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THREE POINT CHARGES ON A FLEXIBLE CONTOUR

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Abstract. Equilibrium configurations of three mutually repelling point charges confined to a flexible contour of fixed length are discussed. For given values of charges and perimeter, we compute all possible equilibrium configurations and critical values of Coulomb energy. Moreover, for any triangle with the given perimeter, we compute the values of three charges such that this triangle is congruent to their equilibrium configuration in isoperimetric setting.

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Introduction

We deal with equilibrium configurations of three mutually repelling point charges confined to a flexible contour of fixed length L. For definiteness, in the sequel we assume that all charges are positive. This problem was formulated in [1] as an analog of the isoperimetric problem for interacting particles studied by P.Exner [2]. In the setting of [1], it is known that equilibrium configurations are realized either by a triangle of perimeter L or by aligned configurations, where all charges belong to a line segment of length $L/2$ [1]. In this paper we aim at describing the exact shapes of equilibrium configurations and consider this problem in the context of an appropriate configuration space.

Namely, for any four positive numbers $(q_1, q_2, q_3; L)$ we study the problem $E(Q; L)$ $(Q = (q_1, q_2, q_3))$ rigorous formulation of which is given below. To this end we consider the set of L-isoperimetric triangles $\triangle(L)$ consisting of all triples of points (p_1, p_2, p_3) in the plane such that not all of these points coincide and the perimeter of triangle $\triangle p_1p_2p_3$ is equal to L. The configuration space $Per_3(L)$ is defined as the factor-space of $\triangle(L)$ over the natural diagonal action of the group $Iso_+(\mathbb{R}^2)$ consisting of all orientation preserving isometries of the plane. It is known that $Per_3(L)$ can be naturally identified with the complex projective line $\mathbb{CP}^1$ (see, e.g., [3]). So one can endow it with a smooth structure and Riemannian metric inherited from $\mathbb{CP}^1$ and consider $Per_3(L)$ as a two-dimensional orientable Riemannian compact smooth manifold isometrically diffeomorphic to the two-dimensional unit sphere S^2 in three-dimensional Euclidean space.

Notice that there is a natural involution S on $Per_3(L)$ which changes the orientation of the triangle $\triangle p_1p_2p_3$ by changing the order of any two vertices, i.e. acts as transposition of indices. Obviously, the fixed points of S in $Per_3(L)$ consist of degenerate (aligned) triangles with all vertices on one line. It is easy to check that the set $\mathtt{Fix}S$ of all fixed points of S is represented by a simple loop (non-intersecting smooth closed curve) Y in $Per_3(L)$.

Remark 1. It can be shown that Y is a metric circle with respect to the Riemannian metric induced on $Per_3(L)$ from the Fubini-Study metric on $\mathbb{CP}^1$. We will not use the latter fact in the sequel.

Notice also that $Per_3(L)$ contains two distinguished points N, S corresponding to a regular triangle of perimeter L taken with the positive and negative orientations, respectively. Moreover, for any $L > 0$, $Per_3(L)$ contains three points A_1, A_2, A_3 corresponding to three degenerate triangles of perimeter L with two coinciding vertices. In our notation A_1 corresponds to coinciding p_2 and p_3, and analogously for A_2, A_3. Obviously, points A_1, A_2, A_3 belong to the loop $Y = \mathtt{Fix}S$ defined above.

For any triple of pairwise distinct points $P = (p_1, p_2, p_3)$ in the plane, the (normalized) *Coulomb potential* of charges (q_1, q_2, q_3) placed at p_1, p_2, p_3 is defined as

$$E(Q, P) = \frac{q_1 q_2}{d_{12}} + \frac{q_2 q_3}{d_{23}} + \frac{q_1 q_3}{d_{13}}, \tag{1}$$

where d_{ij} is the distance between p_i and p_j. This function is obviously invariant with respect to the mentioned isometric action of $Iso_+(\mathbb{R}^2)$ on $\triangle(L)$ so it defines a smooth function E on an open subspace $X = Per_3(L) - \{A_1, A_2, A_3\}$ of configurations consisting of pairwise distinct points.

Since E is a smooth function on X one may consider its critical points and interpret them as the equilibrium configurations of given charges confined to a flexible contour of length L. Accepting terminology used in [1] the problem $E(Q; L)$ will be referred to as *direct problem of electrostatics* (DPE) in isoperimetric setting.

Conversely, given a triangle $\triangle ABC$ with perimeter L one may wonder if there exist values of charges which will be in equilibrium if placed at vertices of $\triangle ABC$. Any three non-zero numbers satisfying this requirement are called *stationary charges* for $\triangle ABC$. By analogy with [1] the problem of finding stationary charges is called the *inverse problem of electrostatics* (IPE) in isoperimetric setting. In the sequel we present complete solutions of these two problems for configurations of three charges and some corollaries of the main results.

1. Isoperimetric equilibria of three charges

To find all critical points of E in isoperimetric setting we need a few more definitions. For each $i = 1, 2, 3$, an *aligned solution* τ_i to the problem $E(Q; L)$ is defined as the following configuration of point charges: charges q_j and q_k (here i, j, k is a cyclic permutation of the numbers $1, 2, 3$) are at the ends of line segment I_i of length $L/2$ and the charge q_i placed in this segment at such point t_i that q_i is in equilibrium in I_i in the electric field created by the end-point charges q_j and q_k (here i, j, k is a cyclic permutation of indices $1, 2, 3$). It is easy to verify that the signed distance between q_i and the charge q_j equals

$$z_i = \frac{q_j - \sqrt{q_j q_k}}{2(q_j - q_k)} L.$$

Notice that each configuration τ_i is a particular case of the mathematical model of linear ion trap considered in [4].

A point in $Per_3(L)$ corresponding to the configuration τ_i is denoted by T_i. It is easy to verify that, for each $i = 1, 2, 3$, T_i is a critical point of E in X. Obviously, each T_i belongs to the loop Y introduced above. Physical considerations show that T_i should be of saddle-point type: stable within the segment I_i and non-stable in the transversal direction. In fact, a direct computation of the hessian matrix of E at T_i given in [4] shows that T_i is a (Morse) non-degenerate critical point of Morse index one, i.e., T_i is indeed a non-degenerate saddle point of E in X. It is then easy to verify that Y does not contain critical points of E different from T_1, T_2, T_3 because any other position of the interior charge in the segment is not stationary within this segment. This follows from the results of [5] and can be verified by direct computation of the differential of E.

Since critical points T_1, T_2, T_3 are non-stable, physical considerations show that X should also contain at least one point M, where E has a minimum, and M has a shape of non-degenerate triangle. To find the exact shape of M we use a version of Lagrange multipliers method described below. Because of S-invariance of E it is clear that the non-degenerate critical points of E come in pairs. Since the minimum should have a shape of non-degenerate triangle there should be at least two such minima differing only by orientation. To find the possible shapes of critical triangles of E let us denote by l_i the lengths of the sides of a critical triangle. We are going to use a version of Lagrange multipliers method adjusted to the context of configuration space $Per_3(L)$.

Coulomb energy takes the form

$$E = \frac{q_1 q_2}{l_3} + \frac{q_2 q_3}{l_1} + \frac{q_1 q_3}{l_2}.$$

As we know, Coulomb energy E is invariant with respect to involution S on X introduced above. It follows that its differential dE at any point of Y vanishes on the direction (tangent vectors) orthogonal to the tangent line of Y in X. For this reason, to apply the Lagrange multipliers method correctly we should separately consider two cases: points in $X - Y$ and points in Y.

Under the assumption that a point does not belong to Y we have to solve a regular constrained optimization problem with target function E and one constraint $l_1 + l_2 + l_3 = L$. Using the standard form of Lagrange multipliers method one easily finds out that there exists exactly one possible triple of lengths of the sides of critical triangle which are given by the following formulas:

$$l_1 = \frac{L\sqrt{q_2 q_3}}{\sqrt{q_1 q_2} + \sqrt{q_2 q_3} + \sqrt{q_1 q_3}}, l_2 = \frac{L\sqrt{q_1 q_3}}{\sqrt{q_1 q_2} + \sqrt{q_2 q_3} + \sqrt{q_1 q_3}},$$

$$l_3 = \frac{L\sqrt{q_1 q_2}}{\sqrt{q_1 q_2} + \sqrt{q_2 q_3} + \sqrt{q_1 q_3}}. \tag{2}$$

Assuming that a point under inspection belongs to Y, i.e., it is a fixed point of involution S, we have to use a version of Lagrange multipliers method applicable to functions invariant with respect to involution. This implies that at fixed points of S we have to search for critical points of the restriction of E to the fixed point set Y. In other words, at each fixed point we should add a constraint given by the local equation of the fixed point set. In our case the complement $Y - \{A_1, A_2, A_3\}$ consists of three arcs Y_i on each of which the triangle inequality for lengths l_i becomes an equality. Thus we have to add a constraint of the form $l_i + l_j = l_k$. Together with the isoperimetric constraint $l_i + l_j + l_k = L$ this implies that $l_k = L/2$ and $l_i = L/2 - l_j$. Thus we are left with the optimization problem for

E as a function of one variable l_i. It is elementary to verify that it has exactly one solution in Y_i given by the configuration τ_i.

Summing up, our considerations imply that we have found the shapes of all critical points of E in the isoperimetric setting. In fact, one can verify that E is non-degenerate (Morse) function, which yields the following result.

Theorem 1. *Function E on X is non-degenerate and has exactly five critical points in X: two minima with the sides given by formulas (2) and three saddles-points at the aligned configurations T_i.*

The types of critical points can be found by computing the so-called bordered hessian of E defined in [6]. Direct computation shows that the bordered hessian is non-degenerate at all critical points so results of [6] imply that E is indeed a Morse function. The indices of bordered hessian are readily found using Sylvester rule and the indices of critical points of E can be found using results of [6], which completes the proof of Theorem 1.

It is instructive to verify Morse formula for the Euler characteristic in this context. Notice that X is diffeomorphic to sphere S^2 with three deleted points corresponding to two coinciding positions of point charges. So the Euler characteristic of X equals -1, which coincides with the sum of Morse indices of the five critical points of E in X.

One can also approach Morse formula in a different way. Function E has isolated poles at three deleted points so one can modify function E in small neighbourhoods of these points so that E has non-degenerate maxima at these points. So their Morse indices are equal to two and the Morse indices of the modified energy function sum up to two which is the Euler characteristic of $Per_3(L)$. Thus our results are consistent with the topological picture.

It is now easy to obtain explicit formulas for all critical values of E. for the reason of space, here we only present the value of absolute minimum which will be used below.

Corollary 1. The absolute minimum of E in the above setting is:

$$E_m(Q) = \frac{(\sqrt{q_1 q_2} + \sqrt{q_2 q_3} + \sqrt{q_1 q_3})^2}{L}. \tag{3}$$

In particular, if all charges are equal to q then

$$E_m(q) = \frac{9q^2}{L}.$$

In the next section we present an explicit solution of DPE in isoperimetric setting.

2. Stationary charges in isoperimetric setting

Suppose we are given three lengths of the sides $l_i = l_i(T)$ of triangle T corresponding to a critical configuration. To solve DPE we need to find values of point charges such that they are in rest in this configuration. Without restricting generality we may fix the sum S of the sought stationary charges. Reversing the arguments used in the above application of Lagrange method we see that to solve DPE one may consider formulas (2) as a system of linear equations for charges q_i. This system admits an explicit solution, which yields the following result.

Theorem 2. *For given lengths l_i of the sides of triangle T and prescribed sum of stationary charges S, the stationary charges for triangle T are given by:*

$$q_1 = \frac{Sl_2^2 l_3^2}{l_1^2 l_2^2 + l_2^2 l_3^2 + l_1^2 l_3^2}, q_2 = \frac{Sl_1^2 l_3^2}{l_1^2 l_2^2 + l_2^2 l_3^2 + l_1^2 l_3^2}, q_3 = \frac{Sl_1^2 l_2^2}{l_1^2 l_2^2 + l_2^2 l_3^2 + l_1^2 l_3^2}. \quad (4)$$

Corollary 2. The absolute minimum of E in the above setting is:

$$E_m(S) = \frac{S^2 L l_1^2 l_2^2 l_3^2}{(l_1^2 l_2^2 + l_2^2 l_3^2 + l_1^2 l_3^2)^2}. \quad (5)$$

In particular, if all side-lengths are equal to $L/3$ then

$$E_m(S) = \frac{S^2}{L}.$$

Thus we have given explicit solutions of the direct and inverse problems of electrostatics for three charges in the isoperimetric setting. In addition to DPE and IPE it is possible to investigate a related problem which seems interesting (at least) from a purely mathematical point of view. Namely, for given $n \in \mathbb{Z}_+, Q > 0, L > 0$, one searches for positive numbers $q_1, \ldots, q_n$ satisfying the condition $\sum q_i = Q$, and n-point configurations P of perimeter L in the plane such that $Q@P$ configuration is a critical point of Coulomb energy in this setting. In other words, in this setting, which we call the $E(n; Q, L)$-problem, one has to solve a constrained optimization problem with target function $E(Q@P)$ and two constraints described above. In other groups, two groups of arguments of $E(Q@P)$ are considered on equal footing.

It turns out that, for $n = 3$, one can easily find the absolute minimum of E in this setting. Namely, the equations provided by Lagrange multipliers method imply that all charges should be equal. From the above results immediately follows the critical configurations are the regular triangle with both orientations and line segment of length $L/2$ with the third charge at the middle. Comparing the values of Coulomb energy given in Corollary we arrive at the following result.

Theorem 3. *For any $Q > 0, L > 0$, the absolute minimum of electrostatic energy in $E(3; Q, L)$-problem is attained at a regular triangle with the side $L/3$ with equal charges $Q/3$ placed at its vertices.*

Analyzing the proof of this result it becomes intuitively plausible that a similar result holds true for $E(n; Q, L)$-problem with arbitrary n.

Conjecture 1. For any $n \in \mathbb{Z}_+, Q > 0, L > 0$, the absolute minimum of electrostatic energy in $E(n; Q, L)$-setting is attained at a convex regular polygon with. the side L/n with equal charges Q/n placed at its vertices.

Attempts to prove this conjecture encounter conceptual problems. The point is that in $E(n; Q, L)$-setting, in addition to convex regular and aligned configurations, there exist other critical configurations and it is not quite clear how to find all of them. As was shown in a recent paper [7], for $n \geq 5$, all regular n-pointed stars with various rotation numbers and charges Q/n at the vertices are also equilibrium configurations in $E(n; Q, L)$-setting. The electrostatic energies of star-like configurations can be explicitly calculated using results of [2], [7] and one sees that these values are bigger than at the regular convex configuration.

This supports Conjecture 1 but does not give a rigorous proof since we do not have a complete list of equilibrium configurations in $E(n; Q, L)$-setting. It also seems worthy of noting that [7] gives some evidence in favour of the following general conjecture.

Conjecture 2. For any $n \in \mathbb{Z}_+, Q > 0, L > 0$, the following is a complete list of the shapes of critical configurations in $E(n; Q, L)$-setting: convex regular, aligned with equal charges and all possible regular n-pointed stars with equally charges vertices.

If Conjecture 2 holds true then Conjecture 1 will follow from the comparison of critical values outlined above. It is interesting to investigate non-degeneracy and Morse indices of star-like equilibrium configurations in $E(n; Q, L)$-setting. The results and approach of the present paper suggest a lot of other open problems and generalizations part of which are given in the last section.

Concluding remarks

It seems worth noting that the problems studied above suggest some purely mathematical developments and physical interpretations. The most obvious mathematical problem is to generalize our results to the cases with more than three charges. Such generalizations are non-obvious already in the case of four charges. It can be shown that not all configurations of four points can be represented as Coulomb equilibria in isoperimetric setting. So one can search for a geometric characterization of Coulomb equilibria of n-point configurations and then try to obtain explicit formulas for stationary charges. Results of [5], [8] suggest that these problems may be more easy for concyclic configurations having all points on the same circle.

In general, for four charges, a relevant configuration space $Per_4(L)$ is homeomorphic to $\mathbb{CP}^2$. The latter space is four-dimensional and, unlike to $Per_3(L)$, its topological structure is hard to describe in a visual way. However, topology still yields some information which is useful in our problem. In particular, it suggests existence of many saddle-points since E has ten poles in $Per_4(L)$ corresponding to coincident vertices of quadrilateral (six from pairwise coincidences and four from triple coincidences). For example, for four equal charges one can easily construct the two minima (regular squares) and six saddles having the shapes of aligned equilibrium configurations. Under assumption that E is a Morse function, the Morse formula for the Euler characteristic implies that there should exist at least four other critical points, whose shapes and sizes remain unclear.

Another purely mathematical research perspective is to obtain analogs of our results for other central forces, specifically, for logarithmic potential. The ion traps corresponding to aligned configurations with logarithmic interaction have been studied for a long time, starting with a seminal paper of Stiltjes [9]. So one may try to use known results to shed some light on equilibria in isoperimetric setting.

Among the arising topics with physical flavour we would like to mention investigation of DPE and IPE in the setting of elastic contour obeying Hooke's law started in [10]. For n equal charges, one can compute the critical energies using results of [2] and [10]. For non-equal charges the problem is practically unexplored.

Physical considerations also suggest several scenarios involving dynamical aspects of the problem. Since configuration space $Per_3(L)$ has a natural Riemannian structure one may consider the gradient flow of Coulomb energy. For example, one can try to calculate the length of the arc connecting two points on the same gradient trajectory and compare it with the length of minimal geodesics joining these two points. One can also try to compute the length of isoenergetic (level) curves of Coulomb energy in $Per_3(L)$.

Another interesting topic suggested by the third named author is to study the evolution of equilibrium configurations after eliminating one of the sides of equilibrium triangle. Informally, this can be described as "cutting" or "burning off" one of the sides. One can obtain explicit formulas describing the dynamics of this process. In particular, using momentum and energy conservation it is possible to find the maximal speed of the opposite vertex and estimate the time of achieving the aligned equilibrium configuration. The arising formulas and computations will be presented elsewhere.

Finally, our results suggest some problems in the spirit of Coulomb control discussed in [7]. Namely, our results show that one can connect any two given points A_1, A_2 in $Per_3(L)$ using the solution of DPE for the linear homotopy connecting the stationary charges of A_1 and A_2. Using the exact formulas for solutions to DPE and IPE one can try to compute the length of the path arising in this way. It is also interesting to study the problem of optimal control of equilibrium configurations in terms of their stationary charges. Similar problems with more than three charges yield a vast and unexplored topic far beyond the scope and aims of the present paper.

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