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ON THE SOLUTION OF SOME NON-CLASSICAL PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE IN A CIRCULAR DOMAIN

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Abstract. In the paper for homogeneous equation of statics of the linear theory of elastic mixture in a circular domain two boundary value problems are considered. In the case of problem I on the boundary of the domain projections of the partial displacements vectors on the normal and rotation are prescribed; and in the case of problem II on the boundary of domain there projections of the partial displacements vectors on the tangend and divergence are prescribed.

The problem are uniquely solvable and the solutions are represented in quadratures.

Keywords and phrases: Elastic mixture, Riemann-Hilbert problems for a circle, Kolosov-Muskhelisvhili type formulas, non-classical problems.

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1. Introduction

The basic two-dimensional boundary value problems statics of the linear theory of elastic mixtures are studied in [1], [3],[5] and also by many other authors.

In the paper we consider the so-called two non-classical boundary value problem for the homogeneous equation of the linear theory of elastic mixture in a circular domain.

On the basis of formulas analogous to Kolosov-Muskhelishvili our problems are reduced to the Riemann-Hilbert problem for a circle.

Uniquenes theorem are proved and the solutions of the problems are represented in quadratures.

2. Basic equation and boundary value problems.

The homogeneous equation of statics of the linear theory of elastic mixture in the two dimensional case has the form [1]

$$\begin{aligned} &a_1\Delta u' + b_1 graddivu' + c\Delta u'' + dgraddivu'' = 0,\\ &c\Delta u' + dgraddivu' + a_2\Delta u'' + b_2 graddivu'' = 0, \end{aligned} \tag{1}$$

where $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$, are partial displacements,

$$a_j = \mu_j - \lambda_j, \quad b_j = \mu_j + \lambda_j + \lambda_5 + (-1)^j \alpha_2 \frac{\rho_{3-j}}{\rho}, \quad j = 1, 2, \quad \rho = \rho_1 + \rho_2,$$

$$\alpha_2=\lambda_3-\lambda_4,\quad c=\mu_3+\lambda_5,\quad d=\mu_3+\lambda_3-\lambda_5-\alpha_2\frac{\rho_1}{\rho}.$$

Here $\mu_1,\mu_2,\mu_3,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5$ are elastic constants, ρ_1 and ρ_2 are partial densities, The above constants satisfy the definite conditions [1].

In [2] M. Basheleishvili obtained the representations (analogous to the Kolosov-Muskhelishvili formulas)

$$\begin{bmatrix} a & c_0\\ c_0 & b\end{bmatrix}\begin{pmatrix}\theta'\\ \theta''\end{pmatrix}+i\begin{bmatrix} a_1 & c\\ c & a_2\end{bmatrix}\begin{pmatrix}\omega'\\ \omega''\end{pmatrix}=2\varphi'(z), \tag{2}$$

$$U=(u_1+iu_2,u_3+iu_4)^T=m\varphi(z)+\frac{1}{2}lz\overline{\varphi'(z)}+\overline{\psi(z)}, \tag{3}$$

where $\varphi=(\varphi_1,\varphi_2)^T$ and $\psi=(\psi_1,\psi_2)^T$are arbitrary analytic vector-functions,

$$\theta'=divu',\quad \theta''=divu'',\quad \omega'=rotu'=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2},$$

$$\omega''=rotu''=\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2},$$

$$m=\begin{bmatrix} m_1 & m_2\\ m_2 & m_3\end{bmatrix},\quad \Delta_0=det|m|>0,\quad m_3>0,\quad l=\begin{bmatrix} l_4,l_5\\ l_5,l_6\end{bmatrix},$$

$$a=a_1+b_1,\quad b=a_2+b_2,\quad c_0=c+d,\quad m_k=l_k+\frac{1}{2}l_{3+k},\quad k=1,2,3,$$

$$l_1=\frac{a_2}{d_2},\quad l_2=-\frac{c}{d_2},\quad l_3=\frac{a_1}{d_2},\quad d_2=a_1a_2-c^2>0,$$

$$l_1+l_4=\frac{b}{d_1},\quad l_2+l_5=-\frac{c_0}{d_1},\quad l_3+l_6=\frac{a}{d_1},\quad d_1=ab-c_0^2>0$$

Let $D^+=\{z:|z|<1\}$, and $L=\{z:|z|=1\}$. Below we assume that $u_k\in C^3(D^+)\bigcap C^2(D^+\bigcup L)$. $\quad k=\overline{1,4}$.

Note that for a regular solution of system (1) we have the Green formula [1,3]

$$\int_{D^+}N(u,u)dx=\int_L uNuds, \tag{4}$$

where $N(u,u)$ is the positively defined quadratic form,

$$u=(u',u'')^T=(u,u_2,u_3,u_4)^T,\quad Nu=[(Nu)_1,(Nu)_2,(Nu)_3,(Nu)_4]^T$$

is the pseudo stress vector;

$$\begin{bmatrix} (Nu)_1 \\ (Nu)_2 \end{bmatrix} = \left[\begin{pmatrix} a \\ c_0 \end{pmatrix}\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix}\right] n + \left[\begin{pmatrix} a_1 \\ c \end{pmatrix}\begin{pmatrix} \omega' \\ \omega'' \end{pmatrix}\right] s^0$$
$$+\frac{1}{\Delta_0}\frac{\partial}{\partial s(x)}\begin{pmatrix} -m_3u_2 + m_2u_4 \\ m_3u_1 - m_2u_3 \end{pmatrix},$$
$$\begin{bmatrix} (Nu)_3 \\ (Nu)_4 \end{bmatrix} = \left[\begin{pmatrix} c_0 \\ b \end{pmatrix}\begin{pmatrix} \theta' \\ \theta'' \end{pmatrix}\right] n + \left[\begin{pmatrix} c \\ a_2 \end{pmatrix}\begin{pmatrix} \omega' \\ \omega'' \end{pmatrix}\right] s^0 \qquad (5)$$
$$+\frac{1}{\Delta_0}\frac{\partial}{\partial s(x)}\begin{pmatrix} m_2u_2 - m_1u_4 \\ -m_3u_1 + m_1u_3 \end{pmatrix},$$

$$\frac{\partial}{\partial s(x)} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}, n = (n_1, n_2)^T \quad and \quad s^0 = (-n_2, n_1)^T$$

denote the unit normal and tangent vectors respectively.

Note also that the equation $N(u, u) = 0$ admints a solution $u_k = c_k = const, \quad k = \overline{1, 4}$

In the present work we consider the folowing problems. Find a regular silution of system (1) in D^+ satusfying one the following boundary conditions

Problem.

$$(P_1)^+ : \quad \begin{pmatrix} u'n \\ u''n \end{pmatrix}^+ (t) = f^{(1)}(t), \qquad \begin{pmatrix} \omega'(t) \\ \omega''(t) \end{pmatrix}^+ = F^{(1)}(t), \quad t \in L,$$

Problem.

$$(P_2)^+; \quad \begin{pmatrix} u's^0 \\ u''s^0 \end{pmatrix}^+ (t) = f^{(2)}(t), \qquad \begin{pmatrix} \theta'(t) \\ \theta''(t) \end{pmatrix}^+ = F^{(2)}(t), \quad t \in L,$$

where $f^{(j)}$ *and* $F^{(j)}$, $j = 1, 2$, are real given vector-functions on L satisfying certain conditions

Now note that we obtain lengthy but elementary calculations

$$\int_L u^0 N u^0 dS = -\frac{1}{\Delta_0 m_3}\int_L \left(\sum_{k=1}^{2}[(m_3u_k^0 - m_2u_{k+2}^0)^2 + \Delta_0 u_{k+2]}^{0^2}\right) ds, \qquad (6)$$

where $u^0 = (u_1^0, u_2^0, u_3^0, u_4^0)^T$ is a regular solution of the homogeneous problem $(P_j)_0^+[f^{(j)} = F^{(j)} = 0, \quad j = 1, 2]$

From the Green formula (4) and equality (6) follows

Theorem. *The problems* $(P_1)^+$ *and* $(P_2)^+$ *are uniquely solvable.*

We start solving the problem $(P_1)^+$. On the basis of formulas (2) and (3) we can reduce the problem $(P_1)^+$ to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in the domain D^+ by the following conditions on L

$$Re\{e^{-i\gamma}[m\varphi(t) + \frac{1}{2}lt\overline{\varphi'(t)} + \overline{\psi(t)}]\}^+$$

$$= Re[mt^{-1}\varphi(t) + \frac{1}{2}l\varphi^{'}(t) + t\psi(t)]^{+} = f^{(1)}(t), \tag{7}$$

$$Re[i\varphi^{'}(t)]^{+} = -\frac{1}{2}\begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix} F^{(1)}(t) = g(t), \tag{8}$$

where $t = e^{i\gamma}$ is the polar equation of L. $(g, f^{(1)}) \in C^{1,\alpha}(L)$, $\quad 0 < \alpha < 1$.

Since $\varphi(z)$ and $\psi(z)$ are arbitrary analytic vector - functions we can suppose that

$$\varphi(0) = \varphi'(0) = 0, \qquad \psi(0) = const \neq 0. \tag{9}$$

Owing to the above reasoning, we can conclude that the boundary conditions (7) and (8) are the Riemann-Hilbert problems for a circle $|z| < 1$.

A solution of problems (7) and (8) owing to (9) can be represented in the form, respectively (see [4] §41)

$$mz^{-1}\varphi(z) + \frac{1}{2}l\varphi'(z) + z\psi(z) = \frac{1}{\pi i}\int_z \frac{zf^{(1)}(t)dt}{t(t-z)}, \tag{10}$$

$$\varphi'(z) = -\frac{1}{\pi}\int_L \frac{zg(t)dt}{t(t-z)}; \qquad \varphi(z) = -\frac{1}{\pi}\int\int_L \frac{zg(t)dt}{t(t-z)}dz. \tag{11}$$

Having found $\varphi(z)$ and $\psi(z)$ (see (10)-(11)) we can obtain from (3) the solution (in quadratures) of the problem $(P_1)^+$

Remark. The $BVP \quad (P_2)^+$ is solved quite analogously.

R E F E R E N C E S

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