CONCYCLIC AND ALIGNED EQUILIBRIUM CONFIGURATIONS OF POINT CHARGES

Giorgadze G., Khimshiashvili G.

Abstract. We discuss equilibrium configurations of point charges with Coulomb interaction confined to a circle or linear segment. Specifically, we aim at characterization of finite configurations of points which can serve as equilibrium configurations of repulsive point charges. We also investigate the stability of arising equilibrium configurations. For concyclic configurations, we present a few general results concerned with these two problems. For aligned configurations, the main results refer to three points in a linear segment with prescribed point charges at its ends. We also discuss the case of several concentric circles and connections with the mathematical theory of electrostatic ion traps.

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Introduction

We discuss equilibrium configurations of point charges with Coulomb interaction confined to a circle or linear segment. Our approach uses the same paradigm as in [1] and [2] we begin by recalling several definitions necessary for rigorous formulation of the main problems.

Assume that we are given a collection of non-zero real numbers $Q = (q_1, ..., q_n)$ and a collection of points $P = (p_1, \ldots, p_n)$ in the plane or three-dimensional $(3d)$ Euclidean space. We consider the pair $S = (Q, P)$ as a system of point charges q_i placed at points p_i. For such a system S, one can define its *Coulomb (electrostatic) potential* $\Phi = \Phi_S$ using well known formulas of electrostatics [3]. Up to a constant, which we omit as irrelevant for our considerations, potential Φ_S can be defined as a real-valued function on the ambient space by the formula

$$\Phi_S(z) = \sum_{i<j} \frac{q_i}{d(p_i, z)}, \tag{1}$$

where z is a point of the ambient space and $d(p_i, z) = |p_i - z|$ is the Euclidean distance between the points p_i and z. Given such a system $S = (Q, P)$, its *electrostatic field* (or *resultant force*) at point z is defined as the gradient Φ_S of Coulomb potential evaluated at point z:

$$F_S(z) = (\nabla \Phi(Q, P))(z). \tag{2}$$

Thus a pair (Q, P) generates a vector field on the ambient space which will be denoted by F_S. The points of ambient space where the resultant force vanishes are called *lacunas* (or *zeros*) of (the electrostatic field generated by) the pair (Q, P).

The *electrostatic (Coulomb) energy $E_Q(P)$* of the pair (Q, P) is defined as a real number

$$E_Q(P) = \sum_{i<j} \frac{q_i q_j}{d_{ij}}, \tag{3}$$

where d_{ij} is the distance between points p_i and p_j. Notice that, for a given n-tuple Q, one can consider E_Q as a function on n-th configuration space of the ambient space. The *resultant force* acting on q_i in position p_i is equal to

$$F_i = \sum Fji = \sum \frac{q_i q_j}{d_{ij}^3}(P_i - P_j), \tag{4}$$

where P_i denotes the radius-vector of point p_i and $F_{ji} = \frac{q_i q_j (P_i - P_j)}{d_{ij}^3}$ is the electrostatic force acting on q_i at p_i due to its interaction with q_j at p_j. In accordance with Newton's third law one has: $F_{ji} = -F_{ij}$. If all charges are of the same sign then forces F_{ji} are repelling and we speak of a system of *repelling charges*.

We are going to deal with situations, where the positions of all charges are confined to a fixed compact differentiable submanifold (maybe with boundary) X, called *conductor*, of the ambient space. In such a situation we work with the restriction E_Q^r of function E_Q to the n-th configuration space X^n. We say that charges $Q = (q_1, ..., q_n)$ placed at $(p_1, ..., p_n) \in X^n$ *stay in rest* in X if the resultant force F_i acting on each charge q_i in position p_i is orthogonal to the tangent space of conductor X at point p_i. In such a case we say that configuration $P = (p_1, ..., p_n)$ is a *Coulomb equilibrium* in X for the collection of charges Q and that collection of charges Q is *stationary* for P in X. As is well known this is equivalent to requiring that P is a constrained critical point of Coulomb potential E_Q considered as a function on X^n [4]. One of our main aims is to investigate and geometrically characterize those configurations of points P for which there exists a collection of stationary charges Q in X. Such configurations of points will be called *Coulomb configurations* in X.

We will basically deal with two closely related problems. The first one is concerned with identifying and calculating the equilibrium configurations of a system of repelling charges. Sometimes it will be sufficient to calculate their *shapes* defined as the collection of pairwise distances between points of equilibrium configurations. For brevity we will refer to this problem as the *direct problem of electrostatics* (DPE). In purely mathematical terms, DPE can be formulated as follows.

(DPE) Given a compact conductor X and collection of n positive numbers $Q = (q_i)$, find all equilibrium configurations of charges q_i in X and determine their types as critical points of Coulomb potential $E_Q|X^n$.

The second problem is concerned with finding a geometrical characterization of Coulomb configurations. Following [5], this problem will be called the *inverse problem of electrostatics* (IPE). More rigorously, IPE can be formulated as follows.

(IPE) Given a finite configuration $P = (p_1, ..., p_n)$ of points in compact conductor X, find out if there exists a collection of non-zero real numbers $Q = (q_1, ..., q_n)$ such that configuration P is a critical point of Coulomb potential E_Q restricted to X^n.

Notice that here we do not require that the stationary charges are repelling. If E_Q has a local minimum at P then it will be called a *stable equilibrium configuration* of charges Q in X. In such a case, a collection of stationary charges Q will be called an *immobilizer* of configuration P in X. If any small displacement of any of charges q_i with all other charges fixed creates a *restoring force* directed toward the original position of q_i then we say that P is a *weakly stable equilibrium*. If E_Q has a global minimum at P then P is called a *ground state configuration* of Q in X. Ground state configuration may not be unique but the energy of ground state is a well-defined notion. In this paper, we discuss problems DPE and IPE, and stability of equilibrium configurations considered, in the cases where X is a circle, system of concentric circles or a finite segment of a straight line.

We begin by considering the case where all charges belong to a circle C in the Euclidean plane $\mathbb{R}^2$, i.e., the charges may freely move along the circle C which remains fixed. In this case, the input of DPE consists of the radius R of circle C, natural number n, and n positive numbers $q_1, \ldots, q_n$ interpreted as values of given point charges. Input for IPE consists of a finite system of points P in C and we need to investigate if system P has non-zero stationary charges.

Next, we consider point charges in a linear segment I equipped with a pair of fixed positive charges (t_1, t_2) placed at its ends. Such a triple will be denoted by $[t_1, I, t_2]$ and called a *linear electrostatic trap* (LET). Our aim is to study the geometry of Coulomb equilibria in a given LET. In this context, DPE takes the following form. For a given set of n positive numbers Q, find all n-point sets P such that the system of given charges q_i placed at points p_i is a Coulomb equilibrium in $[I_T]$. The input of IPE for LET $[t_1, I, t_2]$ consists of a set of n points $P = (p_j)$ in I and we wish to investigate existence of non-zero stationary charges for P in $[I_T]$. If stationary charges exist we wish to calculate their values and verify stability of arising Coulomb configurations.

We add that equilibrium configurations of two-dimensional Coulomb (or logarithmic) potential in a segment have been extensively studied in many papers beginning with the classical results of Stieltjes [6] (see, e.g., [7]-[10]). In particular, some analogs of IPE for logarithmic potential were considered

in [9]-[10]. At the same time in physical problems concerned with the so-called *linear ion traps* it is natural to consider the genuine Coulomb potential [11]. However we are not aware of any paper where Coulomb potential in a segment is discussed in terms of DPE and IPE. One of the aims of this paper is to fill this gap by investigating Coulomb equilibria in the setting of LET motivated by the concept of ion traps.

Remark 1. By the famous Earnshaw theorem, there are no stable equilibria of free point charges in the whole space $\mathbb{R}^3$. At the same time the equilibrium configurations of repelling point charges in compact conductors are stable for physical reasons. Since stable equilibria are especially important from many points of view, we basically deal with equilibrium configurations of repelling point charges in compact conductors.

Remark 2. Notice that there exists an interesting version of DPE on the circle introduced and investigated in [3] under the name of "necklace with interacting beads". Further results on this version of DPE have been obtained in [15] and [16].

Remark 3. There exists vast literature devoted to equilibrium configurations of point charges. However, in most of the papers on this topic all charges are taken equal, as, for example in the famous Thomson problem on equilibria of electrons in the sphere [12]. An essential novelty of IPE formulated in [15] and studied in this paper is that it involves consideration of non-equal charges, which seems relevant for certain problems concerned with ion traps [11]. More comments on such aspects are given in Section 2.

1. Concyclic Coulomb configurations

Recall that a finite set of points in the plane is called *cyclic* if all points belong to a certain circle C. A finite set of points P in the circle C is called *balanced* if no open semi-circle contains all points from P. For a finite set of points P in a circle C, its *shape* is defined as the equivalence class of P with respect to the group of isometries of the circle C. Concerning DPE, we have the following general result of qualitative nature.

Theorem 1. *For any natural number n and ordered system of n repelling charges $Q = (q_i)$, there exists a unique shape of ground state configuration in a given circle C such that the absolute minimum of Coulomb potential E_Q is attained at P. The ground state is balanced in C.*

Equivalently, if we fix a position p_1 of the first charge q_1 in C then the ground state configuration of E_Q is unique and balanced. The order of given charges is important. The problem of calculating the polar angles in equilibrium is in general very difficult and can only be solved numerically. If all charges are equal then by symmetry the regular n-gon is a shape of equilibrium and by the above theorem there no other equilibria of equal charges.

Remark 4. The fact that ground state is balanced follows from a general observation that a non-balanced cyclic configuration P cannot be

a local minimum of Coulomb energy of repelling charges. Indeed, suppose that P is contained in a certain semi-circle S. Then, for any displacement of charges q_1 and q_n towards the ends of S their distances to other charges increase and all other pairwise distances remain unchanged. So the value of Coulomb energy at the new configuration is strictly smaller, which means that P is not a local minimum.

To obtain reasonable results on IPE for cyclic configurations we need to exclude certain atypical situations. The necessary condition is defined inductively. A pair of points on a circle C is called *generic* if it is not a pair of antipodes. Clearly, the set of generic pairs if open and dense in the second configuration space C^2. It is also easy to see that a generic pair of points is not a Coulomb configuration in C. A three-point set in the circle is called *generic* if it does not contain a pair of antipodes. In view of the previous observation it is obvious that the set of generic triples is open and dense in C^3. As was shown in [1], a set of $2k$ points in C can be a Coulomb configuration only if their polar angles satisfy an algebraic relation in which case this set is called *E-compatible*. It follows that the set of E-compatible $2k$-tuples is closed and the set of E-incompatible $2k$-tuples is open and dense in C^{2k}. Taking this into account a subset of $2k$ points is called *generic* if it is E-incompatible.

Now we can present an appropriate inductive definition. Namely, a finite set of points in C is called *generic* if all of its proper subsets are generic in the already defined sense. In particular, a set of $(2k + 1)$ points in C is generic if all of its subsets of odd cardinality are generic and no subset of even cardinality is E-compatible. It is now easy to verify that generic $2k + 1$-tuples of points are open and dense in configuration space C^n. We present first results on cyclic Coulomb configurations of odd cardinality.

Theorem 2. *If $n = 2k + 1$ is an odd natural number then any generic n-configuration P in a circle C has a system of non-zero stationary charges. A Coulomb n-configuration P can be a ground state of repelling charges only if it is balanced.*

For $n = 3$, a detailed proof can be found in [13]. The general case follows by induction. To obtain similar results on IPE for configurations of even cardinality we need to add condition of E-compatibility.

Theorem 3. *If $n = 2k$ is an even natural number then any generic E-compatible concyclic n-tuple $P \subset T$ has a system of non-zero stationary charges. A E-compatible n-configuration P can be a ground state of repelling charges only if it is balanced.*

As an illustration of general results we present some examples on cyclic Coulomb triples. Concerning DPE, calculation of equilibrium coordinates reduces to a system of polynomial equations which in general cannot be solved algebraically. However, this can be done in a few special cases. For example, if all charges are equal then the stable equilibrium has the shape of regular triangle the side and Coulomb energy of which can be explicitly calculated [3]. The same can be done if two charges are equal. We omit the

arising formulae and consider a few examples with three non-equal charges.

Denote by p_1, p_2, p_3 the points on circles with coordinates (u, v), (x, y), (s, t) and charges q_1, q_2, q_3 respectively. Then solving the linear system for stationary charges one has:

$$q_1 = -\frac{q_2(u^2 - 2us + s^2 + v^2 - 2vt + t^2)^{\frac{3}{2}}(tx - ys)}{(x^2 - 2xs + s^2 + y^2 - 2yt + t^2)^{\frac{3}{2}}(sv - tu)},$$

$$q_3 = \frac{q_2(u^2 - 2us + s^2 + v^2 - 2vt + t^2)^{\frac{3}{2}}(xv - yu)}{(x^2 - 2xu + u^2 + y^2 - 2yv + v^2)^{\frac{3}{2}}(sv - tu)}.$$

Using these formulae it is easy to illustrate our results on IPE in the balanced and non-balanced case.

Example 1. (IPE, $n = 3$, cyclic, balanced) Let $p_1 = (1, 0), q_1 = 1, x = 0.1, y = 0.9949874371, s = -0.71, t = -0.7042016757$. This triple is obviously balanced and the above formulas give $q_2 = 1.167633130$, $q_3 = 4.320732247$, so the stationary charges are repelling (positive) and the equilibrium is stable.

Example 2. (IPE, $n = 3$, cyclic, non-balanced) Take now $p_1 = (1, 0)$, $q_1 = 1$, $x = 0.1, y = 0.9949874371$, $s = -0.71, t = 0.7042016757$. This triple is non-balanced and the above formulas give $q_2 = -0.913565248$, $q_3 = 0.3380574536$, so the stationary charges are not repelling and the equilibrium is not stable.

Thus we conclude that, for odd n, generic n-tuples are Coulomb configurations, while for even n, a generic n-tuple is not a Coulomb configuration, but a generic E-compatible n-tuple is a Coulomb configuration. In the next section we present analogs of these results for point charges in a linear segment.

2. Aligned Coulomb configurations

As usual, a finite segment of straight line in the plane will be called a *linear segment*. A configuration of points is called *aligned* if all points belong to a certain linear segment. A linear segment I equipped with a pair of fixed positive charges $T = (t_1, t_2)$ placed at its ends is denoted by $[t_1, I, t_2]$ (or $[I_T]$) and called a *linear electrostatic trap* (LET). All properties of such a triple, which are important in our context, are determined by a triple (L, t_1, t_2), where L is the length of segment I. So without loss of generality we may assume that $I = [0, L]$ and $t_1 = 1$. For a LET, its *lacuna* $z(I_T)$ is the unique point of I where the resultant Coulomb force of charges t_1, t_2 placed at the ends of I vanishes. Obviously, if $t_1 = t_2$ then $z(I_T)$ is the middle point of I. If $t_1 \neq t_2$ then it is easy to verify that lacuna $z(I_T)$ is given by

$$z(I_T) = \frac{t_1 - \sqrt{t_1 t_2}}{t_1 - t_2} L.$$

Given a finite set $P \subset I$ we say that P is z-balanced if there is at least one point of P in each of the open intervals $(0, z(I_T))$ and $(z(I_T), L)$.

We denote by n the number of charges considered and discuss first a few cases with small n. For $n = 1$, the answer is quite simple. Namely, it is easy to see that any point p in I has a positive stationary charge q. If $p \neq z$ then the stationary charge is unique and the arising equilibrium is stable. If $p = z$ then any non-zero number may be taken as a stationary charge. The arising equilibrium is stable if one chooses a positive stationary charge and non-stable if a stationary charge is negative. For $n = 2$, the answer is already more interesting and illustrates some general phenomena. It is easy to verify that if one of the given points coincides with lacuna then non-zero stationary charges do not exist. If both given points are different from lacuna then one has the following result.

Proposition 1. *Any pair of points not containing the lacuna is a Coulomb configuration. Positive stationary charges exist if and only if $p_1 < z(I_T) < p_2$. In this case the arising Coulomb equilibrium is stable. In the contrary case, the stationary charges have different signs and the arising Coulomb equilibrium is unstable.*

The proof follows our general strategy for solving IPE. For further reference, we describe it in the general case. Let N be the number of point charges considered. Then the resultant force on q_i in position x_i is given by

$$F_m = \frac{q_m t_1}{x_m^2} + \sum_{j=1}^{m-1} \frac{q_m q_j}{(x_m - x_j)^2} - \sum_{j=m+1}^{N} \frac{q_m q_j}{(x_j - x_m)^2} - \frac{q_m t_2}{(L - x_m)^2}, \quad m = 1, ..., N.$$

The relations
$$F_1 = 0, F_2 = 0, ..., F_N = 0,$$
give a system of non-homogeneous linear equations
$$MQ = G,$$
for unknowns $q_1, ..., q_N$, where

$$G = \left(\frac{t_2}{(L - x_1)^2} - \frac{t_1}{x_1^2}, \frac{t_2}{(L - x_2)^2} - \frac{t_1}{x_2^2}, ..., \frac{t_2}{(L - x_N)^2} - \frac{t_1}{x_N^2} \right)^T,$$

$M = (m_{ij})_{i,j=1}^{N}$ is an antisymmetric matrix and

$$m_{ij} = (-1)^{\tau_{ij}} (x_j - x_i)^{-2}, \quad i \neq j, \quad \tau_{ij} = 0, \quad \text{if } i > j;$$

and

$$\tau_{ij} = 1, \text{if } i < j.$$

For $N = 2$, the above system takes the form

$$t_1(x_2 - x_1)^2(L - x_1)^2 - t_2 x_1^2(x_2 - x_1)^2 - x_1^2(L - x_1)^2 q_2 = 0,$$

$$t_1(x_2 - x_1)^2(L - x_2)^2 - t_2 x_2^2(x_2 - x_1)^2 - x_2^2(L - x_2)^2 q_1 = 0.$$

Solving the above system for the values of charges one finds out that the stationary charges are given by

$$q_1 = -t_1\frac{(x_2 - x_1)^2}{x_2^2} + t_2\frac{(x_2 - x_1)^2}{(L - x_2)^2}, q_2 = t_1\frac{(x_2 - x_1)^2}{x_1^2} - t_2\frac{(x_2 - x_1)^2}{(L - x_1)^2}.$$

Having these formulae it is easy to verify that $q_1 > 0$ if and only if $x_1 < z(I_T)$, and $q_2 > 0$ if and only if $x_2 > z(I_T)$. The criterion of stability follows from a general result given below.

Theorem 4. *Any equilibrium of repelling charges in linear electrostatic trap is stable. If at least one pair of charges have different signs the equilibrium is unstable.*

The latter theorem is proved by a simple geometric argument. Indeed, if we consider a small displacement of i-th charge, say to the right, then all forces due to the charges on the left of i-th charge will decrease, while all forces due to the charges on the right of i-th charge will increase. So there appears a restoring force directed to the left, which means that the equilibrium is stable. The unstable case is analyzed in a similar manner.

One can also use the above system for constructing solutions of DPE. To this end we use computer algebra to calculate and simplify the resultant of the two equations with respect to x_1. The following examples were obtained in this way.

Example 3. (DPE, aligned, $n = 2$)
Let $t_1 = 1; t_2 = 1; q_1 = 2; q_2 = 5; L = 3$. Then

$\mathtt{resultant} = 160140888x_2^{10} - 50256288x_2^{11} + 11796840x_2^{12} - 444284676x_2^5$

$$+1984608x_2^{13} + 221760x_2^{14} - 14400x_2^{15} + 111071169x_2^4 - 918723708x_2^7$$

$$+400x_2^{16} + 701193753x_2^8 - 387099000x_2^9 + 819009630x_2^6.$$

The roots of resultant in interval $[0, 3]$ are 1.688188447 and 2.079124339. When $x_2 = 1.688188447$, then from the first equation of the system we obtain $x_1 = 0.5138434181$, and there is no other root in $[0, 3]$. When 2.079124339 the single root in $[0, 3]$ is 0.6265615280.

Example 4. (DPE, aligned, $n = 2$)
Let $t_1 = 1; t_2 = 1; q_1 = 2; q_2 = 5; L = 4$. Then the resultant is

$\mathtt{resultant} = 3506438144x_2^4 - 19200x_2^{15} + 37283840x_2^{12} - 4704256x_2^{13}$

$$+394240x_2^{14} + +400x_2^{16} + 14543749120x_2^6 - 10519314432x_2^5 + 7004028928x_2^8$$

$$-2899968000x_2^9 + +899776512x_2^{10} - 211779584x_2^{11} - 12235833344x_2^7.$$

The roots of resultant in $[0, 4]$ are $x_2 = 2.250917929$ and 2.772165785. For $x_2 = 2.250917929$, the first equation of system has a unique solution $x_1 = 0.6851245574$. If $x_2 = 2.772165785$, then $x_1 = 0.8354153706$.

Example 5. (DPE, aligned, $n = 2$)
Let $t_1 = 1; t_2 = 3; q_1 = 2; q_2 = 5; L = 5$. Then the resultant is

$$\texttt{resultant} = 51025390625x_2^4 - 122460937500x_2^5 + 120605468750x_2^6$$

$$-61226562500x_2^7 + +16506640625x_2^8 - 2343125000x_2^9 + 377375000x_2^{10}$$

$$-158300000x_2^{11} + +41570000x_2^{12} - 5016000x_2^{13} + 228800x_2^{14}.$$

The roots in $[0,5]$ are $x_2 = 2.145808518$ and 2.82121. For $x_2 = 2.14581$, the first equation of system has a unique solution $x_1 = 0.6474591186$. For $x_2 = 2.821208270$, then the unique root of the first equation in $[0,5]$ is $x_1 = 0.8337293345$.

For $n \geq 3$, the situation is essentially different. Namely, a typical n-tuple of points in I is not a Coulomb configuration. In fact, as was shown in [13] the set of Coulomb equilibria is a hypersurface in I^n which is given by an explicit equation. For $n = 3$, this equation reads as follows:

$$(t_2x_1^2 - t_1(L - x_1)^2)(L - x_1)^2(x_3 - x_1)^2(L - x_3)^2(x_2 - x_1)^2x_2^2x_3^2$$

$$+(t_2x_3^2 - t_1(L - x_3)^2)(L - x_2)^2(x_3 - x_1)^2(L - x_1)^2(x_3 - x_2)^2x_2^2x_1^2$$

$$-(t_2x_2^2 - t_1(L - x_2)^2)(L - x_1)^2(x_3 - x_2)^2(L - x_3)^2(x_2 - x_1)^2x_1^2x_3^2 = 0.$$

If this condition is fulfilled we say that configuration P is *E-compatible*. The final result for three points can be formulated as follows.

Theorem 5. *For any LET $[I_T]$, any E-compatible triple of points P in I is a Coulomb configuration. If P is z-balanced then the stationary charges are positive.*

Thus we see certain analogy with the results on concyclic Coulomb configurations. In particular, using the exponential map the trap with equal terminal charges can be transformed in a circle with one fixed charge t. Then the definitions of balanced configuration in both cases become identical. However LET has an additional freedom as compared with circular conductor, which enables one to produce more Coulomb configurations. Namely, we can permit to change one of the terminal charges, say t_2, and also the length of segment $[0, L]$. Since only the ratio of terminal charges matters in this issue, we may assume that $t_1 = 1$. It turns out that this setting indeed yields a bigger set of aligned Coulomb equilibria.

Theorem 6. *For any aligned triple of points P, there exists an ambient segment I and terminal charges such that P is a Coulomb configuration in $[t_1, I, t_2]$.*

For $n \geq 4$, a similar result does not hold in general.

3. Concyclic Coulomb equilibria

Our results on cyclic Coulomb configurations can be generalized in various ways some of which have been mentioned in [13], [14]. In this section we present one of such generalizations in which the ambient set (conductor)

X is a system of n concentric circles C_j each containing one of the given n charges. Such a system of charges will be called *concyclic* (or *polycyclic*) with the *frame C_j*. Concyclic configuration of points P is called *generic* if no pair of points from P forms an aligned triple with the center of given circles. For concyclic configurations we have an direct analog of Theorem 2.

Theorem 7. *For odd n and any concentric frame X, each generic polycyclic n-tuple P has nonzero stationary charges.*

For even n, an analog of Theorem 3 is more complicated and we omit it. As to DPE, it is very difficult to calculate the coordinates of equilibrium of given charges even for three concentric circles. However we have obtained some results in the spirit of Morse theory.

Theorem 8. *For a conductor X consisting of three concentric circles, all critical configurations of E_Q on X are non-degenerate. There are 3 saddles, one global maximum and two equal minima corresponding to stable equilibria of E_Q in X. in X.*

The proof is based on the explicit formulae for the stationary charges similar to those given in [14] and [15]. Denote by q_1, q_2, q_3 the sought stationary charges. Following the general strategy of [15] we aim at obtaining a system of linear equations for q_1, q_2, q_3. To this end we write down the analytic expression of the fact that each point is in equilibrium for this system of charges. By Lagrange rule, at an equilibrium the resultant force should be orthogonal to the tangent vector T_i to the corresponding circle at each point P_i, which gives three relations: $(F_i, T_i) = 0$. We consider these relations as a system of three linear equations for three variables q_j and examine its matrix. Since $F_{ji} = \frac{q_i q_j}{d_{ij}^3}(p_i - p_j)$, one easily verifies that the matrix of this system has the form:

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0,$$

where $a_{ij} = \frac{(p_i - p_j, T_i)}{d_{ij}^3}$. Since we deal with the circles we can take $T_i = (-y_i, x_i)$. Then we get $a_{ij} = \frac{q_i q_j A_{ij}}{d_{ij}^3}$, where $A_{ij} = x_i y_j - x_j y_i$ is (two times) the oriented area of the triangle $\Delta OP_i P_j$.

Let p_i be three points on different concentric circles. Then the distance between p_i and p_j, $i, j = 1, 2, 3$, in polar coordinates is equal to $d_{ij}^2 = r_i^2 + r_j^2 - r_i r_j \cos(\varphi_i - \varphi_j)$. Moreover, $T_i = (-r_i \sin \varphi_i, r_2 \cos \varphi_i)$, $p_i - p_j = (r_i \cos \varphi_i - r_j \cos \varphi_j, r_i \sin \varphi_i - r_j \sin \varphi_j$ and so $\langle p_i - p_j, T_i \rangle = r_i r_j \sin(\varphi_i - \varphi_j)$.

Thus for the sought charges we get a system of linear equations

$$\begin{pmatrix} 0 & r_1 r_2 \sin(\varphi_{12}) d_{31}^3 & r_1 r_3 \sin(\varphi_{13}) d_{21}^3 \\ r_1 r_2 \sin(\varphi_{21}) d_{32}^3 & 0 & r_2 r_3 \sin(\varphi_{23}) d_{12}^3 \\ r_1 r_3 \sin(\varphi_{31}) d_{23}^3 & r_2 r_3 \sin(\varphi_{31}) d_{13}^3 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0,$$

where $\varphi_{ij} = \varphi_i - \varphi_j$, $i, j = 1, 2, 3, i \neq j$. The matrix of this system is

$$A = \begin{pmatrix} 0 & r_1 r_2 \sin(\varphi_{12}) d_{31}^3 & r_1 r_3 \sin(\varphi_{13}) d_{21}^3 \\ r_1 r_2 \sin(\varphi_{21}) d_{32}^3 & 0 & r_2 r_3 \sin(\varphi_{23}) d_{12}^3 \\ r_1 r_3 \sin(\varphi_{31}) d_{23}^3 & r_2 r_3 \sin(\varphi_{31}) d_{13}^3 & 0 \end{pmatrix}.$$

It is now easy to see that $\det A = 0$. Therefore this system has nontrivial solutions. Moreover, $\text{rank} A = 2$ if and only if there exists a pair of polar angles such that $\varphi_i \neq \varphi_j$. It follows that, for any non-degenerate triple, the rank of this matrix is two, so stationary charges are defined up to a constant multiple. It is also easy to see that condition that $\text{rank} A = 1$ is equivalent to the condition $\text{rank} A = 0$, which happens if and only if all three polar angles are equal. By examining the signs of the above expressions in terms of the introduced angles one easily verifies that the three critical charges are positive if and only if the origin (centre of the circles) is inside the triangle $\Delta P_1 P_2 P_3$, which exactly means that the triple is balanced. To establish stability one can use a well known formula on the index of a constrained critical point [4]. The coefficients of the bordered Hessian of E_Q can be found by a straightforward routine computation which is omitted. Substituting the above values of stationary charges, calculating the signs of the principal minors of the bordered Hessian one verifies that the Hessian of the restriction E_Q is positive definite for balanced non-degenerate triples, which completes the proof.

Remark 5. In Cartesian coordinates the system of equations for stationary charges has the form

$$\begin{pmatrix} 0 & (x_1 y_1 - x_2 y_1) d_{31}^3 & (x_1 y_3 - x_3 y_1) d_{21}^3 \\ (x_2 y_1 - x_1 y_2) d_{32}^3 & 0 & (x_2 y_3 - x_3 y_2) d_{12}^3 \\ (x_3 y_1 - x_2 y_3) d_{23}^3 & (x_3 y_2 - x_2 y_3) d_{13}^3 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0,$$

where d_{ij} are the distances as above. It seems that this form may appear appropriate for generalization to the case of many concentric circles.

Having in mind applications to Coulomb control of variable constrained charges in the spirit of [17] one may wish to investigate the topological structure of Coulomb equilibria in our setting. An obvious necessary condition for the existence of complete Coulomb control in the sense of [17], is the connectedness of the set of Coulomb equilibria. This suggests that one should try to obtain some information on the topology of this set. Denote by $E(A, B, C)$ the set of all Coulomb equilibria of three point charges on three nested circles A, B, C.

Theorem 9. *For a triple of concentric circles A, B, C, the set $E(A, B, C)$ of Coulomb triples is connected.*

To prove this notice first that the complement of Coulomb triples in $M_3(A, B, C)$ is contained in the union of two simple loops in T^2 intersecting at two points. More precisely, this complement consists exactly of those triples of points where exactly two of the points belong to the same diameter of A. Fixing a point on A and considering the polar angles φ, χ of the

two remaining points as the coordinates on $M_3(A, B, C) = T^2$ we conclude that this complement consists of the points where $\varphi = \chi$ or $\varphi = \chi + \pi$ except the two points where $\varphi = 0$ or $\varphi = 0$, $\chi = \pi$. Indeed, in the latter two configurations all three points belong to the same (horizontal) diameter of A and such configurations are Coulomb equilibria. Thus the set $E(A, B, C)$ consists of two simple loops intersecting at two points with the latter two points removed. It is now easy to see that this set is connected. In fact, given two points in $E(A, B, C)$ one can explicitly describe a path in $E(A, B, C)$ connecting these two points. Moreover, it is now clear that the set $E(A, B, C)$ is homotopic to the union of two circles with two common points. In other words, for concentric circles we have a quite detailed description of the topology of Coulomb triples.

The above considerations can be used to show that the set of stable Coulomb triples is not connected. For non-concentric nested circles, an explicit description of $E(A, B, C)$ in terms of polar angles becomes rather complicated and we cannot prove that this set is connected in general. However, if the three centers of circles A, B, C lie on the same line, an easy modification of the above reasoning shows that $E(A, B, C)$ is connected. We conjecture that the set of Coulomb triples is connected for any triple of nested circles. These results and observations suggest that it should be possible to describe an explicit algorithm for Coulomb control of Coulomb triples in the spirit of [16]. However, we do not yet have explicit formulae for the values of charges realizing a path connecting two Coulomb triples.

4. Concluding remarks

In conclusion we wish to mention that there are many research perspectives suggested by our results. More or less complete description of arising problems requires a separate publication so here we only mention a few.

A quite obvious generalization arises if one considers Coulomb equilibria of $n > 3$ point charges assuming that each charge is confined to one of n nested circles. For odd n and concentric circles, a number of case studies enables us to formulate two plausible conjectures. The first one is concerned with IPE.

Conjecture 1. *If n is odd then any generic n-tuple of points P on a system X of n concentric circles has a system of non-zero stationary charges.*

To prove this one may try to use the the same argument as above. The condition of equilibrium can be rewritten as a linear system on values of sought charges q_j. The structure of the matrix of this system suggests that it can be factored as a product of two matrices one of which is skew-symmetric, which would imply vanishing of determinant and existence of non-zero stationary charges. However we were not able to prove existence of the mentioned factorization. An interesting concrete problem is to work out the details in the case of five concentric circles. For even n, situation is

much more complicated as is clear from the presented results on Coulomb equilibria on one circle.

The second conjecture gives some qualitative information on DPE.

Conjecture 2. *For any system of n concentric circles and any system of positive charges, all Coulomb configurations are non-degenerate critical points of electrostatic energy. The Morse index can be expressed by combinatorial data of Coulomb configuration.*

As a first step, it is interesting to verify this conjecture for aligned polycyclic configurations.

The above problems and conjectures are meaningful for a system of non-intersecting closed loops in the plane. As a first step one could consider the case of three confocal ellipses. The general case of nested ellipses also deserves closer investigation since it may serve as a model of some situations encountered in astronomy. As was already mentioned, another interesting perspective is to find generalizations of our results to the case where point charges are confined to a flexible contour of fixed length studied in [3].

Finally, it is natural to search for analogs of our results for the logarithmic potential and compare them with the similar results in [9], [10]. Some developments in the latter direction will be presented in a forthcoming paper of the authors.

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Authors' addresses:

G. Giorgadze
I. Javakhishvili Tbilisi State University
I. Vekua Institute of Applied Mathematics
2, University str., Tbilisi 0186
Georgia
E-mail: gia.giorgadze@tsu.ge

G. Khimshiashvili
Ilia State University
32, I. Chavchavadze Ave., Tbilisi 0162
Georgia
E-mail: giorgi.khimshiashvili@iliauni.edu.ge