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EQUILIBRIA OF POINT CHARGES IN A LINE SEGMENT

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Abstract. We present several results on equilibria of point charges in a line segment with charged end-points obtained in the framework of inverse problems approach to linear ion traps. In particular, we give a solution of inverse electrostatic problem for four and five point charges.

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Introduction

Equilibria of point charges confined to a line segment with charged end-points have been studied from various points of view (see, e.g. [1], [2], [3], [4]). In particular, the case of equal charges interacting by logarithmic potential have been studied in great detail in a seminal paper of T.J. Stieltjes [1]. The results of Stieltjes served as a paradigm for a vast topic which is nowadays called electrostatic interpretation of configurations of points on the line (see, e.g., [5]). A more general setting for such developments has recently been suggested in [6] in the case of Coulomb potential in the plane.

In last decades the interest towards these topics increased in connection with the mathematical aspects of *linear ion traps* (Paul traps) [7], [5], [8], [9], [10]. In this context it is natural to consider both the logarithmic potential and Coulomb potential. Some results in the case of a few charges interacting by Coulomb potential have been obtained in previous papers of the authors [11], [12], [13], [14]. In particular, the inverse electrostatic problem for two and three point charges in a linear ion trap has been solved in [11]. A spectacular evidence of importance of such studies is the demonstration of five trapped ion based quantum computer, which is programmable and reconstructible, by a research group from the Joint Quantum Institute in the USA [15]. This suggests that a natural and useful next step in the same direction is to study the cases where the number of ions (point charges) is bigger than three. Along these lines, in the present paper we present some results for four and five point charges in a line segment with charged end-points.

We begin by discussing some aspects of this topic in the case of the logarithmic potential. After that we present several results on equilibria of four point charges in a linear ion trap.

1. Electrostatic interpretation of zeros of orthogonal polynomials

The equilibrium points of n free unit charges in the interval $(-1,1) \subset \mathbf{R}$ in the field generated by two fixed charges $\frac{\alpha+1}{2}$ at 1 and $\frac{\beta+1}{2}$ at -1, where the charges repel each other according to the law of logarithmic potential, are zeros of classical *Jacobi polynomials* $\left(P_n^{\alpha,\beta}(x)\right)_{n=0}^{\infty}$, orthogonal on $[-1,1]$ with respect to weight function $(1-x)^{\alpha}(1+x)^{\beta}$. This means that the energy of the field has a local minimum at the zeros of the Jacobi polynomial of degree n. This deep result belongs to Stieltjes (see [1]). Szegö proved that the energy has a unique minimum, thus establishing the stability of the equilibrium [3].

Theorem 1. [3] *Let $\alpha > -1$ and $\beta > -1$. Then differential equation*

$$(1-x^2)y^{''} + (\beta - \alpha - (\alpha+\beta+2)x)y^{'} + \gamma y = 0, \tag{1}$$

where γ is a parameter, has a polynomial solution not identically zero if and only if γ has the form $n(n+\alpha+\beta+1)$, $n = 0, 1, \ldots$. Moreover, if $\gamma = n(n+\alpha+\beta+1)$, the polynomials $cP_n^{\alpha,\beta}(x)$, where c is any constant, are solutions of this equation, and there are no other polynomial solutions.

Consider more general problem: given $m+1$ positive charges q_j at p_j, $p_0 < \ldots < p_m$, and all possible equilibrium locations x_k of m free unit charges. This problem is closely related to the question of characterizing the polynomial solutions of the differential equation

$$A(x)y^{''} + 2B(x)y^{'} + C(x)y = 0, \tag{2}$$

where $A(x) = (x-p_0)\ldots(x-p_m)$, $B(x)$ and $C(x)$ are polynomials of degree m and $m-1$, and

$$\frac{B(x)}{A(x)} = \sum_{j=0}^{m} \frac{q_j}{x-p_j}.$$

The equation (2) is called a *generalized Lamé equation* in algebraic form.

Theorem 2. [3] *For given $A(x)$ and $B(x)$, there exist exactly $\frac{(n+m-1)!}{n!(p-1)!}$ polynomials $C(x)$, such that, for each of them, there exists a polynomial solution $y(x)$ of degree n with only real zeros of the Lamé equation (2).*

The polynomial $C(x)$ is called a *Van Vleck polynomial* and the corresponding polynomial solution $y(x)$ of (2) is called a *Stieltjes polynomial.*

The *Jacobi polynomials* from Theorem 1 can be given explicitly by the expression

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2}\sum_{j=0}^{n} \frac{(n+\alpha)!}{(n-j)!}\frac{(n+\beta)!}{j!}(x-1)^j(x+1)^{n-j}.$$

From the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!}(x-1)^{-\alpha}(x+1)^{-\beta}\left(\frac{d}{dx}\right)^n \left[(x-1)^{n+\alpha}(x+1)^{n+\beta}\right]$$

it follows, that $P_n^{(\alpha,\beta)}(x)$ are analytic functions of the parameters $\alpha, \beta \in \mathbf{C}$ and that $\deg P_n^{(\alpha,\beta)}(x) \leq n$.

We now describe a more detailed *electrostatic model* of n unit point charges on the line.

Let two positive fixed charges of mass $\frac{\beta+1}{2}$ and $\frac{\alpha+1}{2}$ at -1 and $+1$, respectively and allow n positive unit charges $X = \{x_1, \ldots, x_m\}$ to move freely in $(-1,1)$. The total energy $E(X)$ of this system if the interaction obeys the logarhitmic potential law equal to

$$E(X) = E_{int} + E_{ext},$$

where

$$E_{int} = -\sum_{1\le k\le j\le n} \ln|x_k - x_j|,$$

and

$$E_{ext} = \sum_{k=1}^{n} \varphi(x_k)$$

with the external field $\varphi(x)$ created by the fixed charges:

$$\varphi(x) = -\frac{\beta+1}{2}\ln|x+1| - \frac{\alpha+1}{2}\ln|x-1|. \tag{3}$$

Theorem 1 means that there exists a unique configuration $X^* = \{x_1^*, ..., x_m^*\}$ providing the global minimum of $E(X)$ in $[-1,1]^n$, corresponding to the unique equilibrium position for given free charges, the points x_j^* are the zeros of the polynomial $P_n^{(\alpha,\beta)}$.

The critical points of the energy functional $E(X)$ as the function of x_j is the solutions of the equation

$$\frac{\partial}{\partial x_k}E(X) = 0.$$

Suppose X^* is critical configuration, then

$$\frac{\partial}{\partial x_k}E_{int}(X)|_{X=X^*} + \varphi'(x_k) = 0 \tag{4}$$

Suppose $y(x) = (x-x_1^*)(x-x_2^*)...(x-x_n^*)$ is monic polynomial with zeros at x_k^*'s, then

$$\frac{\partial}{\partial x_k}E_{int}(X)|_{X=X^*} = -\sum_{1\le j\le n, j\neq k}\frac{1}{x_k^*-x_j^*} = -\frac{1}{2}\frac{y''(x_k^*)}{y'(x_k^*)}$$

and

$$\varphi'(x) = -\frac{\beta+1}{2(x+1)} - \frac{\alpha+1}{2(x-1)}.$$

From (4) we obtain

$$y''(x) + \left(\frac{\beta+1}{x+1} + \frac{\alpha+1}{x-1}\right)y' = 0 \tag{5}$$

for all $x \in X^*$.

From the equation (5) we obtain that the polynomial

$$(1-x^2)y''(x) + (x(\alpha+\beta+2) + (\alpha-\beta))y'$$

of degree n is equal to zero at the zeros of polynomial $y(x)$ and therefore equal to `const` $\times\, y(x)$. Denote by γ this constant, we obtain a second order differential equation (1) (see [3], sect. 4.2 and 6.7).

The Lamé equation is a Fuchsian type differential equation on the extended complex plane with four regular singular points $-1, 0, s$ and ∞, with exponents

$(0, 1/2), (0, 1/2), (0, 1/2)$ and $(l/2, -(l+1)/2)$. The canonical form of the second order differential equation of this kind is

$$y''(x) + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-s}\right)y'(x) - \frac{l(l+1)x + 4q}{4x(x-1)(x-s)}y(x) = 0. \quad (6)$$

Equation (6) is a particular case of the second order Fuchsian differential equation with four singular points $0, 1, s, \infty$ and with exponents $(0, 1-\gamma), (0, 1-\delta), (0, 1-\epsilon), (\alpha, \beta)$ at singular points:

$$y''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-s}\right)y'(z) - \frac{\alpha\beta z - q}{z(z-1)(z-s)}y(z) = 0, \quad (7)$$

where the constants $\alpha, \beta, \gamma, \delta, \epsilon$ satisfy the Fuchs relation

$$\alpha + \beta + 1 = \gamma + \delta + \epsilon.$$

The equation (7) is known as the *Heun equation* [17] and relation between Lamé and Heun equations is similar the relation hypergeometric (Euler-Gauss) and Riemann equations: both are the second order differential equations with three singular points and by a conformal transformation singular points s_1, s_2, s_3 pass to $0, 1, \infty$. At the same time, the exponents are changed. The parameter q in equation (7) is called *accessory parameter* and arises as additional parameter from monodromy group of Fuchsian equation. The Schwarz-Cristoffel parameter problem [18] for four points, investigation of spherical quadrilaterals [19] are closely related to Heun equation and its generalizations and have many applications for study different physical problems (see [20]).

2. Mathematics of linear ion traps

It is known that ion traps are good candidates to produce quantum computational processes because they are one of the proposals that fulfill all the criteria. The ion traps that run with static and dynamic electric fields are usually called *Paul* or *radio frequency traps.*

Ion traps can produce logical operations through the use of quantum gates, as was firstly proposed by Cirac and Zoller [8] (see also [9], [10]). Equilibrium positions of ions in a linear trap were investigated in [16].

Let us consider the following model. Suppose a chain of N ions in a trap is given. The ions are assumed to be strongly bound in the y and z directions but weakly bound in an harmonic potential in the x direction. The position of the m-th ion, where the ions are numbered from left to right, will be denoted $x_m(t)$. The motion of each ion will be influenced by an overall harmonic potential due to the trap electrodes and by the Coulomb force exerted by all of the other ions. We will assume that the binding potential in the y and z directions is sufficiently strong that motion along these axes can be neglected. Hence the potential energy of the ion chain is given by the following expression:

$$V = \sum_{j=1}^{N} \frac{1}{2}M\nu^2 x_m(t)^2 + \sum_{i,j=1, i\neq j}^{N} \frac{Z^2e^2}{8\pi\epsilon_0}\frac{1}{|x_i(t) - x_j(t)|}, \quad (8)$$

where M is the mass of each ion, e is the electron charge, Z is the degree of ionization of the ions, ϵ_0 is the permitivity of free space, and ν is the trap frequency, which characterizes the strength of the trapping potential in the axial direction.

Assume that the ions are sufficiently cold so that the position of the m-th ion can be approximated by the formula

$$x_m(t) \approx x_m^{(0)} + q_m(t) \tag{9}$$

where $x_m^{(0)}$ is the equilibrium position of the ion, and $q_m(t)$ is a small displacement. The equilibrium positions will be determined by the following equation:

$$\left[\frac{\partial V}{\partial x_m}\right]_{x_m = x_m^{(0)}} = 0 \tag{10}$$

and the dimensionless equilibrium position $u_m = \frac{x_m^{(0)}}{l}$, then (10) may be rewritten as the following system of N algebraic equations for the values of u_m :

$$u_m - \sum_{m=1}^{N-1} \frac{1}{(x_m - x_n)^2} + \sum_{n=m+1}^{N} \frac{1}{(x_m - x_n)^2} = 0, \quad m = 1, ..., N. \tag{11}$$

The above described model is called the direct problem of electrostatics (see [6], [12]): given a compact conductor X and collection of n positive numbers $Q = (q_i)$, find all equilibrium configurations of charges q_i in X and determine their types as critical points of Coulomb potential $E_Q|X^n$.

3. Inverse problem of electrostatics for point charges in line segment

Let us consider now an inverse problem called the inverse problem of electrostatics (see [6], [13]): given a finite configuration $P = (p_1, ..., p_n)$ of points in compact conductor X, find out if there exists a collection of non-zero real numbers $Q = (q_1, ..., q_n)$ such that configuration P is a critical point of Coulomb potential E_Q restricted to X^n.

Let N be the number of point charges considered. Then the resultant force on q_i in position x_i is given by

$$F_m = \frac{q_m t_1}{x_m^2} + \sum_{j=1}^{m-1} \frac{q_m q_j}{(x_m - x_j)^2} - \sum_{j=m+1}^{N} \frac{q_m q_j}{(x_j - x_m)^2} - \frac{q_m t_2}{(L - x_m)^2}, \quad m = 1, ..., N.$$

The relations

$$F_1 = 0, F_2 = 0, ..., F_N = 0,$$

give a system of non-homogeneous linear equations

$$MQ = G, \tag{12}$$

for unknowns $q_1, ..., q_N$, where

$$G = \left(\frac{t_2}{(L - x_1)^2} - \frac{t_1}{x_1^2}, \frac{t_2}{(L - x_2)^2} - \frac{t_1}{x_2^2}, ..., \frac{t_2}{(L - x_N)^2} - \frac{t_1}{x_N^2}\right)^T,$$

$M = (m_{ij})_{i,j=1}^{N}$ is an antisymmetric matrix and $m_{ij} = (-1)^{\tau_{ij}}(x_j - x_i)^{-2}$, $i \neq j$, $\tau_{ij} = 0$, if $i > j$; and $\tau_{ij} = 1$, if $i < j$.

For $N = 4$ we have:

$$F_1 = \frac{t_1 q_1}{x_1^2} - \frac{q_1 q_2}{(x_2 - x_1)^2} - \frac{q_1 q_3}{(x_3 - x_1)^2} - \frac{q_1 q_4}{(x_4 - x_1)^2} - \frac{q_1 t_2}{(L - x_1)^2},$$

$$F_2 = \frac{t_1 q_2}{x_2^2} + \frac{q_1 q_2}{(x_2 - x_1)^2} - \frac{q_2 q_3}{(x_3 - x_2)^2} - \frac{q_2 q_4}{(x_4 - x_2)^2} - \frac{q_2 t_2}{(L - x_2)^2},$$

$$F_3 = \frac{t_1 q_3}{x_3^2} + \frac{q_1 q_3}{(x_3 - x_1)^2} + \frac{q_2 q_3}{(x_3 - x_2)^2} - \frac{q_3 q_4}{(x_4 - x_3)^2} - \frac{q_3 t_2}{(L - x_3)^2},$$

$$F_4 = \frac{t_1 q_4}{x_4^2} + \frac{q_1 q_4}{(x_4 - x_1)^2} + \frac{q_2 q_4}{(x_4 - x_2)^2} + \frac{q_3 q_4}{(x_4 - x_3)^2} - \frac{q_4 t_2}{(L - x_4)^2}.$$

$$M_4 = \begin{pmatrix} 0 & -\frac{1}{(x_2-x_1)^2} & -\frac{1}{(x_3-x_1)^2} & -\frac{1}{(x_4-x_1)^2} \\ \frac{1}{(x_2-x_1)^2} & 0 & -\frac{1}{(x_3-x_2)^2} & -\frac{1}{(x_4-x_2)^2} \\ \frac{1}{(x_3-x_1)^2} & \frac{1}{(x_3-x_2)^2} & 0 & -\frac{1}{(x_4-x_3)^2} \\ \frac{1}{(x_4-x_1)^2} & \frac{1}{(x_4-x_2)^2} & \frac{1}{(x_4-x_3)^2} & 0 \end{pmatrix},$$

$$G = \left(\frac{t_2}{(L - x_1)^2} - \frac{t_1}{x_1^2}, \frac{t_2}{(L - x_2)^2} - \frac{t_1}{x_2^2}, \frac{t_2}{(L - x_3)^2} - \frac{t_1}{x_3^2}, \frac{t_2}{(L - x_4)^2} - \frac{t_1}{x_4^2} \right)^T,$$

$$Q = (q_1, q_2, q_3, q_4)^T.$$

Denote by

$$P = \frac{1}{(x_4 - x_3)^2 (x_4 - x_2)^2 (x_4 - x_1)^2 (x_3 - x_2)^2 (x_3 - x_1)^2 (x_2 - x_1)^2}.$$

Then

$$\begin{aligned} & DetM_4 \\ &= 2P \left[(x_4 - x_2)^2 (x_3 - x_1)^2 - (x_4 - x_3)^2 (x_2 - x_1)^2 - (x_4 - x_1)^2 (x_3 - x_2)^2 \right] \\ &+ \frac{1}{(x_4 - x_3)^4 (x_2 - x_1)^4} + \frac{1}{(x_4 - x_2)^4 (x_3 - x_1)^4} + \frac{1}{(x_4 - x_1)^4 (x_3 - x_2)^4}. \end{aligned}$$

Here we used the known formula for the antisymmetric matrix $(PfM_4)^2 = DetM_4$, where PfM_4 is the Pfaffian of the matrix M_4 and

$$PfM_4 = \frac{1}{(x_2 - x_1)^2} \frac{1}{(x_4 - x_3)^2} - \frac{1}{(x_3 - x_1)^2} \frac{1}{(x_4 - x_2)^2} + \frac{1}{(x_3 - x_2)^2} \frac{1}{(x_4 - x_2)^2}.$$

Proposition 1. $DetM_4 \neq 0$. $P \neq 0$, since $x_i \neq x_j$, $i \neq j$ and $i, j = 1, .., 4$. Consider the following expression

$$\begin{aligned} \Delta &= P^{-2} DetM_4 \\ &= 2(x_4 - x_3)^2 (x_4 - x_2)^4 (x_4 - x_1)^2 (x_3 - x_2)^2 (x_3 - x_1)^4 (x_2 - x_1)^2 \\ &-2(x_4 - x_3)^4 (x_4 - x_2)^2 (x_4 - x_1)^2 (x_3 - x_2)^2 (x_3 - x_1)^2 (x_2 - x_1)^4 \\ &-2(x_4 - x_3)^2 (x_4 - x_2)^2 (x_4 - x_1)^4 (x_3 - x_2)^4 (x_3 - x_1)^2 (x_2 - x_1)^2 \\ &+(x_4 - x_2)^4 (x_4 - x_1)^4 (x_3 - x_2)^4 (x_3 - x_1)^4 \end{aligned}$$

$$+(x_4-x_3)^4(x_4-x_1)^4(x_3-x_2)^4(x_2-x_1)^4$$
$$+(x_4-x_3)^4(x_4-x_2)^4(x_3-x_1)^4(x_2-x_1)^4.$$

Denote by $x_{ij}=(x_i-x_j)^2$, $j<i$, $x_{ij}>0$. Then

$$\Delta = 2x_{43}x_{42}^2x_{41}x_{32}x_{31}^2x_{21} - 2x_{43}^2x_{42}x_{41}x_{32}x_{31}x_{21}^2 - 2x_{43}x_{42}x_{41}^2x_{32}^2x_{31}x_{21} +$$
$$+x_{42}^2x_{41}^2x_{32}^2x_{31}^2 + x_{43}^2x_{41}^2x_{32}^2x_{21}^2 + x_{43}^2x_{42}^2x_{31}^2x_{21}^2.$$

Using the obvious inequalities $x_{21}<x_{31}<x_{41}$, we obtain $\Delta>0$.

Denote by $M_4^{-1}=\frac{1}{DetM_4}\begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} \\ -m_{12} & 0 & m_{23} & m_{24} \\ -m_{13} & -m_{23} & 0 & m_{34} \\ -m_{14} & -m_{24} & -m_{34} & 0 \end{pmatrix}$, where

$$m_{12}=\frac{1}{(x_2-x_1)^2(x_4-x_3)^2(x_4-x_2)^2}-\frac{1}{(x_3-x_2)^2(x_4-x_3)^2(x_4-x_1)^2}$$
$$-\frac{1}{(x_4-x_3)^4(x_2-x_1)^2},$$

$$m_{13}=\frac{1}{(x_3-x_1)^2(x_4-x_2)^4}-\frac{1}{(x_4-x_2)^2(x_3-x_2)^2(x_4-x_1)^2}$$
$$-\frac{1}{(x_4-x_3)^2(x_4-x_2)^2(x_2-x_1)^2},$$

$$m_{14}=\frac{1}{(x_2-x_1)^2(x_3-x_2)^2(x_4-x_3)^2}+\frac{1}{(x_3-x_1)^2(x_4-x_2)^2(x_3-x_2)^2}$$
$$-\frac{1}{(x_3-x_2)^4(x_4-x_1)^2},$$

$$m_{23}=\frac{1}{(x_3-x_1)^2(x_4-x_2)^2(x_4-x_1)^2}+\frac{1}{(x_4-x_1)^2(x_2-x_1)^2(x_4-x_3)^2}$$
$$-\frac{1}{(x_3-x_2)^4(x_4-x_1)^2},$$

$$m_{24}=\frac{1}{(x_3-x_1)^4(x_4-x_2)^2}+\frac{1}{(x_4-x_1)^2(x_3-x_2)^2(x_3-x_1)^2}$$
$$-\frac{1}{(x_3-x_1)^2(x_4-x_3)^2(x_2-x_1)^2},$$

$$m_{34}=\frac{1}{(x_2-x_1)^4(x_3-x_1)^2}-\frac{1}{(x_4-x_1)^2(x_2-x_1)^2(x_3-x_2)^2}$$
$$-\frac{1}{(x_4-x_3)^2(x_2-x_1)^4}.$$

One may calculate M_4^{-1} using the so called Cayley-Hamilton method:

$$M_4^{-1}=\frac{1}{DetM_4}\left(\frac{1}{6}((trM_4)^3-3trM_4\,trM_4^2+2trM_4^3)\mathbf{I}\right)$$
$$-\frac{1}{2DetM_4}M_4((trM_4)^2-trM_4^2+M_4\,trM_4-M_4^2),$$

where $\mathbf{I}$ is 4×4-identity matrix.

Proposition 2. *For $N = 4$, the unique balancing charges are as follows*

$$q_1 = n_{12}\left(\frac{t_2}{(L-x_2)^2} - \frac{t_1}{x_2^2}\right) + n_{13}\left(\frac{t_2}{(L-x_3)^2} - \frac{t_1}{x_3^2}\right) + n_{14}\left(\frac{t_2}{(L-x_4)^2} - \frac{t_1}{x_4^2}\right),$$

$$q_2 = -n_{12}\left(\frac{t_2}{(L-x_1)^2} - \frac{t_1}{x_1^2}\right) + n_{23}\left(\frac{t_2}{(L-x_3)^2} - \frac{t_1}{x_3^2}\right) + n_{24}\left(\frac{t_2}{(L-x_4)^2} - \frac{t_1}{x_4^2}\right),$$

$$q_3 = -n_{13}\left(\frac{t_2}{(L-x_1)^2} - \frac{t_1}{x_1^2}\right) - n_{23}\left(\frac{t_2}{(L-x_2)^2} - \frac{t_1}{x_2^2}\right) + n_{24}\left(\frac{t_2}{(L-x_4)^2} - \frac{t_1}{x_4^2}\right),$$

$$q_4 = -n_{14}\left(\frac{t_2}{(L-x_1)^2} - \frac{t_1}{x_1^2}\right) - n_{24}\left(\frac{t_2}{(L-x_2)^2} - \frac{t_1}{x_2^2}\right) - n_{34}\left(\frac{t_2}{(L-x_3)^2} - \frac{t_1}{x_3^2}\right),$$

where $n_{ij} = \frac{m_{ij}}{DetM}$, $i = 1, 2, 3$ and $j = 2, 3, 4$.

Analogous results for $N = 5$ can be obtained using the following system of equations for balancing charges.

$$F_1 = \frac{t_1 q_1}{x_1^2} - \frac{q_1 q_2}{(x_2-x_1)^2} - \frac{q_1 q_3}{(x_3-x_1)^2} - \frac{q_1 q_4}{(x_4-x_1)^2} - \frac{q_1 q_5}{(x_5-x_1)^2} - \frac{q_1 t_2}{(L-x_1)^2},$$

$$F_2 = \frac{t_1 q_2}{x_2^2} + \frac{q_1 q_2}{(x_2-x_1)^2} - \frac{q_2 q_3}{(x_3-x_2)^2} - \frac{q_2 q_4}{(x_4-x_2)^2} - \frac{q_2 q_5}{(x_5-x_2)^2} - \frac{q_2 t_2}{(L-x_2)^2},$$

$$F_3 = \frac{t_1 q_3}{x_3^2} + \frac{q_1 q_3}{(x_3-x_1)^2} + \frac{q_2 q_3}{(x_3-x_2)^2} - \frac{q_3 q_4}{(x_4-x_3)^2} - \frac{q_3 q_5}{(x_5-x_3)^2} - \frac{q_3 t_2}{(L-x_3)^2},$$

$$F_4 = \frac{t_1 q_4}{x_4^2} + \frac{q_1 q_4}{(x_4-x_1)^2} + \frac{q_2 q_4}{(x_4-x_2)^2} + \frac{q_3 q_4}{(x_4-x_3)^2} - \frac{q_4 q_5}{(x_5-x_4)^2} - \frac{q_4 t_2}{(L-x_4)^2},$$

$$F_5 = \frac{t_1 q_5}{x_5^2} + \frac{q_1 q_5}{(x_5-x_1)^2} + \frac{q_2 q_5}{(x_5-x_2)^2} + \frac{q_3 q_5}{(x_5-x_3)^2} + \frac{q_4 q_5}{(x_5-x_4)^2} - \frac{q_5 t_2}{(L-x_5)^2}.$$

From this

$$M_5 = \begin{pmatrix} 0 & -\frac{1}{(x_2-x_1)^2} & -\frac{1}{(x_3-x_1)^2} & -\frac{1}{(x_4-x_1)^2} & -\frac{1}{(x_5-x_1)^2} \\ \frac{1}{(x_2-x_1)^2} & 0 & -\frac{1}{(x_3-x_2)^2} & -\frac{1}{(x_4-x_2)^2} & -\frac{1}{(x_5-x_2)^2} \\ \frac{1}{(x_3-x_1)^2} & \frac{1}{(x_3-x_2)^2} & 0 & -\frac{1}{(x_4-x_3)^2} & -\frac{1}{(x_4-x_3)^2} \\ \frac{1}{(x_4-x_1)^2} & \frac{1}{(x_4-x_2)^2} & \frac{1}{(x_4-x_3)^2} & 0 & -\frac{1}{(x_5-x_4)^2} \\ \frac{1}{(x_5-x_1)^2} & \frac{1}{(x_5-x_2)^2} & \frac{1}{(x_5-x_3)^2} & \frac{1}{(x_5-x_4)^2} & 0 \end{pmatrix}.$$

$DetM_5 = 0$, since M_5 is an antisymmetric matrix. One gets $rankM_5 = 4$, because M_5 contains M_4 as minor and $detM_4 \neq 0$. Let

$$\begin{pmatrix} 0 & -\frac{1}{(x_2-x_1)^2} & -\frac{1}{(x_3-x_1)^2} & -\frac{1}{(x_4-x_1)^2} & -\frac{1}{(x_5-x_1)^2} & \frac{t_2}{(L-x_1)^2} - \frac{t_1}{x_1^2} \\ \frac{1}{(x_2-x_1)^2} & 0 & -\frac{1}{(x_3-x_2)^2} & -\frac{1}{(x_4-x_2)^2} & -\frac{1}{(x_5-x_2)^2} & \frac{t_2}{(L-x_2)^2} - \frac{t_1}{x_2^2} \\ \frac{1}{(x_3-x_1)^2} & \frac{1}{(x_3-x_2)^2} & 0 & -\frac{1}{(x_4-x_3)^2} & -\frac{1}{(x_4-x_3)^2} & \frac{t_2}{(L-x_3)^2} - \frac{t_1}{x_3^2} \\ \frac{1}{(x_4-x_1)^2} & \frac{1}{(x_4-x_2)^2} & \frac{1}{(x_4-x_3)^2} & 0 & -\frac{1}{(x_5-x_4)^2} & \frac{t_2}{(L-x_4)^2} - \frac{t_1}{x_4^2} \\ \frac{1}{(x_5-x_1)^2} & \frac{1}{(x_5-x_2)^2} & \frac{1}{(x_5-x_3)^2} & \frac{1}{(x_5-x_4)^2} & 0 & \frac{t_2}{(L-x_5)^2} - \frac{t_1}{x_5^2} \end{pmatrix}.$$

The system of equations (12) for $N=5$ is solvable if $rank\widetilde{M_5}=4$, where

$$\widetilde{M_5}=\begin{pmatrix} 0 & -\frac{1}{(x_2-x_1)^2} & -\frac{1}{(x_3-x_1)^2} & -\frac{1}{(x_4-x_1)^2} & \frac{t_2}{(L-x_1)^2}-\frac{t_1}{x_1^2} \\ \frac{1}{(x_2-x_1)^2} & 0 & -\frac{1}{(x_3-x_2)^2} & -\frac{1}{(x_4-x_2)^2} & \frac{t_2}{(L-x_2)^2}-\frac{t_1}{x_2^2} \\ \frac{1}{(x_3-x_1)^2} & \frac{1}{(x_3-x_2)^2} & 0 & -\frac{1}{(x_4-x_3)^2} & \frac{t_2}{(L-x_3)^2}-\frac{t_1}{x_3^2} \\ \frac{1}{(x_4-x_1)^2} & \frac{1}{(x_4-x_2)^2} & \frac{1}{(x_4-x_3)^2} & 0 & \frac{t_2}{(L-x_4)^2}-\frac{t_1}{x_4^2} \\ \frac{1}{(x_5-x_1)^2} & \frac{1}{(x_5-x_2)^2} & \frac{1}{(x_5-x_3)^2} & \frac{1}{(x_5-x_4)^2} & \frac{t_2}{(L-x_5)^2}-\frac{t_1}{x_5^2} \end{pmatrix}.$$

Lemma 1. *$Det\widetilde{M_5}=\sum_{i=1}^{5}M_5^{i5}$, where*

$$M_5^{i5}=\left(\frac{t_2}{(L-x_i)^2}-\frac{t_1}{x_i^2}\right)\widetilde{M_5^{i5}},\quad i=1,2,3,4$$

and

$$M_5^{55}=\left(\frac{t_2}{(L-x_5)^2}-\frac{t_1}{x_5^2}\right)DetM_4.$$

Here

$$\widetilde{M_5^{15}}=\frac{1}{(x_5-x_2)^2}\left(-\frac{1}{(x_3-x_2)^2(x_4-x_2)^2(x_4-x_1)^2}\right.$$
$$\left.-\frac{1}{(x_4-x_2)^2(x_4-x_3)^2(x_3-x_1)^2}-\frac{1}{(x_4-x_2)^2(x_4-x_3)^2(x_2-x_1)^2}\right)$$
$$+\frac{1}{(x_5-x_3)^2}\left(-\frac{1}{(x_4-x_2)^2(x_3-x_1)^2(x_4-x_2)^2}\right.$$
$$\left.+\frac{1}{(x_4-x_2)^2(x_3-x_2)^2(x_4-x_1)^2}-\frac{1}{(x_4-x_2)^4(x_2-x_1)^2}\right)$$
$$+\frac{1}{(x_5-x_4)^2}\left(-\frac{1}{(x_2-x_1)^2(x_3-x_2)^2(x_4-x_3)^2}\right.$$
$$\left.+\frac{1}{(x_3-x_2)^2(x_3-x_1)^2(x_4-x_2)^2}-\frac{1}{(x_3-x_2)^4(x_4-x_1)^2}\right);$$
$$\widetilde{M_5^{25}}=\frac{1}{(x_2-x_1)^2}\left(-\frac{1}{(x_3-x_1)^2(x_4-x_3)^2(x_5-x_5)^2}\right.$$
$$\left.-\frac{1}{(x_4-x_3)^2(x_4-x_1)^2(x_5-x_3)^2}+\frac{1}{(x_4-x_3)^4(x_5-x_1)^2}\right)$$
$$+\frac{1}{(x_3-x_1)^2}\left(-\frac{1}{(x_3-x_1)^2(x_4-x_2)^2(x_5-x_4)^2}\right.$$
$$-\frac{1}{(x_4-x_3)^2(x_5-x_2)^2(x_4-x_1)^2}+\frac{1}{(x_4-x_3)^2(x_4-x_2)^2(x_5-x_1)^2}$$
$$\left.-\frac{1}{(x_3-x_2)^2(x_4-x_1)^2(x_5-x_4)^2}\right)$$
$$+\frac{1}{(x_4-x_1)^2}\left(-\frac{1}{(x_3-x_1)^2(x_4-x_2)^2(x_5-x_3)^2}\right.$$
$$+\frac{1}{(x_3-x_2)^2(x_4-x_3)^2(x_5-x_1)^2}-\frac{1}{(x_4-x_3)^2(x_5-x_2)^2(x_3-x_1)^2}$$

$$- \frac{1}{(x_3-x_2)^2(x_4-x_1)^2(x_5-x_3)^2}\Bigg);$$

$$\frac{1}{(x_4-x_2)^2(x_4-x_1)^2(x_5-x_3)^2} - \frac{1}{(x_5-x_1)^2(x_4-x_3)^2(x_4-x_2)^2}$$

$$+ \frac{1}{(x_4-x_1)^2(x_3-x_2)^2(x_5-x_4)^2}\Bigg)$$

$$-\frac{1}{(x_3-x_1)^2}\Bigg(-\frac{1}{(x_2-x_1)^2(x_4-x_2)^2(x_5-x_4)^2}$$

$$+ \frac{1}{(x_4-x_1)^2(x_5-x_2)^2(x_4-x_2)^2} - \frac{1}{(x_4-x_2)^4(x_5-x_1)^2}\Bigg)$$

$$+\frac{1}{(x_4-x_1)^2}\Bigg(-\frac{1}{(x_2-x_1)^2(x_4-x_2)^2(x_5-x_3)^2}$$

$$-\frac{1}{(x_3-x_2)^2(x_4-x_1)^2(x_4-x_2)^2} + \frac{1}{(x_4-x_2)^2(x_5-x_1)^2(x_3-x_2)^2}$$

$$- \frac{1}{(x_2-x_1)^2(x_4-x_2)^2(x_4-x_3)^2}\Bigg);$$

$$\widetilde{M}_5^{45} = \frac{1}{(x_2-x_1)^2}\Bigg(-\frac{1}{(x_3-x_2)^2(x_4-x_3)^2(x_5-x_1)^2}$$

$$+\frac{1}{(x_3-x_1)^2(x_4-x_2)^2(x_5-x_3)^2} + \frac{1}{(x_5-x_3)^2(x_4-x_3)^2(x_2-x_1)^2}$$

$$+ \frac{1}{(x_3-x_1)^2(x_3-x_2)^2(x_5-x_1)^2}\Bigg)$$

$$-\frac{1}{(x_3-x_1)^2}\Bigg(-\frac{1}{(x_2-x_1)^2(x_3-x_2)^2(x_5-x_1)^2}$$

$$+ \frac{1}{(x_4-x_2)^2(x_5-x_2)^2(x_3-x_1)^2} - \frac{1}{(x_4-x_2)^2(x_3-x_2)^2(x_5-x_1)^2}$$

$$+ \frac{1}{(x_4-x_3)^2(x_2-x_1)^2(x_5-x_2)^2}\Bigg)$$

$$+\frac{1}{(x_4-x_1)^2}\Bigg(-\frac{1}{(x_2-x_1)^2(x_3-x_2)^2(x_5-x_3)^2}$$

$$-\frac{1}{(x_3-x_1)^2(x_5-x_2)^2(x_3-x_2)^2} + \frac{1}{(x_5-x_1)^2(x_3-x_2)^4}\Bigg);$$

Proposition 3. *If $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$ is the root of polynomial equation $DetM_5 = 0$, then for given t_1, t_2, L and fixed $q_1 > 0$, there exist balancing charges q_2, q_3, q_4, q_5.*

Remark 1. $DetM_5 = 0$ is a polynomial equation of degree 14 with 5 variables. The computer experiment shows that, in the general case, for fixed real numbers $0 < x_1, x_2, x_3, x_4 < L$ there always exists a real solution x_5 of this equation which satisfies the condition $0 < x_5 < L$.

Remark 2. The solution of direct problem of electrostatic is reduced to the analysis of the system of equations (11) for $N = 5$:

$$u_1 + \frac{1}{(u_1-u_2)^2} + \frac{1}{(u_1-u_3)^2} + \frac{1}{(u_1-u_4)^2} + \frac{1}{(u_1-u_5)^2} = 0,$$

$$u_2 - \frac{1}{(u_2-u_1)^2} + \frac{1}{(u_2-u_3)^2} + \frac{1}{(u_2-u_4)^2} + \frac{1}{(u_2-u_5)^2} = 0,$$

$$u_3 - \frac{1}{(u_3-u_1)^2} - \frac{1}{(u_3-u_2)^2} + \frac{1}{(u_3-u_4)^2} + \frac{1}{(u_3-u_5)^2} = 0,$$

$$u_4 - \frac{1}{(u_4-u_1)^2} - \frac{1}{(u_4-u_2)^2} - \frac{1}{(u_4-u_3)^2} + \frac{1}{(u_4-u_5)^2} = 0,$$

$$u_5 - \frac{1}{(u_5-u_1)^2} - \frac{1}{(u_5-u_2)^2} - \frac{1}{(u_5-u_3)^2} - \frac{1}{(u_5-u_4)^2} = 0.$$

It is difficult to solve this system without the assumption that the roots are located symmetrically with respect to 0. In [16] it the numerical solution of this system up to $N = 10$ is given. For $N = 5$, i.e. solution of the above system is $(-1.7429, -0.8221, 0, 0.8221, 1.7429)$. On the other hand, using Proposition 3, solutions of the inverse problem may be obtained by computer algebra system for PC.

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