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THE PROBLEM OF FINDING AN EQUALLY STRONG CONTOUR FOR A RECTANGULAR PLATE WEAKENED BY A RECTILINEAR CUT, WHOSE ENDS ARE CUT OUT BY CONVEX SMOOTH ARCS

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Abstract. The problem of finding an equally strong contour for a rectangular plate weakened by a rectilinear cut which ends are cut out by convex smooth arcs (we will call the set of these arcs the unknown part of the boundary) is considered. It is assumed that absolutely smooth rigid punches are applied to every link of the rectangular. The punches are under the action of normal stretching forces with the given principal vectors and the internal part of the boundary is free from external forces. Our problem is to find an elastic equilibrium of the plate and analytic form of the unknown contour under the condition that the tangential normal stress on it takes the constant value (the condition of the unknown contour full-strength). The problem is solved by the method of complex analysis. The complex potentials of N. Muskhelishvili and equations of an unknown contour are constructed effectively (analytically).

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Introduction

The boundary value problems of the plane theory of elasticity and plate bending with a partially unknown boundary (or, which is the same, the problems of finding an equally stable contour) belong to the broad problems of optimization of elastic bodies - the problems of optimization of stress concentration for elastic plates with a hole. The study of these problems was started by G.P. Cherepanov [1], N.V. Banichuk [2], R.D. Bantsuri [3, 4] and always was in the focus of attention of many scientists. Different methods were introduced for researching these problems and among them one important is the method of complex analysis. Analogous problems of plane elasticity are considered in [5-10].

1. Statement of the problem

Let a middle surface of the homogeneous Isotropic plate on a plane z of a complex variable occupy a doubly-connected domain S_0 whose external boundary is a rectangle and the internal boundary is a rectilinear cut which ends are cut out by convex smooth arcs (the unknown part of the boundary, see Fig. 1). It is assumed that the sides of the rectangle are under the action of constant normal tensile forces with a given principal vector $2P$ and $2Q$

(we consider the symmetrical case), and the interior boundary is free from stresses.

Consider the problem: Find an elastic equilibrium of the rectangular and analytic form of an unknown contour under the condition that the tangential normal stress takes on that contour value $\sigma_\vartheta = k = const$.

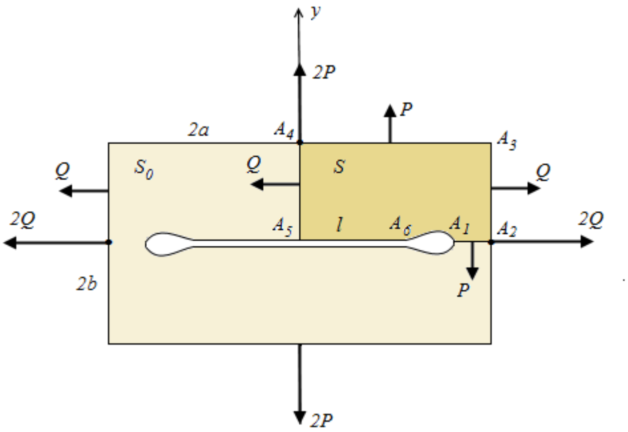

Fig. 1.

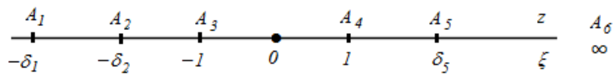

Fig. 2.

2. Solution of the problem

Due to the axial symmetry, we restrict ourselves to the consideration of elastic equilibrium on a quarter of the plate only and denote it by S. By $L = L_1 \cup L_2$ we denote its boundary consisting of rectilinear segments $L_1 = L^{(1)} \cup L^{(0)}$ $(L^{(1)} = \bigcup\limits_{k=1}^{4} L_k^{(1)},\ L_k^{(1)} = A_k A_{k+1};\ L^{(0)} = A_5 A_6)$ and arc $L_2 = A_6 A_1$.

It is not difficult to see that in this case the tangential stresses $\tau_{ns} = 0$ on the whole boundary $L = L_1 \cup L_2$ of the domain S, the normal displacements $v_n(t) = v_n^{(k)} = const$, $t \in L_k^{(1)}$ $(k = 2,3)$ and $v_n(t) = 0$ on $L_1^{(1)} \cup L_1^{(4)} \cup L_2$.

On the basis of the well-known Kolosov-Muskhelishvili formulas [11] the problem under consideration is reduced to finding two holomorphic in S functions $\varphi(z)$ and $\psi(z)$ with the following boundary conditions:

$$\operatorname{Re}\left[e^{-\alpha(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})\right] = C(t), \quad t \in L_1, \tag{1}$$

$$\operatorname{Re}\left[e^{-i\alpha(t)}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}\right] = 2\mu v_n(t), \quad t \in L_1, \tag{2}$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = 0;\ t \in L^{(0)} \cup L_2, \tag{3}$$

$$\mathrm{Re}\,[\varphi'(t)] = \frac{k}{4},\quad t \in L_2. \tag{4}$$

where $\alpha(t)$ is the angle lying between the ox-axis and the external normal to the boundary L_1 at the point $t \in L_1$, and $\alpha(t) = \alpha_k = const$, $t \in L_k^{(1)}$, $(k = \overline{1,4})$. $C(t)$ and $v_n(t)$ are the piecewise constant functions $C(t) = \mathrm{Re} \int\limits_{A_1}^{t} iN(s_0) \exp[\alpha(t_0) - \alpha(t)] ds_0 = \sum\limits_{j=1}^{k} \int\limits_{L_j^{(1)}} N(s_0) \sin[\alpha(k) - \alpha_j] ds_0$, $N(s_0)$ is a normal stress.

Summing up the equalities (1) and (2), differentiating with respect to the arc abscissa s and taking into account the fact that the functions $c(t)$ and $v_n(t)$ are piecewise constant, we obtain

$$\mathrm{Im}\,\varphi'(t) = 0,\quad t \in L_1. \tag{5}$$

We can write conditions (4) and (5) in the following form

$$\mathrm{Re}\left[\varphi'(t) - \frac{k}{4}\right] = 0,\quad t \in L_2;\quad \mathrm{Im}\left[\varphi'(t) - \frac{k}{4}\right] = 0,\quad t \in L_1. \tag{6}$$

After the conformal mapping of the domain S onto the circular ring, this problem for the function $\phi^0(\chi) = \varphi'[\omega_0(\chi)] - \frac{k}{4}$ reduces to the Riemann-Hilbert problem which has a unique solution

$$\varphi(z) = \frac{k}{4} z \tag{7}$$

(an arbitrary constant of integration is assumed to be equal to zero).

By virtue of the relation (7), the boundary condition (1) and (3), for the functions

$$\Phi_1(z) = \frac{k}{2} z - \psi(z) \tag{8}$$

we obtain the following boundary value problem:

$$\begin{aligned} &\mathrm{Im}\Phi_1(t) = 0,\quad t \in L_1^{(1)} \cup L^{(0)} \cup L_2; \\ &\mathrm{Re}\Phi_1(t) = P + ka,\quad t \in A_2A_3; \\ &\mathrm{Im}\,\Phi_1(t) = Q,\quad t \in A_3A_4; \\ &\mathrm{Re}\Phi_1(t) = 0,\quad t \in A_4A_5. \end{aligned} \tag{9}$$

Analogously, from (1), (3), (7) and (9) for the functions

$$\Phi_2(z) = i\left[\frac{k}{2} z + \psi(z)\right] \tag{10}$$

we obtain the following boundary value problem:

$$\begin{aligned} &\mathrm{Im}\Phi_2(t) = 0, \quad t \in L_4^{(1)} \cup L^{(0)} \cup L_2; \\ &\mathrm{Re}\Phi_2(t) = 0, \quad t \in A_2A_3; \\ &\mathrm{Im}\ \Phi_2(t) = P, \quad t \in A_2A_3; \\ &\mathrm{Re}\Phi_2(t) = Q - kb, \quad t \in A_3A_4, \end{aligned} \tag{11}$$

where $2a$ and b are the length of the sides of the rectangle S_0.

Consider the problem (9). Let the function $z = \omega(\zeta)$ map conformally the upper half-plane ($\mathrm{Im}\zeta > 0$) onto the domain S. By a_k we denote preimages of the points A_k ($k = \overline{1,6}$) and assume that $a_6 = \infty$; $a_1 = -\delta_1$; $a_2 = -\delta_2$; $a_3 = -1$; $a_4 = 1$; $a_5 = \delta_5$ (see Fig. 1), moreover δ_1, δ_2 and δ_5 are unknown parameters

We easily observe that problem (9) for the function $\Phi_{10}(\zeta) = \Phi_1[\omega(\zeta)]$ reduces to the Keldysh-Sedov problem (see [12], [13])

$$\begin{aligned} &\mathrm{Im}\Phi_{10}(t) = 0, \quad t \in (-\infty; -\delta_1] \cup [-\delta_1; -\delta_2] \cup [\delta_5; \infty); \\ &\mathrm{Re}\Phi_{10}(t) = -P + ka, \quad t \in [-\delta_2; -1]; \\ &\mathrm{Im}\Phi_{10}(t) = Q, \quad t \in [-1; 1]; \\ &\mathrm{Re}\Phi_{10}(t) = 0, \quad t \in [1; \delta_5]. \end{aligned} \tag{12}$$

We will seek for a bounded at infinity solution of problem (12) of the class $h(a_1, ..., a_6)$ (regarding this class see [12]). The indices of these problems of the given class are equal to -2.

The necessary and sufficient condition for the existence of a bounded at infinity solution of problem (10) has the form (see [12], [13])

$$(-P + ka) \int_{-\delta_2}^{-1} \frac{dt}{\chi_1(t)} + iQ \int_{-1}^{1} \frac{dt}{\chi_1(t)} = 0 \tag{13}$$

and the solution itself is given by the formula

$$\Phi_{10}(\xi) = \frac{\chi_1(\zeta)}{\pi i} \left[(-P + ka) \int_{-\delta_2}^{-1} \frac{dt}{\chi_1(t)(t - \zeta)} + iQ \int_{-1}^{1} \frac{dt}{\chi_1(t)(t - \zeta)} \right], \tag{14}$$

where $\chi_1(\zeta) = \sqrt{(\zeta + \delta_2)(\zeta + 1)(\zeta - 1)(\zeta - \delta_5)}$.

We have analogous results for the problem (11). Namely, for the function $\Phi_{20}(\zeta) = \Phi_2[\omega(\zeta)]$, the problem (11) is reduced in the Keldysh-Sedov problem:

$$\begin{aligned}
&\mathrm{Im}\Phi_{20}(t) = 0, \ \ t \in (-\infty; -\delta_1] \cup [1; \delta_5] \cup [\delta_5; \infty);\\
&\mathrm{Re}\Phi_{20}(t) = 0, \ \ t \in [-\delta_1; -\delta_2];\\
&\mathrm{Im}\Phi_{20}(t) = P, \ \ t \in [-\delta_2; -1];\\
&\mathrm{Re}\Phi_{20}(t) = Q - kb, \ \ t \in [-1; 1].
\end{aligned} \tag{15}$$

For the class $h(-\delta_1; -\delta_2; -1; 1)$ the necessary and sufficient condition for the existence of a bounded at infinity solution of problem (15) has the form

$$iP \int_{-\delta_2}^{-1} \frac{dt}{\chi_2(t)} - (Q - kb) \int_{-1}^{1} \frac{dt}{\chi_2(t)} = 0, \tag{16}$$

and the solution itself is given by the formula

$$\Phi_{20}(\zeta) = \frac{\chi_2(\zeta)}{\pi i} \left[iP \int_{-\delta_2}^{-1} \frac{dt}{\chi_2(t)(t-\zeta)} + (Q - kb) \int_{-1}^{1} \frac{dt}{\chi_2(t)(t-\zeta)} \right], \tag{17}$$

where $\chi_2(\zeta) = \sqrt{(\zeta + \delta_1)(\zeta + \delta_2)(\zeta + 1)(\zeta - 1)}$.

Having found the fuctions $\Phi_{10}(\zeta)$ and $\Phi_{20}(\zeta)$, by virtue of (8) and (10), we can define the fuctions $\omega(\zeta)$ and $\psi_0(\zeta) = \psi[\omega(\zeta)]$ by the formulas

$$\omega(\zeta) = \frac{1}{k}[\Phi_{10}(\zeta) - i\Phi_{20}(\zeta)], \tag{18}$$

$$\psi_0(\xi) = -\frac{1}{2}[\Phi_{10}(\zeta) + i\Phi_{20}(\zeta)]. \tag{19}$$

The equation for the part $L_2 = A_6 A_1$ of the unknown contour can be obtained from the image of the function (18) for $\zeta = \xi \in [-\infty; \ -\delta_1]$.

For determination of the parameters δ_1, δ_2, δ_5 and k with conditions (13) and (16), we have two conditions from the equality $\omega(\infty) = l$, where $2l$ is length of the rectilinear cut. From $\omega(\infty) = l$ we have

$$(-p + ka) \int_{1}^{\delta_2} \frac{t\, dt}{|\chi_1(t)|} + Q \int_{-1}^{1} \frac{t\, dt}{|\chi_1(t)|} = k\pi\ell, \tag{20}$$

$$P \int_{1}^{\delta_2} \frac{t\, dt}{|\chi_2(t)|} + (Q - kb) \int_{-1}^{1} \frac{t\, dt}{|\chi_2(t)|} = 0. \tag{21}$$

Notice that the integrals appearing in formula (13), (14), (16), (17), (19) and (20) are expressed by elliptic integrals of the first and third kind (see [14]).

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