\overline{U}$.

Problem RH. *Find a piecewise-regular solution of system* (16) *on the whole plane $\mathbb{C}$ equal to zero at infinity and satisfying on Γ the following boundary condition*

$$W^+ = G(t)W^-, \tag{19}$$

where $detG(t) \neq 0$ on Γ.

This problem is solvable using the methods of the singular integral equations and it is known that the number of linearly independent solutions on $\mathbb{R}$ is finite. Denote this number by l. Let $k = \frac{1}{2\pi}\Delta_\Gamma \mathtt{argdet} G(t)$, as above, be the index of the problem RH. It is known that $l \geq max(0, 2k)$ and it is possible to choose a matrix function $G(t)$, such that $l(G) = s$, for every given number $s \geq max(0, 2k)$, and therefore it is possible to consider the number l as a function of $G(t)$. The index of the problem is a topological invariant and in the one dimensional case it is a complete invariant. It is known also that in the multi-dimensional case the index is not a complete invariant, but in the *stable* case, the index defines all invariants of the problem. The number l as the function of $G(t)$ is stable, if $l(G) = l(G_1)$ for all nondegenerate matrix functions on Γ, which are sufficiently close to $G(t)$.

Theorem 9. *The number l is stable iff $l = max(0, 2k)$.*

Let $C(t)$ be any matrix function on Γ and $C(t) \in \Omega$, which has a holomorphic extension to U^+, not necessarily nonsingular everywhere, and let

$$\frac{1}{2\pi}\Delta_\Gamma \mathtt{argdet}(G^{-1}C) = 0,$$

then there exists an extension of $G^{-1}C$ to U^+. Denote by $P(z)$ this extension and let $\Phi(z)$ be some holomorphic solution of the RHbvp. Consider the substitution

$$w(z) = P(z)\Psi(z) \text{ on } z \in U^+; w(z) = \Psi(z) \text{ on } z \in U^-.$$

Proposition 1. *The matrix function $\Psi(z)$ is holomorphic in $U^+ \cup U^-$ iff w is a solution of the system*

$$\partial_{\overline{z}} w = Aw, \tag{20}$$

where $A(z) = \partial_{\overline{z}} PP^{-1}$, for $z \in U^+$ and $A(z) = 0$, for $z \in U^-$. Let the index of the problem be k and let $C(t)$ be a diagonal matrix function with diagonal entries $diagC(t) = (t^p, ..., t^p)$.

Theorem 10. *The matrix function $G(t) \in \Omega^k$ iff the Liouville theorem holds for the system (20).*

Proof of this theorem follows from the following arguments from the theory of singular integral equations. Fulfilment of the Liouville theorem for the solution of the system (20) is equivalent to the existence of a solution of the following matric system of singular integral equations

$$B(z) + \frac{1}{\pi}\int\int_U \frac{WB}{t-z} dU_t = I, \tag{21}$$

where I is an identity matrix. On the other hand $G(t) \in \Omega^k$ iff the system (21) is solvable with respect to $B(z)$.

The modern language of the boundary value problems is given in papers [15], [32], where geometric and algebraic character of the subject is explained and the linkage is outlined to related topics.

4. Real analysis and mechanics

One of the most important research directed by I. Vekua and his school concerns the deformation of surfaces and corresponding problems of mechanics. Besides, it is known that the Beltrami equation is the equation that has to be solved in order to construct isothermal coordinate systems on surfaces. Therefore, Bojarski's interest towards the problems of mechanics is natural. Some of his works are in this direction, among them the first is a joint work with I. Vekua [22] and the second is with the famous geometer V. Efimov [23]. As is well known, for a regular (twice differentiable) surface $z = z(x, y)$, no part of which is planar, the component ζ (along the z-axis) of an infinitesimal bending field $\tau(\xi, \eta, \zeta)$ attains its maximum and its minimum on the boundary of the surface. In [12],[22],[23] the rigidity of piecewise regular closed convex surfaces of non negative curvature is proved, the rigidity of closed convex piecewise twice differentiable surfaces also is proved.

The equation (3) and system of equations (11) give a very rich theory when the partial derivatives are considered in a Sobolev sense. Thus the main functional space which is naturally related to these equations and their partial case is a Sobolev space. The area of Bojarski's interest during his whole scientific activity was the investigation of the interior structure of a Sobolev space. He dedicated some remarkable works to the investigation of analytic and geometric properties of this space. Obtained results are very deep and elegant [16], [20], [21], [26-27], [30], [34-35]. They naturally fill up the theory of elliptic differential equations.

Last years of life

In recent years Bogdan Bojarski visited Georgia several times and participated in various scientific events in Tbilisi and Batumi. In 2011 and 2013 he was a plenary speaker and board committee member at international conferences held in Tbilisi, among which was the one dedicated to the 50th anniversary of founding of the Georgian Academy of Sciences (2011) and the 110th anniversary of its first president Nikoloz Muskhelishvili. Another event was dedicated to the 95th anniversary of founding of the Tbilisi State University and the 45th anniversary of the I. Vekua Institute of Applied Mathematics (2013).

In 2011 during the international conference held in Tbilisi on the initiative of the Department of Complex Analysis together with B. Bojarski we designed the long-term work plan for the department with the purpose of further development of the generalized analytic functions theory and expanding its methodological application. Our last meeting took place in 2016 in the suburb of Warsaw in the villa of Prof. Bojarski. Despite the fact that he was already ill at that time, we still held a very meaningful conversation concerning the Atiyah-Singer Index Theorem, namely we touched upon the interest and contribution that he and Georgian mathematicians at an the early stage of the development of this theorem. We also talked about our old topics such as the "complex geometry of the real world". Since then we hoped we would have the opportunity to continue this conversation in Georgia.

To our question, *Pan Bogdan, when will you come to Tbilisi?* he replied with his sincere smile.

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