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ON THE CAUCHY INTEGRALS WITH THE WEIERSTRAß KERNEL

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Abstract. In the work the integral with the Weierstraß kernel and its properties is considered. Some problems of hydrodynamics associated with this integral are presented. Linear and nonlinear singular integral equations with the Weierstraß kernel arising in planar problems of hydrodynamics are given. The results of the author connected with the linear singular integral equation with the Weierstraß kernel are discussed. The properties of solutions of the nonlinear singular integral equation with the Weierstraß kernel associated with the Planar Stokes waves are analyzed. The sufficient condition of the existence of solutions of these non-linear integral equation is obtained.

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1. Introduction

The work deals with the integrals with the Weierstraß kernel and their applications.

In a complex z-plane ($z = x + iy$) we consider the integral of the following type

$$W(z) = \frac{1}{2\pi i}\int_{L_0} \phi(t)\zeta(t-z)dt, \tag{1}$$

where L_0 is a piece-wise smooth line [1], ζ is the Weierstraß zeta-function [2, 3, 4], $\phi(t)$ is the function of Muskhelishvili-Kveselava H^* class [1]

Definition 1. If the function $\phi(t)$ given on L_0 , satisfies the Hölder condition on every closed part of L_0 not containing the finite number of points c_i (i=1,2,...,p) of L_0, and if at those points the following condition holds

$$\phi(t) = \frac{\phi^*(t)}{(t-c_i)^\alpha},\ 0 < \alpha < 1,$$

where $\phi^*(t) \in H$ on L_0, then $\phi(t)$ will be said to belong to the class H^* on L_0.

If $(t-c_i)^\epsilon\phi(t)$ is Hölder continuous near the point c_i for any arbitrary small $\epsilon > 0$ then $\phi(t)$ is sad to belong to the class H^*_ϵ.

We now recall some definitions from the theory of elliptic functions.

Definition 2. Weierstraß zeta-function is the quasi-periodic function given by the double series

$$\zeta(z) = \frac{1}{z} + \sum_{\substack{|m|+|n|\neq 0 \\ m,n=-\infty}}^{\infty}\left(\frac{1}{z-T_{mn}} + \frac{1}{T_{mn}} + \frac{z}{T_{mn}^2}\right),\ T_{mn} = 2m\omega_1 + 2ni\omega_2, \tag{2}$$

ω_1 and ω_2 are the given complex numbers satisfying the condition

$$Im\left(\frac{i\omega_2}{\omega_1}\right) > 0.$$

The series (2) converges uniformly in every closed region of z-plane not containing the points T_{mn}.

The Weierstraß zeta-function has the following properties:

1) It is a meromorphic function with the simple poles T_{mn} $m, n = 0, \pm 1, \pm 2, \dots,$

2) $\zeta(-z) = -\zeta(z)$,

3) $\zeta(z)$ is a double quasi-periodic function i.e.,

$$\zeta(z + 2\omega_1) = \zeta(z) + \delta_1, \;\; \zeta(z + 2i\omega_2) = \zeta(z) + \delta_2,$$

where δ_1 and δ_2 are the addends of the zeta-function

$$\delta_1 = 2\zeta(\omega_1), \;\;\; \delta_2 = 2\zeta(i\omega_2), \;\; i\omega_2\delta_1 - \omega_1\delta_2 = \pi i,$$

4) $[\ln\sigma(z)]' = \zeta(z)$, where $\sigma(z)$ is the Weierstraß sigma-function.

5) Zeta-function is represented by the series [2, 3, 4]

$$\zeta(z) = \frac{\delta_1 z}{2\omega_1} + \frac{\pi}{2\omega_1} ctg \frac{\pi z}{2\omega_1} + \frac{2\pi}{\omega_1}\Sigma_{r=1}^{\infty}\frac{h^{2r}}{1 - h^{2r}} \sin\frac{r\pi z}{\omega_1}, \tag{3}$$

where $h = \exp\frac{-\pi\omega_2}{\omega_1}$.

Definition 3. Weierstraß sigma-function is the holomorphic function defined as the following infinite product [2, 3, 4]

$$\sigma(z) = z \prod_{\substack{|m|+|n|\neq 0 \\ m,n=-\infty}}^{\infty} \Big(1 - \frac{z}{T_{mn}}\Big) \exp\Big(\frac{1}{T_{mn}} + \frac{z^2}{2T_{mn}^2}\Big), \tag{4}$$

σ- function is the holomorphic function with the simple zeros at the points $T_{mn}, \;\; m, n = \pm 1, \pm 2, \cdots$ and has the following properties

$$1.\ \sigma(z + 2\omega_1) = -\sigma(z)\exp(\delta_1 z + \delta_1\omega_1),$$

$$2.\ \sigma(z + 2i\omega_2) = -\sigma(z)\exp(\delta_2 z + i\delta_2\omega_2),$$

$$3.\ \sigma(z) = \sin\frac{\pi z}{2\omega_1}\exp\left[\frac{\delta_1}{4\omega_1}(z^2 - \omega_1^2)\right]\prod_{r=1}^{\infty}\exp\left[-\frac{2}{r}\frac{h^{2r}}{(1 - h^{2r})}\cos\frac{r\pi z}{\omega_1}\right]. \tag{5}$$

σ-function is not doubly quasi-periodic, but by means of it any elliptic function can be constructed [2, 3, 4].

Theorem 1. *Every elliptic function F(z) of n-th order with zeros* $\alpha_2, ..., \alpha_n$ *and poles* $\beta_1, \beta_2, ..., \beta_n$ *in the fundamental parallelogram (parallologram with the vertices* $0, 2\omega_1, 2\omega_1 + 2i\omega_2, 2i\omega_2$*) can be represented in the form*

$$F(z) = C_0\frac{\sigma(z - \alpha_1)\sigma(z - \alpha_2)...\sigma(z - \alpha_n)}{\sigma(z - \beta_1)\sigma(z - \beta_2)...\sigma(z - \beta_n)}, \tag{6}$$

where $\alpha_1 = (\beta_1 + \beta_2 + ... + \beta_n) - (\alpha_2 + ... + \alpha_n)$*,* C_0 *is the definite constant.*

According to formula (2) the function given by (1) represents the Cauchy type integral and is the sectionally holomorphic double quasi-periodic function with the jump line L [5-10], where L is a union of a countable number of smooth non-intersected contours $L^j_{mn}; j = 1, 2, \dots, k; m, n = 0, \pm 1, \pm 2, \dots$ double-periodically distributed in the z-plane with periods $2\omega_1$ and $2i\omega_2$

$$L = \bigcup_{m,n=-\infty}^{\infty} L_{mn}, \;\; L_{00} = L_0,$$

$$L_{mn} = \bigcup_{j=1}^{k} L^j_{mn}, \;\; L^{j_1}_{mn} \bigcap L^{j_2}_{mn} = \emptyset, \;\; j_1 \neq j_2, \;\; j_1, j_2 = 1, 2, \dots, k.$$

The integral (1) has various applications in double-quasi periodic problems of hydrodynamics [5, 7, 11, 12].

The properties of the integral (1) was first studied by Sedov [5]. He has studied the following boundary value problem arising in planar hydrodynamics

Problem 1. In the stripe $0 < y < \omega_2$ find a periodic analytic function $F(z)$ with the period $2\omega_1$, satisfying the boundary conditions

$$ImF(z)|_{y=0} = v_1; \;\; ImF(z)|_{y=\omega_2} = v_2; \;\; 0 < x < 2\omega_1,$$

where v_1 and v_2 are the given functions of the Hölder class subject to the condition

$$\int_{i\omega_2}^{i\omega_2+2\omega_1} v_2(t)dt = \int_0^{2\omega_1} v_1(t)dt.$$

By the modification of Villat's formula [13] Sedov has obtained the solution of Problem 1

$$F(z) = \frac{1}{\pi}\int_{i\omega_2}^{i\omega_2+2\omega_1} v_2(t)[\zeta(t-z) - \zeta(t)]dt - \frac{1}{\pi}\int_0^{2\omega_1} v_1(t)[\zeta(t-z) - \zeta(t)]dt + K,$$

K is an arbitrary real constant.

By means of the integral (1) Sedov has also solved double-periodic planar problems of the theory of hydroturbins [5].

The integrals of the type (1) were used in [6] for studying the linear conjugation problem with the double-periodic jump line for the class of double-periodic functions.

The following theorem is true [6]

Theorem 2. *If $\phi \in H^*$ on L_0, then $W(z)$ is a sectionally holomorphic double quasi- periodic function with the jump line L and with the addends*

$$\gamma_1 = -\frac{\delta_1}{2\pi i}\int_{L_0} \phi(t)dt, \;\; \gamma_2 = -\frac{\delta_2}{2\pi i}\int_{L_0} \phi(t)dt,$$

there exists the limiting values of $W(z)$ from the left and from the right of L and the following formula is valid

$$W^{\pm}(t_0) = \pm\frac{\phi(t_0)}{2} + \frac{1}{2\pi i}\int_{L_0} \phi(t)\zeta(t - t_0)dt, \;\; t_0 \in L,$$

where $W^{\pm}(t_0)$ denotes boundary values from the left and from the right of L respectively.

By means of the function $W(z)$ in [7-10, 14, 15] the effective solutions of the linear conjugation problem with the double-periodic jump line for the class of polynomially and exponentially double-quasi periodic functions are obtained.

In [16] the following theorem is proved

Theorem 3. *If*

$$\int_{L_0} \phi(t)dt = 0,$$

then $W(z)$ is a double-periodic sectionally holomorphic function representable as the double series

$$W(z) = \frac{1}{2\pi i}\int_L \left[\frac{\phi(t)}{t-z} - \frac{\phi(t)}{t}\right] dt$$

$$= \sum_{m,n=-\infty}^{\infty} \frac{1}{2\pi i}\int_{L_{mn}} \left[\frac{\phi(t)}{t-z} - \frac{\phi(t)}{t}\right] dt + \frac{1}{2\pi i}\int_{L_0} \phi(t)\zeta(t)dt.$$

In double-quasi periodic problems of hydrodynamics we have the linear singular integral equation of the type [7, 11, 12]

$$\frac{1}{2\pi i}\int_{L_0} \phi(t)\zeta(t-t_0)dt = f(t_0),\ f \in H^*,\ t_0 \in L_0. \qquad (7)$$

The singular integral equation (7) when the line of integration is the union of a countable number of segments distributed along the axis ox was studied in [20, 21] and in the case of countable number of closed contours in [22].

The equation (7) was solved completely in [16- 19] for the different types of the line L_0. In [19] it is proved that the solution of equation (7) of the Muskhelishvili-Kveselava class always exists and the effective solutions are obtained.

In the future we will use the results of Muskhelishvili [1] and Melnik [32].

Theorem 4. *If the function $\phi(t) \in H^*$ at the line $L = [0,1]$, then the following formula is true*

$$\int_0^1 \frac{\phi(t)}{t-\tau}dt = ctg\alpha\pi\frac{\phi^*(0)}{(t-c_i)^\alpha} + f^*(\tau),\ 0<\alpha<1,\ \tau \in [0,1], i = 1,2,...,p, \qquad (8)$$

where the function $(t-c_i)_1^\alpha f^*$; $0 < \alpha_1 < \alpha$ is Hölder continuous function [1, Chapter 1].

By the notation $t^2 = t'$ it is easy to obtain

Corollary 4.1 If the function $\phi(t) \in H$ at the line $L = [0,1]$, then the following formula is true

$$\int_0^1 \frac{\phi(t)}{t^2-\tau^2}dt = f^*(\tau),\ \tau \in [0,1],$$

where the function $(t)^{1/2}f^*$ is Hölder continuous function and

$$\lim_{t_0 \rightarrowtail 0} (t)^{1/2} f^* = 0.$$

Theorem 5. *If the function $\phi(t) \in H$ at the line $L = [0,1]$, then the following formula is true*

$$\int_0^1 \frac{\phi(t)[\ln t]}{t-\tau}dt = -\frac{\phi(0)}{2}[\ln^2(\tau) - 2\pi i \ln(\tau)] + f^{**}(\tau), \tag{9}$$

where the function f^{**} is Hölder continuous on $[0,1]$.

Let us consider the following integral

$$\int_0^1 \frac{[\ln t]}{t+\tau}dt.$$

Putting the notation $-\tau = \tau'$; $\tau' \in [-1,0]$, according to Theorem 5 we obtain

$$\int_0^1 \frac{[\ln t]}{t+\tau}dt = -\frac{\ln^2(-\tau)}{2} + \pi i[\ln(\tau) + \pi i] + f^{**}(-\tau).$$

Corollary 5.1 For $\tau \in [0,1]$ the following formula is true

$$\int_0^1 \ln t[\frac{1}{t-\tau} - \frac{1}{t+\tau}]dt = \frac{\pi^2}{2} + f^{**}(\tau) - f^{**}(-\tau),\ \tau \in [0,1].$$

2. Analysis of the nonlinear integral equation with the Weierstraß kernel

In the present work we consider the nonlinear integral equation with the Weiersraß kernel

$$\begin{aligned} v(t_0) = 3g \textstyle\int_0^{t_0} \sin \Big[\frac{1}{3\pi} \int_0^{2\omega_1} [\ln v(t)][\zeta(t-\tau) - \zeta(t-\tau-i\omega_2) \\ -\zeta(t) + \zeta(t-i\omega_2)]dt \Big] d\tau,\ t_0 \in [0, 2\omega_1], \end{aligned} \tag{10}$$

where g is gravity acceleration, the function $v(t_0)$ is an unknown function of the H^* class, $\ln z$ is the branch for which $\ln 1 = 0$.

The equation (10) is associated with the planar Stokes waves and was obtained by the author in [11, 12]. By these results the profile of Stokes wave is given by the formula

$$f_0(t_0) = \frac{1}{2g}(2A - v^{2/3}(t_0)), \tag{11}$$

A is the definite constant. The function $f_0(t_0)$ is one period of Stokes wave with the periods $2n\omega_1; n = \pm 1, \pm 2, \ldots$. These waves are gravity waves in incompressible Euler fluid [11, 12, 23, 27].

In [28] by means of the conformal mapping method equation (10) is reduced to the integral equation with the weakly singular kernel and the existence of solution of this equation is shown. In [29,30] the approximate solution of this equation is obtained.

Here we will analyze the behavior of the solution of equation (10) near the point $t_0 = 0$ and construct the approximate solution in the case of the symmetric wave i. e. $f_0(-t_0) = f_0(t_0)$. Consequently $v(-t_0) = v(t_0)$. For this purpose we rewrite equation (10) in the different form.

Let us consider the following integral

$$K_0(\tau) = \int_0^{2\omega_1} [\ln v(t)][\zeta(t-\tau) - \zeta(t-\tau-i\omega_2) \\ -\zeta(t) + \zeta(t-i\omega_2)]dt, \ \tau \in [-2\omega_1, 2\omega_1]. \tag{12}$$

Substituting in (12) the variable $t = 2\omega_1 - t'$ and using the property of zeta-function 2 one obtains

$$K_0(\tau) = -\int_0^{2\omega_1} [\ln v(t)][\zeta(t+\tau) - \zeta(t+\tau+i\omega_2) \\ -\zeta(t) + \zeta(t+i\omega_2)]dt, \ \tau \in [-2\omega_1, 2\omega_1]. \tag{13}$$

By using the representations (12),(13) and the formula $\zeta(t+i\omega_2) = \zeta(t - i\omega_2 + 2i\omega_2) = \zeta(t-i\omega_2) + \delta_2$ equation (10) can be rewritten in the form

$$v(t_0) = 3g \int_0^{t_0} \sin \frac{1}{6\pi} A[v(\tau)]d\tau, \tag{14}$$

where

$$A[v(\tau)] = \int_0^{2\omega_1} [\ln v(t)]K(t,\tau)dt, \tag{15}$$

and the kernel $K(t,\tau)$ is representable in the form

$$K(t,\tau) = \zeta(t-\tau) - \zeta(t+\tau) - \zeta(t-\tau-i\omega_2) + \zeta(t+\tau-i\omega_2) \\ = \zeta(t-\tau) - \zeta(t+\tau) - 1/2\zeta(t-\tau-i\omega_2) - 1/2\zeta(t-\tau+i\omega_2 \\ +1/2\zeta(t+\tau-i\omega_2) + 1/2\zeta(t+\tau+i\omega_2). \tag{16}$$

The function given by the formula (16) is the function of two variables t and τ and is double-periodic with respect to this variables with the periods $2\omega_1$ and $2i\omega_2$. By using the properties of zeta-function it is easy to check, that zeros of $K(t,\tau)$ are $0; \omega_1; i\omega_2; -\omega_1 - i\omega_2$ with respect to τ and $\frac{\omega_1}{2}; -\frac{i\omega_2}{2}; \frac{i\omega_2}{2} + \omega_1; \frac{3i\omega_2}{2} - \omega_1$ with respect to t.The poles of $K(t,\tau)$ are $-t; t; t - i\omega_2; -t + i\omega_2$ with respect to τ and $-\tau; \tau; \tau + i\omega_2; -\tau + i\omega_2$ with respect to t. By using the Theorem 1 after simple transformations we obtain

$$K(t,\tau) = C_0 \times \frac{\sigma(\tau)\sigma(\tau-\omega_1)\sigma(\tau-i\omega_2)\sigma(\tau+\omega_1+i\omega_2)}{\sigma(t-\tau)\sigma(t+\tau)} \\ \times \frac{\sigma(t-i\omega_2/2)\sigma(t+i\omega_2/2)\sigma(t-i\omega_2/2-\omega_1)\sigma(t-3i\omega_2/2+\omega_1)}{\sigma(t-\tau-i\omega_2)\sigma(t+\tau-i\omega_2)}, \tag{17}$$

where

$$C_0 = \frac{2\zeta(\omega_1 - i\omega_2/2) + 2\zeta(\omega_1 + i\omega_2/2)}{\sigma^4(i\omega_2/2)}.$$

Taking into the account the formulaes (3) and (5) in (17) we can rewrite the function $K(t,\tau)$ in the form

$$K(t,\tau) = 2C_0 \frac{sin\frac{\pi\tau}{2\omega_1} cos\frac{\pi\tau}{2\omega_1}}{sin^2\frac{\pi t}{2\omega_1} - sin^2\frac{\pi\tau}{2\omega_1}} \sigma^*(t,\tau), \tag{18}$$

where

$$\sigma^*(t,\tau)=\frac{\sigma(\tau-i\omega_2)\sigma(\tau+\omega_1+i\omega_2)}{\sigma(t-\tau-i\omega_2)}$$

$$\times\frac{\sigma(t-i\omega_2/2)\sigma(t+i\omega_2/2)\sigma(t-i\omega_2/2-\omega_1)\sigma(t-3i\omega_2/2+\omega_1)}{\sigma(t+\tau-i\omega_2)} \qquad (19)$$

$$\times\exp[\frac{\delta_1}{4\omega_1}(\omega_1^2-2\tau\omega_1-2t^2)]\Pi_{r=1}^{\infty}\exp[-\frac{4}{r}\frac{h^{2r}}{(1-h^{2r})}(\cos\frac{r\pi\tau}{\omega_1}-\cos\frac{r\pi t}{2\omega_1}\cos\frac{r\pi\tau}{2\omega_1})].$$

By using properties 1, 2, and 3 of “zeta-function” after simple transformations we obtain

$$A[v(\tau)]=2\int_0^{\omega_1}[\ln v(t)][K(t,\tau)]dt. \qquad (20)$$

From the viewpoint of hydrodynamics we suppose, that the solution of the equation (14) is reperesentable in the form

$$v=v_0\sin\frac{\pi t_0}{2\omega_1}, \qquad (21)$$

where v_0 is unknown bounded function of the H^* class satisfying the inequality $0<m\le v_0\le 1$ (m is the definite constant).

According to (2) and (16) the function $K(t,\tau)$ is representable in the form

$$K(t,\tau)=\frac{1}{t-\tau}-\frac{1}{t+\tau}+K_1(t,\tau), \qquad (22)$$

where $K_1(t,\tau)$ is the analytic function having definite limit at the point $\tau=0$.

According to the Corollaries 4.1, 5.1 and the representations (20), (21), (22) the function $A[v(\tau)]$ satisfies the following

$$\lim_{\tau\rightarrowtail 0}A[v(\tau)]=\pi^2. \qquad (23)$$

If in the formula (14) we use (23) and the representation

$$\sin\frac{1}{6\pi}A[v(\tau)]=\sin\frac{1}{6\pi}[A[v(\tau)]-\pi^2+\pi^2]$$

$$=\sin\frac{1}{6\pi}[A[v(\tau)]-\pi^2]\cos\pi/6+\sin\pi/6\cos\frac{1}{6\pi}[A[v(\tau)]-\pi^2],$$

we obtain

$$\lim_{t_0\rightarrowtail 0}\frac{v(t_0)}{t_0}=3g\sin\pi/6,\quad \lim_{t_0\rightarrowtail 0}\frac{v(t_0)}{\sin\pi t_0/2\omega_1}=3g\omega_1/\pi. \qquad (24)$$

Hence, we conclude

Theorem 6. *If the solution of equation (14) is representable by formula (21) then formula (24) is true.*

Now, let us suppose

$$\sin\frac{1}{6\pi}[A[v(\tau)]-\pi^2]\approx\frac{1}{6\pi}[A[v(\tau)]-\pi^2] \qquad (25)$$

and

$$\ln\frac{v_0(t_0)}{M_0}\approx\frac{v_0(t_0)}{M_0}-1;M_0=3g\omega_1/\pi;t_0\in[0,\omega_1]. \qquad (26)$$

Taking into the account (25) and (26) we can rewrite equation (14) in the form

$$\begin{aligned} v(t_0) &= \frac{\sqrt{3}g}{4\pi}\int_0^{t_0} A[v(\tau)]d\tau + \frac{gt_0}{4}(3-\sqrt{3}\pi) \\ &= \frac{\sqrt{3}g}{4\pi}\int_0^{\omega_1}[\ln v(t)]K_0(t_0,t)dt + \frac{gt_0}{4}(3-\sqrt{3}\pi), \end{aligned} \tag{27}$$

where

$$\begin{aligned} &K_0(t_0,t) = \textstyle\int_0^{t_0} K(t,\tau)d\tau \\ &= 1/2\ln\Big|\frac{\sigma(t_0-t-i\omega_2)\sigma(t_0-t+i\omega_2)\sigma(t_0+t-i\omega_2)\sigma(t_0+t+i\omega_2)}{\sigma^2(t_0-t)\sigma^2(t_0+t)} \\ &\times\frac{\sigma^4(t)}{\sigma^2(t+i\omega_2)\sigma^2(t-i\omega_2)}\Big|. \end{aligned} \tag{28}$$

Taking into the account

$$\int_0^{\omega_1} K_0(t_0,t)dt = \delta_1 t_0^2,$$

we can rewrite equation (27) in the form (the equation with respect to $v_0; t_0 \in [0,\omega_1]$)

$$v_0(t_0) = \frac{\sqrt{3}g}{4\pi u_0(t_0)}\int_0^{\omega_1}\Big[\frac{v_0(t)}{M_0}-1\Big]K_0(t_0,t)dt + f_0(t_0),\ u_0(t) = \sin\frac{\pi t_0}{2\omega_1}, \tag{29}$$

where

$$f_0(t_0) = \frac{\sqrt{3}g}{2\pi u_0(t_0)}\int_0^{\omega_1}[\ln u_0(t)]K_0(t_0,t)dt + \frac{\sqrt{3}g\delta_1 t_0^2}{4\pi u_0(t_0)}\ln M_0 + \frac{gt_0}{4u_0(t)}(3-\sqrt{3}\pi).$$

Let us rewrite equation (29) in the form

$$v_0(t_0) = \frac{\sqrt{3}g\omega_1}{4\pi u_0(t_0)}\int_0^1\Big[\frac{v_0(\omega_1 t)}{M_0}-1\Big]K_0(t_0,\omega_1 t)dt + f_0(t_0), \tag{30}$$

where

$$f_0(t_0) = \frac{\sqrt{3}g\omega_1}{2\pi u_0(t_0)}\int_0^1[\ln u_0(t)]K_0(t_0,\omega_1 t)dt + \frac{\sqrt{3}g\delta_1 t_0^2}{4\pi u_0(t_0)}\ln M_0 + \frac{gt_0}{4u_0(t)}(3-\sqrt{3}\pi).$$

By Theorem 6 for the function v_0 we have

$$\lim_{t_0\rightarrowtail 0} v_0(t_0) = 3g\omega_1/\pi.$$

Let us denote the operator on the right hand side of (30) by $B[v_0]$. This operator is completely continuous in the area $|t_0| \le \omega_1;\ |v_0 - M_0| \le \epsilon_0,\ \epsilon_0 > 0$, [31, 33].

By Theorems 4 and 5 functions $f_0(t_0)$ and $\frac{\sqrt{3}g\omega_1}{4\pi u_0(t_0)}\int_0^1 K_0(t_0,\omega_1 t)dt$ are Hölder continues for $t_0 \in [0,\omega_1]$, hence

$$\begin{aligned} &|f_0 - M_0| \le C_1\omega_1{\omega_1}^{\alpha},\ 0<\alpha\le 1,\ 0<\beta\le 1, \\ &\left|\frac{\sqrt{3}g\omega_1}{4\pi u_0(t_0)}\int_0^1 K_0(t_0,\omega_1 t)dt - \frac{\sqrt{3}g\omega_1}{4\pi u_0(t_0)}\int_0^1 K_0(t_0,\omega_1 t)dt\right| \le C_2\omega_1{\omega_1}^{\beta}, \end{aligned} \tag{31}$$

where C_1 and C_2 are the definite constants.

According to (29) and (31) we obtain

$$|B[v_0] - M_0| \leq C_1\omega_1{\omega_1}^{\alpha} + C_2\frac{\epsilon_0}{M_0}\omega_1{\omega_1}^{\beta} = C_1\omega_1{\omega_1}^{\alpha} + C_2\frac{\epsilon_0\pi}{3g}{\omega_1}^{\beta}. \qquad (32)$$

By (32) the following theorem is true [31, 33]

Theorem 7. *If*

$$\omega_1 \leq \min\left\{\epsilon_0, \left(\frac{1}{2C_1}\right)^{1/\alpha}, \left(\frac{1}{2C_3}\right)^{1/\beta}, C_3 = \frac{C_2\pi}{3g}\right\},$$

there exists the solution of the equation (29) for $t_0 \in [0, \omega_1]$.

Remark 1. If there exists the solution of the equation (29), it belongs to the class H^* [33].

Taking into the consideration Theorem 7 we conclude:

If the conditions (25) and (26) hold, there exists the solution of the equation (27) and hence there exists the Stokes wave of the form

$$f_0(t_0) = A/g - \left[v_0 \sin\frac{\pi t_0}{2\omega_1}\right]^{2/3},$$

where v_0 is the function of the H^* class, $v_0 \neq 0$.

Remark 2. If the Stokes wave is not symmetric, it is of the form

$$f_0(t_0) = \frac{A}{g} - \frac{v_0^{2/3}}{\cos^{4/3}\left(\frac{\pi t_0}{2\omega_1}\right)}\sin^{2/3}\left(\frac{\pi t_0}{2\omega_1}\right)\ln^{2/3}\left(\sin\frac{\pi t_0}{2\omega_1}\right),$$

where v_0 is the function of the H^* class, $v_0 \neq 0$.

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