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CONFORMAL MODULUS OF QUADRILATERALS

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Abstract. In this paper we consider dependence of conformal modulus of quadrilateral on the special conformal map and it is shown that this function as a parameter is monotonic.

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A domain is an open connected set in the complex place $\mathbf{C}$. A closed curve is the continuous image of a circle. A simple closed curve, or a Jordan curve, is a one-to-one continuous image of a circle. The Jordan curve theorem asserts that every Jordan curve Γ divides the plane into two domains with the common boundary Γ, the interior (or bounded domain) and the exterior (or unbounded domain) of the curve. The interior of a Jordan curve is called a Jordan domain.

A domain D in $\mathbf{C}$ is called simple-connected if all closed curves in D are null-homotopic.

Riemann Mapping Theorem. *The simple-connected domain which is a proper subset of the complex place can be mapping conformally onto the unit disk* $\Delta = \{z \in \mathbf{C} : |z| < 1\}$.

Caratheodory Extension Theorem. Let D be a Jordan domain bounded by a Jordan curve Γ, and let f map D conformally onto the unit disk Δ. Then f can be extended to a homeomorphism of $\overline{D} = D \cup \Gamma$ onto the closed disk $\overline{\Delta}$.

One obvious corollary is that a conformal mapping of a Jordan domain D_1 onto another Jordan domain D_2 can be extended to a homeomerpism of $\overline{D}_1$ onto $\overline{D}_2$.

A Jordan domain D with four distinct points q_1, q_2, q_3, q_4 on the boundary curve of D, which occur in this order, when traversing the boundary in the positive direction, is called a quadrilateral.

Let the bounded Jordan curve in the complex plane divide the extended complex plane into two domains $\mathbf{I}$ and $\mathbf{E}$, whose common boundary is a Jordan curve. Let $\mathbf{I}$ be the bounded domain.

By Caratheodory theorem there exists conformal mapping from the domain $\mathbf{I}$ into the rectangle $f : \mathbf{I} \to [-D_x, D_x] \times [0, iD_y]$ such that the four distinct points are mapped into the vertices of the rectangle $D_x, D_x + iD_y, -D_x + iD_y, -D_x$. The ratio of the lenght of the line segments of the rectangle $M = 2D_x/D_y$ is called the modulus of the rectangle(see [1]). Obviously the similar rectangles have the same moduli.

In order to find modulus of some quadrilateral it has to be mapped into a rectangle. Let $z = h(q)$ be the mapping from some quadrilateral in the Q complex plane into the upper half plane. This maps the interior of the quadrilateral to the upper half plane, and maps the boundary of the quadrilateral to the real line, such that the four points on the boundary are mapped somewhere in the real line: $h(q_1) = x_1, h(q_2) = x_2, h(q_3) = x_3, h(q_4) = x_4$.

The mapping

$$f(z) = \frac{z - x_1}{z - x_4}\frac{x_2 - x_4}{x_2 - x_1}$$

is well known as a cross-ratio of four points. It is known that it maps the upper half plane into itself. Indeed, from the identity

$$f(x, y) = u(x, y) + iv(x, y)$$

$$= \frac{x_2 - x_4}{x_2 - x_1}\frac{y^2 + x^2 - x(x_1 + x_4) + x_1x_4}{y^2 + (x - x_4)^2} + i\frac{x_2 - x_4}{x_2 - x_1}\frac{y(x_1 - x_4)}{y^2 + (x - x_4)^2}$$

$$f(x_3) = \xi$$

it follows, that if $y > 0$ then $Im(f) > 0$ and vice versa.

Below we use the properties of f, in particular $f(x_3) = \xi$ notation.

Let

$$f_1(z) = \frac{z - x_1}{z - x_4}\frac{x_2 - x_4}{x_2 - x_1}$$

and

$$f_2(\omega) = \frac{\omega + \eta}{\omega - \eta}\frac{1 + \eta}{1 - \eta},$$

where $\eta > 1$ is real. f_2 maps the points $-\eta, -1, \eta$ to $0, 1, +\infty$ and $f_2(1) = \xi' > 1$. The inverse of f_2 maps the upper half plane into itself and the four points $0, 1, \xi', +\infty$ to $-\eta, -1, 1, \eta$. Suppose $\xi' = \xi$, then $f_2^{-1}(f(z))$ maps the upper half of the plane z into the upper half of the plane ω such that the any four points on the real line x_1, x_2, x_3, x_4 are mapped to four symmetric points on the real line $-\eta, -1, 1, \eta$, where η is found by the transformation $\xi' = \xi$:

$$\xi' = \frac{(\eta + 1)^2}{(\eta - 1)^2} = \frac{x_3 - x_1}{x_3 - x_4}\frac{x_2 - x_4}{x_2 - x_1} = \xi \Rightarrow \eta = \frac{\sqrt{\xi} + 1}{\sqrt{\xi} - 1} > 1$$

The Schwarz-Christoffel Transformation maps the upper half plane into the polygon, such that the real axis is mapped into the boundary of the polygon:

$$G(\omega_0) = A + B\int_0^{\omega_0} \frac{d\omega}{(\omega - a)^{1-\frac{\alpha}{\pi}}(\omega - b)^{1-\frac{\beta}{\pi}}\ldots} \tag{1}$$

where **A** and **B** are in general some complex constants. a, b (and so on), are the real numbers and they are mapped into the vertices of the polygon. α, β (and so on), are the angels, respectively, of those vertices.

Consider the transformation of the upper half plane, such that fixed four points $(1, \eta, -\eta, -1)$ map into the rectangle $(D_x, D_x+iD_y, -D_x+iD_y, -D_x)$. Then the point 0_ω will be mapped into "itself". It means, that $\mathbf{A}$ is zero. From (1) we obtain

$$D_x = Bk\int_0^1 \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}}, \tag{2}$$

$$D_x + iD_y = Bk\int_0^\eta \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}}, \tag{3}$$

where $k = \frac{1}{\eta}$.

From (3) we have:

$$\begin{aligned} D_x + iD_y &= Bk\int_0^1 \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}} + Bk\int_1^\eta \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}} \\ &= Bk\Psi(k) + Bk\int_1^\eta \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}}. \end{aligned} \tag{4}$$

Let $k' = \sqrt{1-k^2}$. After changing the variable in (4) we obtain

$$\zeta = \frac{\sqrt{\omega^2-1}}{k'\omega} \Rightarrow \omega = \frac{1}{\sqrt{1-k'^2\zeta^2}}; d\omega = \frac{k'^2\zeta d\zeta}{(1-k'^2\zeta^2)^{3/2}}.$$

Therefore,

$$\begin{aligned} \int_1^\eta \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}} &= \int_0^1 \frac{\sqrt{1-k'^2\zeta^2}\, i\sqrt{1-k'^2\zeta^2}\, k'^2\zeta d\zeta}{k'\sqrt{1-\zeta^2}\, k'\zeta(1-k'^2\zeta^2)^{3/2}} \\ &= i\int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k'^2\zeta^2)}}. \end{aligned} \tag{5}$$

From (2), (4) and (5) it follows that the vertices of the rectangle can be written as

$$[Bk\Psi(k); Bk\Psi(k) + iBk\Psi(k'); -Bk\Psi(k) + Bk\Psi(k'); -Bk\Psi(k)].$$

Finally for moduli M we have the explicit formula

$$\mathbf{M} = 2\mathbf{B}k\Psi(k)/\mathbf{B}k\Psi(k') = \frac{2\Psi(k)}{\Psi(k')}. \tag{6}$$

Next, we present $\Psi(k)$ by the power series.

From identities

$$\frac{1}{\sqrt{1-k^2\omega^2}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} k^{2n}\omega^{2n},$$

$$\frac{1}{\sqrt{1-\omega^2}} = \sum_{m=0}^{\infty} \frac{(2m)!}{4^m (m!)^2}\omega^{2m}$$

follows

$$\Psi(k) = \int_0^1 \sum_{n,m=0}^{\infty} \frac{(2n-1)!!(2m)!k^{2n}}{4^m(2n)!!(m!)^2}\omega^{2n+2m}$$

$$= \sum_{n,m=0}^{\infty} \int_0^1 \frac{(2n-1)!!(2m)!k^{2n}}{4^m(2n)!!(m!)^2}\omega^{2n+2m}.$$

So

$$\Psi(k) = \sum_{n,m=0}^{\infty} \frac{(2n-1)!!(2m)!k^{2n}}{4^m(2n)!!(m!)^2(2n+2m+1)}.$$

Similarly

$$\Psi(k') = \sum_{p,q=0}^{\infty} \frac{(2p-1)!!(2q)!k'^{2p}}{4^q(2p)!!(q!)^2(2p+2q+1)}.$$

By the conditions $k' = \sqrt{1-k^2}$, $k = 1/\eta = \frac{\sqrt{\xi}-1}{\sqrt{\xi}+1}$ and (6) we obtain the modulus as the function of ξ

$$M(\xi) = 2\frac{\sum_{n,m=0}^{\infty} \frac{(2n-1)!!(2m)!}{4^m(2n)!!(m!)^2(2n+2m+1)}\left(\frac{\sqrt{\xi}-1}{\sqrt{\xi}+1}\right)^{2n}}{\sum_{p,q=0}^{\infty} \frac{(2p-1)!!(2q)!}{4^q(2p)!!(q!)^2(2p+2q+1)}\left(\frac{2\sqrt[4]{\xi}}{\sqrt{\xi}+1}\right)^{2p}}. \tag{7}$$

Denote by

$$f_k(\xi) = \left(\frac{\sqrt{\xi}-1}{\sqrt{\xi}+1}\right)^{2k},$$

$$g_k(\xi) = \left(\frac{2\sqrt[4]{\xi}}{\sqrt{\xi}+1}\right)^{2k},$$

$$a_{kj} = \frac{(2k-1)!!(2j)!}{4^j(2k)!!(j!)^2(2k+2j+1)}.$$

Then

$$\mathbf{M} = 2\frac{\sum_{n,k=0}^{\infty} a_{nk} f_n(\xi)}{\sum_{m,i=0}^{\infty} a_{mi} g_m(\xi)}.$$

The derivative of $M(\xi)$ is equal to

$$\frac{d\mathbf{M}}{d\xi} = 2\frac{\sum_{n,m=0}^{\infty} a_{nk}a_{mi}(f_n' g_m - f_n g_m')}{(\sum_{m=0}^{\infty} a_{mi} g_m(\xi))^2},$$

where

$$\Big(\sum_{m=0}^{\infty} a_m g_m(\xi)\Big)^2$$

is always positive.

It is clear that

$$K_{n,m}(\xi) = f_n' g_m - f_n g_m'$$
$$= n(\sqrt{\xi} - 1)^{2n-1} 2^{2m+1} \xi^{\frac{2m+1}{4}} + m(\sqrt{\xi} - 1)^{2n+1} 2^{2m-1} \xi^{\frac{2m-1}{4}}$$

is positive, when $\xi > 1$. It means that

$$\frac{d\mathbf{M}}{d\xi} > 0$$

when $\xi > 1$. Therefore the modulus monotonously increases.

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