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ON THE IMBEDDING OF THE SURFACE IN THE 3–D RIEMANNIAN MANIFOLD

Meunargia T.

Abstract. In the paper it is shown that any regular surface can be imbedded in the 3-dimensional Riemannian manifold.

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The surface with non-zero Gaussian curvature is the 2-dimensional Riemannian manifold, which is imbedded in the 3-D Euclidean space. Therefore, for these varieties of properties it is possible to construct quite clear representations. Further, it is shown that any regular surface can be imbedded in the 3-D Riemannian manifold.

Let S be the regular surface. The radius vector $\vec{R}$ of any point M may be expressed by the formula [1]

$$\vec{R}(x^1, x^2, x^3) = \vec{r}(x^1, x^2) + x^3\vec{n}(x^1, x^2), \tag{1}$$

where x^1, x^2 are Gaussian parameters of the surface $S, \vec{r}(x^1, x^2)$ and $\vec{n}(x^1, x^2)$ are, respectively, radius vector and unit vector of the normal to S at the point $(x^1, x^2) \in S$ fig. 1.

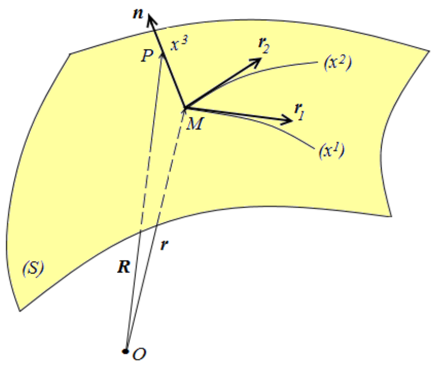

Fig. 1.

Differentiating equality (1) we have

$$\partial_\alpha \vec{R} = \vec{R_\alpha} = (a_\alpha^\beta - x^3 b_\alpha^\beta)\vec{r_\beta}, \quad \vec{R_3} = \partial_3 \vec{R} = \vec{n},$$

where

$$a_\alpha^\beta = \vec{r_\alpha}\vec{r^\beta}, \quad \vec{n_\alpha} = -b_\alpha^\beta \vec{r_\beta}, \quad \vec{r_\alpha} = \partial_\alpha \vec{r}, \quad (\alpha, \beta = 1, 2).$$

For the coordinate system $\hat{S} : x^3 = const$, the coefficients of the first quadratic form are expressed by the formulae

$$g_{\alpha\beta} = \vec{R}_\alpha \vec{R}_\beta = a_{\alpha\beta} - 2x^3 b_{\alpha\beta} + (x^3)^2 b_\alpha^\lambda b_{\lambda\beta} \quad (\alpha, \beta, \lambda = 1, 2)$$

$$g_{\alpha 3} = \vec{R}_\alpha \vec{R}_3 = 0, \quad g_{33} = \vec{R}_3 \vec{R}_3 = 1, \quad (\vec{R}_3 = \vec{n}).$$

Let $x^1 = const, x^2 = const$ be coordinate lines of curvature on the surface S, then we have

$$\vec{R}_1 = (1 - k_1 x^3)\vec{r}_1, \quad \vec{R}_2 = (1 - k_1 x^3)\vec{r}_2, \quad \vec{R}_3 = \vec{n},$$

$$\vec{R}^1 = (1 - k_1 x^3)^{-1}\vec{r}^1, \quad \vec{R}^2 = (1 - k_1 x^3)^{-1}\vec{r}^2, \quad \vec{R}^3 = \vec{n},$$

where k_1 and k_2 are principal curvatures of the surface S.

Let us consider the 3-D Riemannian manifold M_s, constrained with the surface $S : \vec{r} = \vec{r}(x^1, x^2)$ by the formulas [2]

$$dS^2 = A_{ik} dx^i dx^k, \quad (i, k = 1, 2, 3) \tag{2}$$

where

$$A_{ik} = \vec{R}_i \vec{r}_k = a_{ik} - x^3 b_{ik}, \tag{3}$$

$$a_{ik}, a^{ik}, a_k^i = \begin{cases} a_{\alpha\beta}, a^{\alpha\beta}, a_\beta^\alpha = \delta_\beta^\alpha, & if \;\; i = \alpha, \; k = \beta, \\ \delta_{ij}, a^{ij}, a_j^i, & if \;\; i = 3 \;\; or \;\; k = 3, \end{cases} \tag{4}$$

$$b_{ik}, b^{ik}, b_k^i = \begin{cases} b_{\alpha\beta}, b^{\alpha\beta}, b_\beta^\alpha & if \;\; i = \alpha, \; k = \beta, \\ 0, & if \;\; i = 3 \;\; or \;\; k = 3, \end{cases} \tag{5}$$

$$a_{\alpha\beta} = \vec{r}_\alpha \cdot \vec{r}_\beta, \;\; a^{\alpha\beta} = \vec{r}^\alpha \cdot \vec{r}^\beta, \;\; \delta_j^i = \vec{r}^i \cdot \vec{r}_j.$$

In view of (4) and (5) the quadratic form (2) has the form

$$dS^2 = (1 - x^3 k_0) dS_0^2 + (dx^3)^2,$$

where

$$dS_0^2 = a_{\alpha\beta} dx^\alpha dx^\beta,$$

and k_0 is the normal curvature of the surface S.

For the surface $x^3 = C = const$ we have

$$dS^2 = (1 - Ck_0) dS_0^2.$$

Let M_S^0 denote the 3-D manifold with the metric

$$dS^2 = dS_0^2 + (dx^3)^2 = a_{ij} dx^i dx^j, \quad (i, j = 1, 2, 3). \tag{6}$$

In this case $x^3 = C$ and $dS = dS_0$.

For discriminants of the form (2) and (6) we have

$$g = a\vartheta^2,$$

where

$$a = a_{11}a_{22} - a_{12}^2 > 0$$

$$\vartheta = (1 - k_1x^3)(1 - k_2x^3) = 1 - 2Hx^3 + k(x^3)^2,$$

$$H = k_1 + k_2, \quad K = k_1k_2.$$

Here H and K are, respectively, middle and Gaussian curvatures of the surfaces. k_1 and k_2 are principal curvatures.

The Christoffel symbols of the first and second kinds are expressed by the following formulas

$$\overline{\Gamma}_{ij,k} = \frac{1}{2}(\partial_i A_{jk} + \partial_j A_{ik} - \partial_k A_{ij}),$$

$$\overline{\Gamma}_{ij}^k = A^{km} \cdot \overline{\Gamma}_{ij,m},$$

where

$$A^{\alpha\beta} = \vartheta^{-1}(a^{\alpha\beta} - x^3k^{-1}R^{\nu\alpha\beta\gamma}b_{\nu\gamma}),$$

$$A^{33} = 1, \ \ A^{3\alpha} = A^{\alpha,3} = 0.$$

Here $R^{\nu\alpha\beta\gamma}$ are contravariant components of the Riemann-Christoffel tensor of the surface S

$$R^{\nu\alpha\beta\gamma} = b^{\alpha\beta}b^{\nu\gamma} - b^{\alpha\gamma}b^{\nu\beta}.$$

The Riemann-Christoffel tensor has also the form

$$R^{\nu\cdots}_{\cdot\alpha\beta\gamma} = \partial_\gamma\Gamma^\nu_{\alpha\beta} - \partial_\beta\Gamma^\nu_{\alpha\gamma} + \Gamma^\lambda_{\alpha\beta}\Gamma^\nu_{\lambda\gamma} - \Gamma^\lambda_{\alpha\gamma}\Gamma^\nu_{\lambda\beta}.$$

In view of (3) we get

$$\overline{\Gamma}_{ij,k} = \Gamma_{ij,k} - x^3\widetilde{\Gamma}_{ij,k} - \frac{1}{2}(\delta_i^3 b_{jk} + \delta_j^3 b_{ik} - \delta_k^3 b_{ij}),$$

where Christoffel symbols have the forms

$$\Gamma_{ij,k} = \frac{1}{2}(\partial_i a_{jk} + \partial_j a_{ik} - \partial_k a_{ij}),$$

$$\widetilde{\Gamma}_{ij,k} = \frac{1}{2}(\partial_i b_{jk} + \partial_j b_{ik} - \partial_k b_{ij}).$$

The minimum intended for the indices $\Gamma_{ij,k}$ and $\widetilde{\Gamma}_{ij,k}$ has the forms

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}(\partial_\alpha a_{\beta\gamma} + \partial_\beta a_{\alpha\gamma} - \partial_\gamma a_{\alpha\beta}),$$

$$\widetilde{\Gamma}_{\alpha\beta,\Gamma} = \frac{1}{2}(\partial_\alpha b_{\beta\gamma} + \partial_\beta b_{\alpha\gamma} - \partial_\gamma b_{\alpha\beta}). \tag{7}$$

Using now the formula

$$\partial_\gamma b_{\alpha\beta} = \nabla_\gamma b_{\alpha\beta} + b_{\nu\beta}\Gamma^\nu_{\gamma\alpha} + b_{\alpha\nu}\Gamma^{\cdot\cdot\nu}_{\gamma\beta\cdot}.$$

we can write equalities (7) as

$$\widetilde{\Gamma}_{\alpha\beta,\gamma} = \frac{1}{2}(\nabla_\gamma b_{\alpha\beta} + 2\Gamma^\nu_{\alpha\beta} b_{\nu\gamma})$$

$$\overline{\Gamma}_{\alpha\beta,3} = \frac{1}{2} b_{\alpha\beta}, \ \ \overline{\Gamma}_{3\alpha,\beta} = \overline{\Gamma}_{\alpha 3,\beta} = -\frac{1}{2} b_{\alpha\beta}.$$

Further

$$\overline{\Gamma}_{\alpha\beta,\gamma} = \Gamma_{\alpha\beta,\gamma} - \frac{1}{2} x^3 (\nabla_\alpha b_{\beta\gamma} + 2\Gamma^\nu_{\alpha\beta} b_{\gamma\nu}),$$
$$\overline{\Gamma}_{\alpha\beta,3} = -\Gamma_{3\alpha,\beta} = -\Gamma_{\alpha 3,\beta} = \tfrac{1}{2} b_{\alpha\beta},$$
$$\overline{\Gamma}_{33,k} = \overline{\Gamma}_{3k,3} = -\Gamma_{k3,3}.$$

In view of

$$A^{\alpha\beta} = \vartheta^{-1}(a^{\alpha\beta} - x^3 k^{-1} R^{\nu\alpha\beta\gamma} b_{\nu\gamma}),$$
$$A^{33} = 1, \quad A^{3\alpha} = A^{\alpha 3} = 0.$$

We have

$$\overline{\Gamma}^\nu_{\alpha\beta} = A^{\gamma\sigma}\overline{\Gamma}_{\alpha\beta,\sigma} = \frac{1}{\vartheta}\{\Gamma^\gamma_{\alpha\beta} - (\nabla^\gamma_{\alpha\beta} + \Gamma^\mu_{\alpha\beta} b^\gamma_\mu + k^{-1} R^{\nu\gamma\sigma\lambda} b_{\nu\lambda}\Gamma_{\alpha\beta,\sigma}) x^3$$

$$+ \frac{(x^3)^2}{2k}[R^{\nu\gamma\sigma\lambda} b_{\nu\lambda}\nabla_\alpha b_{\alpha\beta} + 2\Gamma^\mu_{\alpha\beta} R^{\nu\gamma\sigma\lambda} b_{\nu\lambda} b_{\sigma\mu}]\},$$

where

$$\Gamma^\mu_{\alpha\beta} b^\nu_\mu + \frac{1}{k} R^{\nu\gamma\sigma\lambda} b_{\nu\lambda}\Gamma_{\alpha\beta,\sigma} = 2H\Gamma^\gamma_{\alpha\beta},$$
$$\frac{1}{k} R^{\nu\gamma\sigma\lambda} b_{\nu\lambda} b_{\sigma\mu}\Gamma^\mu_{\alpha\beta} = K\Gamma^\gamma_{\alpha\beta}.$$

Now we have

$$\overline{\Gamma}^\nu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} - \frac{1}{2} x^3 B^{\nu\cdots}_{\cdot\alpha\beta},$$

where

$$B^{\nu\cdots}_{\cdot\alpha\beta} = \frac{1}{\vartheta}(\nabla^\gamma b_{\alpha\beta} - x^3 K^{-1} R^{\mu\nu\sigma\lambda} b_{\mu\lambda}\nabla_\sigma a_{\alpha\beta})$$

and

$$B^{1\cdots}_{\cdot\alpha\beta} = \frac{1}{1 - k_1 x_3}\nabla^1 b_{\alpha\beta}, \quad B^{2\cdots}_{\cdot\alpha\beta} = \frac{1}{1 - k_2 x_3}\nabla^2 b_{\alpha\beta}.$$

Let S be a spherical surface of the radius R. Then

$$\vec{n} = -\frac{1}{R}\vec{r}$$

and

$$b_{\alpha\beta} = -\vec{n}_\alpha \vec{r}_\beta = \frac{1}{R}\vec{r}_\alpha \vec{r}_\beta = \frac{1}{R} a_{\alpha\beta}.$$

Hence

$$A_{\alpha\beta} = \left(1 - \frac{x^3}{R}\right) a_{\alpha\beta}, \quad A_{33} = 1, \ \ A_{3\alpha} = A_{\alpha 3} = 0,$$

$$A^{\alpha 3} = \frac{a^{\alpha\beta}}{1 - \dfrac{x^3}{R}} \quad A^{33} = 1, \quad A^{3\alpha} = A^{\alpha 3} = 0.$$

Then

$$\overline{\Gamma}_{\alpha\beta,\gamma} = \left(1 - \frac{x^3}{R}\right)\Gamma_{\alpha\beta,\gamma} \quad \overline{\Gamma}_{\alpha\beta,3} = \frac{1}{2R}a_{\alpha\beta},$$

$$\overline{\Gamma}_{\alpha 3,\beta} = \overline{\Gamma}_{3\alpha,\beta} = -\frac{1}{2R}a_{\alpha\beta}, \quad \overline{\Gamma}_{33,k} = \overline{\Gamma}_{3k,3} = \overline{\Gamma}_{k3,3} = 0,$$

$$\overline{\Gamma}^{\gamma}_{\alpha\beta}, \quad \Gamma^{\gamma}_{\alpha\beta}, \quad \overline{\Gamma}^{3}_{\alpha\beta} = \frac{1}{2R}a_{\alpha\beta}, \quad \overline{\Gamma}^{\beta}_{\alpha 3} = \overline{\Gamma}^{\beta}_{3\alpha} = \frac{1}{2R\left(1 - \dfrac{x^3}{R}\right)^3}a_{\alpha\beta}.$$

The Riemann-Christoffel tensor has the form

$$\overline{\Gamma}^{\lambda\cdots}_{\cdot\alpha\beta\gamma} = \left(1 - \frac{1}{4\left(1 - \dfrac{x^3}{R}\right)}\right) R^{\lambda\cdots}_{\cdot\alpha\beta\gamma}, \tag{8}$$

where $R^{\lambda\cdots}_{\cdot\alpha\beta\gamma}$ is a Riemann-Christoffel tensor of the spherical surface, which has the form

$$R^{\lambda\cdots}_{\cdot\alpha\beta\gamma} = \frac{1}{R^2}C^{\cdot\lambda}\alpha\cdot C_{\beta\gamma},$$

where

$$C_{11} = C_{22} = 0, \quad C_{12} = C_{21} = \sqrt{a},$$

$$C^{\cdot\beta}_{\alpha\cdot} = a^{\beta\gamma}C_{\alpha\gamma}, \quad C^{\beta\cdot}_{\cdot\alpha} = a^{\beta\gamma}C_{\gamma\alpha},$$

$$C^{11} = C^{22} = 0, \quad C^{12} = -C^{21} = \frac{1}{\sqrt{a}}.$$

From (8) we have

$$\overline{R}^{\nu\cdots}_{\cdot\alpha\beta\gamma} \neq 0, \quad if \quad x^3 < \frac{3R}{4}.$$

Thus for outside the spherical surface with the radius $\frac{3R}{4}$ the M_S is a strictly Riemann surface.

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R E F E R E N C E S

1. Vekua I.N. Shell Theory: General Methods of Construction. *Pitman Advanced Publishing Program, Boston-London-Melbourne*, 1985.
2. Vekua I.N. Foundations of Tensor Analysis and the Theory of Ovariants (Russian). *Nauka, Moscow*, 1978, 296 pp.

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Author's address:

T. Meunargia
I. Vekua Institute of Applied Mathematics of
I. Javakhishvili Tbilisi State University
2, University str., Tbilisi 0186
Georgia
E-mail: tengizmeunargia37@gmail.com