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# ON THE SCHWARZ-CHRISTOFFEL PARAMETERS PROBLEM

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**Abstract**. In the paper the Schwarz-Christoffel parameter problem is analyzed and the method for their computation is given.

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In integrated circuitry and elsewhere in electronics, the electrical resistance of a polygonal circuit element or pathway is often physically important. In the simplest problem of resistor *analysis* and *design*, a polygon is given and its resistance must be determined through selection of a polygon, from a family of candidates parametrized by some geometric quantity (see [1],[2]). It depends on the solution of a boundary value problem for Laplace's equation, and as a result it is invariant under a conformal map. Therefore, we can construct a conformal map onto a new domain where the problem is trivial.

Conformal mapping methods for resistor analysis on polygons requires the numerical solution of a Schwarz-Christoffel *parameter problem* [3], which has only recently become feasible for general polygons.

In this paper we will consider Schwarz-Christoffel formula

$$f^{-1}(z) = w_c + C\int\limits_0^z \prod_{k=1}^N \left(1-\frac{\zeta}{z_k}\right)^{-\beta_k} d\zeta$$

and find its constants through given parameters.

**Theorem 1** (Schwarz-Christoffel). *Let $P$ be the interior of a polygon $\Gamma$ having vertices $w_1, \ldots, w_n$ and external angles $\beta_1\pi, \ldots, \beta_n\pi$ i n counter-clockwise order. If $f$ is conformal mapping from $P$ to the upper half plane $\mathbb{H}$, then its inverse can be written in the form:*

$$f^{-1}(z) = w_c + C\int\limits_0^z \prod_{k=1}^N \left(1-\frac{\zeta}{z_k}\right)^{-\beta_k} d\zeta,$$

*where $z_k = f(w_k)$ and $w_c$, $C$ are some constants.*

Our goal is to solve the Schwarz-Christoffel *parameter problem.* For this we will study the structure of the integration part.

If we have $g(z)$ and its inverse $g^{-1}(z)$, then inverse of $f(z) = a + bg(z)$ is

$$f^{-1}(z) = g^{-1}\left(\frac{z-a}{b}\right)$$

Therefore, if manage to find the inverse of

$$g(z) = \int_0^z \prod_{k=1}^N \left(1 - \frac{\zeta}{z_k}\right)^{-\beta_k} d\zeta,$$

we will find $f^{-1} : P \to \mathbb{H}^+$ conformal mapping.

To study the equation we will need to interpret expression, inside integration, as the sum series. By using Taylor series it easy to show the following proposition.

**Proposition 1.** *Let* $z \in \mathbb{C} \setminus \{z = 0 \vee |z| = 1\}$ *and* $\alpha \in \mathbb{Q}$. *Then*

$$(1 + z)^\alpha = e^{2\pi ic\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k, \quad \textit{when} \quad |z| < 1$$

*and*

$$(1 + z)^\alpha = e^{2\pi ic\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} z^{\alpha-k}, \quad \textit{when} \quad |z| > 1.$$

*Here the infinite power series gives only one of the values of the* $(1 + z)^\alpha$, *while* $e^{2\pi ic\alpha}$, $c \in \mathbb{Z}$, *rotates on unit circle, giving all branch points and since* $\alpha$ *is rational, these points are finite.*

The coefficients are called binomial and they are used in combinatorics and they are generalization of Newton's binomial series. However, if we follow Taylor formula step-by-step we will see that the coefficients are constructed, like

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \prod_{j=1}^{k} \frac{\alpha - j + 1}{j}.$$

Furthermore, since factorials can be generalized by $\Gamma$ function, we will use the following expression

$$n! := \Gamma(n + 1)$$

for $\forall n \in \mathbb{R}$.

To simplify visualization of operation on combinatorial structures, we'll be using expressions like

$$\sum_{k_1+k_2=3} a_{k_1} b_{k_2} = a_3b_0 + a_2b_1 + a_1b_2 + a_0b_3,$$

$$\sum_{k_1+2k_2+3k_3=4} a_{k_1} b_{k_2} c_{k_3} = a_4b_0c_0 + a_2b_1c_0 + a_0b_2c_0 + a_1b_0c_1.$$

In algebra it is common to use this kind of expression, since it allows to better visualise operation on sets.

**Proposition 2.** *If* $x \neq 0$, *then*

$$\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^{-j}\right) = \sum_{r=-n}^{n}\left(\sum_{i-j=r} a_i b_j\right) x^r,$$

*where $i-j=r$ expression means that $i$ and $j$ are non-negative integer numbers that satisfy this equation.*

**Example 1.** If $n=2$, we have

$$\left(\sum_{i=0}^{2} a_i x^i\right)\left(\sum_{j=0}^{2} b_j x^{-j}\right) = \sum_{r=-2}^{2}\left(\sum_{i-j=r} a_i b_j\right) x^r =$$

$$= \frac{a_0 b_2}{x^2} + \frac{a_0 b_1 + a_1 b_2}{x} + (a_0 b_0 + a_1 b_1 + a_2 b_2) + (a_1 b_0 + a_2 b_1)x + a_2 b_0 x^2.$$

If we organize coefficients as a matrix, we'll get

$$\begin{matrix} a_0b_0 & a_1b_0 & a_2b_0 \\ a_0b_1 & a_1b_1 & a_2b_1 \\ a_0b_2 & a_1b_2 & a_2b_2 \end{matrix}$$

$(n+1)\times(n+1)$ dimensional square matrix and it is easy to see that for $|r|\le n$

$$\sum_{i-j=r} a_i b_j = \sum_{i=r}^{n} a_i b_{i-r}.$$

**Definition 1.** For any $z_1,\dots,z_N \in \mathbb{C}-\{0\}$ and $\alpha_1,\dots,\alpha_N \in \mathbb{Q}$ points, we define $\lambda_n$ and $\mu_n$ as

$$\lambda_n(\alpha_1,\dots,\alpha_N; z_1,\dots,z_N) = \sum_{k_1+\dots+k_N=n}\left(\prod_{j=1}^{N}\binom{\alpha_j}{k_j}\frac{1}{z_j^{k_j}}\right),$$

$$\mu_n(\alpha_1,\dots,\alpha_N; z_1,\dots,z_N) = \sum_{k_1+\dots+k_N=n}\left(\prod_{j=1}^{N}\binom{\alpha_j}{k_j}\frac{1}{z_j^{\alpha_j-k_j}}\right).$$

The functions $\lambda_n$ and $\mu_n$ are coefficients of power series which we are constructing for

$$\prod_{k=1}^{N}\left(1-\frac{\zeta}{z_k}\right)^{-\beta_k}$$

product series.

**Proposition 3.** *For the sequence $\{(z_j,\alpha_j)\}_{j=1}^{N}$, such that $0<|z_j\zeta|<1$, have the identity*

$$\prod_{j=1}^{N}(1+\zeta z_j)^{\alpha_j} = \sum_{n=0}^{\infty}\left(\sum_{k_1+\dots+k_N=n}\left(\prod_{j=1}^{N}\binom{\alpha_j}{k_j} z_j^{k_j}\right)\right)\zeta^n$$

*and if $|z_j\zeta|>1$, we have*

$$\prod_{j=1}^{N}(1+\zeta z_j)^{\alpha_j} = \sum_{n=0}^{\infty}\left(\sum_{k_1+\dots+k_N=n}\left(\prod_{j=1}^{N}\binom{\alpha_j}{k_j} z_j^{\alpha_j-k_j}\right)\right)\zeta^{\alpha_1+\dots+\alpha_N-n}.$$

◁ Indeed, if $0 < |z_j\zeta| < 1$, then

$$\prod_{j=1}^{N}(1+\zeta z_j)^{\alpha_j} = \prod_{j=1}^{N}\sum_{k=0}^{\infty}\binom{\alpha_j}{k}(\zeta z_j)^k = \sum_{k_1,\dots,k_N}\binom{\alpha_1}{k_1}\cdots\binom{\alpha_N}{k_N}z_1^{k_1}\cdots z_N^{k_N}\zeta^{k_1+\dots+k_N}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k_1+\dots+k_N=n}\binom{\alpha_1}{k_1}\cdots\binom{\alpha_N}{k_N}z_1^{k_1}\cdots z_N^{k_N}\right)\zeta^n$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k_1+\dots+k_N=n}\left(\prod_{j=1}^{N}\binom{\alpha_j}{k_j}z_j^{k_j}\right)\right)\zeta^n.$$

If $|z_j\zeta| > 1$, then

$$\prod_{j=1}^{N}(1+\zeta z_j)^{\alpha_j} = \left(\prod_{j=1}^{N}(\zeta z_j)^{\alpha_j}\right)\left(\prod_{j=1}^{N}\left(1+\frac{1}{\zeta z_j}\right)^{\alpha_j}\right)$$
$$= \left(\prod_{j=1}^{N}(\zeta z_j)^{\alpha_j}\right)\left(\sum_{n=0}^{\infty}\left(\sum_{k_1+\dots+k_N=n}\left(\prod_{j=1}^{N}\binom{\alpha_j}{k_j}\frac{1}{z_j^{k_j}}\right)\right)\zeta^{-n}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k_1+\dots+k_N=n}\left(\prod_{j=1}^{N}\binom{\alpha_j}{k_j}z_j^{\alpha_j-k_j}\right)\right)\zeta^{\alpha_1+\dots+\alpha_N-n}.$$

▷

**Remark 1.** We can formulate Proposition 3 in the following manner:

$$\prod_{j=1}^{N}\left(1+\frac{\zeta}{z_j}\right)^{\alpha_j} = e^{i\theta}\sum_{n=0}^{\infty}\lambda_n(\alpha_1,\dots,\alpha_N;z_1,\dots,z_N)\,\zeta^n, \text{ when } \left|\frac{\zeta}{z_j}\right| < 1$$

and

$$\prod_{j=1}^{N}\left(1+\frac{\zeta}{z_j}\right)^{\alpha_j} = e^{i\theta}\zeta^{\alpha_1+\dots+\alpha_N}\sum_{n=0}^{\infty}\mu_n(\alpha_1,\dots,\alpha_N;z_1,\dots,z_N)\,\zeta^{-n}, \text{ when } \left|\frac{\zeta}{z_j}\right| > 1,$$

where $0 \le \theta < 2\pi$ is some constant.

At this point we have power series, which allow us to make approximation of a product, and all the difficulties are moved to coefficients $\lambda_n$ and $\mu_n$, which don't depend on $\zeta$.

**Example 2.** For better visualization, why we study this problem by power series, let's take fixed points on $\mathbb{R}$ and see how it works

$$f(\zeta) = \left(1+\frac{\zeta}{3}\right)^{7/3}\left(1+\frac{\zeta}{17}\right)^{4/5}\left(1+\frac{\zeta}{35}\right)^{23/11}$$

and

$$f_s(\zeta) = \sum_{n=0}^{s}\lambda_n\left(\frac{7}{3},\frac{4}{5},\frac{23}{11};3,17,35\right)\zeta^n.$$

From the theory we know that each component *absolute convergence* when $\zeta \in (0,3)$ interval, and our $f_s$ is a result of multiplying these components. Suppose $\zeta = 2$, then

$$\begin{aligned} f(2) &\approx 4.04345073813124671, \\ f_4(2) &\approx 4.04262394108668742, \\ f_8(2) &\approx 4.04344234608117199, \\ f_{50}(2) &\approx 4.04345073813124596. \end{aligned}$$

As we can see, $f_s$ is making approximation of $f$ and for rational arguments it returns rational numbers. Since we showed how this series works, let's integrate both $f$ and $f_s$

$$\int_0^2 f(\zeta)d\zeta \approx 4.57756,$$
$$\int_0^2 f_8(\zeta)d\zeta \approx 4.57756.$$

Therefore, we can use this power series to approximately calculate this type of integrals.

**Lemma 1.** *Let $z_1, \ldots, z_N \in \mathbb{C} \setminus \{0\}$ be on arbitrary set of points and*

$$0 < |z_1| < |z_2| < \cdots < |z_N| < \infty,$$

*$\alpha_1, \ldots, \alpha_N \in \mathbb{Q}$. Then for any*

$$\zeta \in \left\{ z \in \mathbb{C} \setminus \{0\} \,\middle|\, \left|\frac{z}{z_j}\right| \neq 1,\ j = \overline{1, N} \right\}$$

*we have three cases*

1. *If $|\zeta| < |z_1|$, then*

$$\prod_{j=1}^{N} \left(1 - \frac{\zeta}{z_j}\right)^{-\beta_j} = e^{i\theta} \sum_{n=0}^{\infty} (-1)^n \lambda_n \left(-\beta_1, \ldots, -\beta_N; z_1, \ldots, z_N\right) \zeta^n$$

2. *If $|\zeta| > |z_N|$, then*

$$\prod_{j=1}^{N} \left(1 - \frac{\zeta}{z_j}\right)^{-\beta_j} = e^{i\theta} \zeta^{-(\beta_1 + \cdots + \beta_N)} \times$$

$$\times \sum_{n=0}^{\infty} (-1)^n \mu_n \left(-\beta_1, \ldots, -\beta_N; z_1, \ldots, z_N\right) \zeta^{-n}$$

*3. If* $|z_M| < |\zeta| < |z_{M+1}|$*, then*

$$\prod_{j=1}^{N}\left(1-\frac{\zeta}{z_j}\right)^{-\beta_j} = \prod_{j=1}^{M}\left(1-\frac{\zeta}{z_j}\right)^{-\beta_j}\prod_{j=M+1}^{N}\left(1-\frac{\zeta}{z_j}\right)^{-\beta_j}$$

$$= e^{i\theta}\zeta^{-(\beta_1+\dots+\beta_M)}\sum_{n\in\mathbb{Z}}(-1)^n\xi_{n,M}\left(-\beta_1,\dots,-\beta_N;z_1,\dots,z_N\right)\zeta^n$$

*where*

$$\xi_{n,M}\left(\alpha_1,\dots,\alpha_N;z_1,\dots,z_N\right) =$$

$$\sum_{k=n\vee 0}^{\infty}\lambda_k\left(\alpha_{M+1},\dots,\alpha_N;z_{M+1},\dots,z_N\right)\mu_{k-n}\left(\alpha_1,\dots,\alpha_M;z_1,\dots,z_M\right)$$

*where* $n\vee 0 := max\,\{n,0\}$*, since if* $k<0$*, then* $\lambda_k=\mu_k=0$*.*

◁ First two are proved in Proposition 3. The third follows from Proposition 2. ▷

**Example 3.** Let's take a $f$ function

$$f(\zeta) = \left(1+\frac{\zeta}{3}\right)^{7/3}\left(1+\frac{\zeta}{17}\right)^{4/5}\left(1+\frac{\zeta}{35}\right)^{8/11}\left(1+\frac{\zeta}{39}\right)^{5/17}$$

and let $\zeta = 19$, then

$$f(\zeta)\approx 293.3450707$$

unlike, when we hade only one side, when coefficients where calculated from finite sum, in this case we have an infinite sum. For coefficients we write

$$\xi_n^m\left(\alpha_1,\alpha_2,\alpha_3,\alpha_4;z_1,z_2,z_3,z_4\right) = \sum_{k=n}^{n+m}\lambda_k\left(\alpha_3,\alpha_4;z_3,z_4\right)\mu_{k-n}\left(\alpha_1,\alpha_2;z_1,z_2\right),$$

$$f_n^m(\zeta) = \zeta^{7/3+4/5}\sum_{k=-n}^{n}\xi_k^m\left(\frac{7}{3},\frac{4}{5},\frac{8}{11},\frac{5}{17};3,17,35,39\right)\zeta^k$$

and approximations are

$$f_{20}^{30}(19)\approx 293.3439706,$$
$$f_{25}^{30}(19)\approx 293.3454928.$$

If integrate both on $(19,23)$ interval, we get

$$\int_{19}^{23} f(\zeta)d\zeta \approx 1570.876256,$$

$$\int_{19}^{23} f_{30}^{30}(\zeta)d\zeta \approx 1570.875901.$$

If we have the formal power series

$$f(x) = 1 + a_1x + a_2x^2 + \cdots,$$

then

$$\frac{1}{f(x)} = 1 + b_1x + b_2x^2 + \cdots,$$

where

$$b_n = \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+k_2+\cdots+k_n} (k_1 + k_2 + \cdots + k_n)! \prod_{j=1}^{n} \frac{a_j^{k_j}}{k_j!}.$$

Before we begin to prove, we need to mention that by

$$\frac{1}{f(x)} = 1 + b_1x + b_2x^2 + \cdots,$$

we mean that right side is such that

$$\left(1 + a_1x + a_2x^2 + \cdots\right)\left(1 + b_1x + b_2x^2 + \cdots\right) = 1 + c_1x + c_2x^2 + \cdots,$$

where all $c_n = 0$. For example, first four $b_n$ are

$$\begin{aligned} b_1 &= -a_1, \\ b_2 &= a_1^2 - a_2, \\ b_3 &= -a_1^3 + 2a_1a_2 - a_3, \\ b_4 &= a_1^4 - 3a_1^2a_2 + 2a_1a_3 + a_2^2 - a_4. \end{aligned}$$

It is easy to justify that

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}, \quad a_0 = b_0 = 1$$

and since $c_n = 0$, we get recursive interpretation

$$b_n = -\sum_{i=1}^{n} a_i b_{n-i}.$$

To solve the recursion, we'll assume that

$$b_n = \sum_{k_1+2k_2+\cdots+nk_n=n} (-1)^{k_1+k_2+\cdots+k_n} \phi(k_1, k_2, \ldots, k_n) \prod_{j=1}^{n} a_j^{k_j},$$

where $\phi$ doesn't depend on $n$ and has the following properties

1. $\phi(0) = 1$;
2. $\phi(k_1, \ldots, k_n) = \phi(k_1, \ldots, k_n, 0, 0, \ldots)$;

3. If any argument is negative, then $\phi = 0$.

$$\sum_{z_1+2z_2+\dots+nz_n=n} (-1)^{z_1+\dots+z_n}\phi(z_1,z_2,\dots,z_n)\prod_{j=1}^{n} a_j^{z_j} = b_n = -\sum_{s=1}^{n} a_s b_{n-s}$$

$$= -\sum_{s=1}^{n}\left(\sum_{k_1+2k_2+\dots+nk_n=n-s} (-1)^{k_1+\dots+k_n}\phi(k_1,\dots,k_n)\prod_{j=1}^{n} a_j^{k_j}\right) a_s$$

$$= -\sum_{s=1}^{n}\left(\sum_{k_1+2k_2+\dots+nk_n=n-s} (-1)^{k_1+\dots+k_n}\phi(k_1,\dots,k_n)\prod_{j=1}^{n} a_j^{k_j}\right) \times$$

$$\left(\sum_{m_1+2m_2+\dots+nm_n=s} \chi(\forall i\neq s: m_i=0)\,\chi(m_s=1)\prod_{j=1}^{n} a_j^{m_j}\right)$$

$$= -\sum_{s=1}^{n}\sum_{k_1+2k_2+\dots+nk_n=n-s}\ \sum_{m_1+2m_2+\dots+nm_n=s}$$

$$(-1)^{k_1+\dots+k_n}\phi(k_1,\dots,k_n)\,\chi(\forall i\neq s: m_i=0)\,\chi(m_s=1)\prod_{j=1}^{n} a_j^{k_j+m_j}$$

$$= \sum_{z_1+2z_2+\dots+nz_n=n}\ \sum_{s=1}^{n}\ \sum_{k_1+2k_2+\dots+nk_n=n-s}$$

$$(-1)^{1+k_1+\dots+k_n}\phi(k_1,\dots,k_n)\,\chi(\forall i\neq s: k_i=z_i)\,\chi(z_s-k_s=1)\prod_{j=1}^{n} a_j^{z_j}$$

Here we made few important steps, the first one is following interpretation

$$a_s = \sum_{m_1+2m_2+\dots+nm_n=s} \chi(\forall i\neq s: m_i=0)\,\chi(m_s=1)\prod_{j=1}^{n} a_j^{m_j},$$

where $\chi(\mathcal{A})$ is a Boolean function, which is 1 if $\mathcal{A}$ statement is true and 0 if it is false.

The second step was changing summation set, for visualisation let's write it like

$$\sum_{s=1}^{n}\ \sum_{k_1+2k_2+\dots+nk_n=n-s}\ \sum_{m_1+2m_2+\dots+nm_n=s} u(k_1,\dots,k_n)\,v(m_1,\dots,m_n)$$

$$= \sum_{s=1}^{n}\ \sum_{k_1+2k_2+\dots+nk_n=n-s}\ \sum_{z_1+2z_2+\dots+nz_n=n} u(k_1,\dots,k_n)\,v(z_1-k_1,\dots,z_n-k_n)$$

$$= \sum_{z_1+2z_2+\dots+nz_n=n}\left(\sum_{s=1}^{n}\ \sum_{k_1+2k_2+\dots+nk_n=n-s} u(k_1,\dots,k_n)\,v(z_1-k_1,\dots,z_n-k_n)\right).$$

here we need the third property of $\phi$, that if the argument is negative, then the function is 0.

Since $a_j$ coefficients are arbitrary we get, when $z_1 + 2z_2 + \cdots + nz_n = n$

$$(-1)^{z_1+\cdots+z_n}\phi(z_1,\ldots,z_n) =$$

$$= \sum_{s=1}^{n} \sum_{k_1+2k_2+\cdots+nk_n=n-s} (-1)^{1+k_1+\cdots+k_n}\phi(k_1,\ldots,k_n) \times$$

$$\times\chi(\forall i \neq s : k_i = z_i)\,\chi(z_s - k_s = 1)$$

$$= \sum_{s=1}^{n}(-1)^{1+z_1+\cdots+z_{s-1}+(z_s-1)+z_{s+1}+\cdots+z_n}\phi(z_1,\ldots,z_{s-1},z_s-1,z_{s+1},\ldots,z_n)$$

Therefore

$$\phi(z_1,\ldots,z_n) = \sum_{s=1}^{n}\phi(z_1,\ldots,z_{s-1},z_s-1,z_{s+1},\ldots,z_n).$$

For example

$$\phi(2,1,0) = \phi(1,1,0) + \phi(2,0,0) = \phi(1,0,0) + \phi(1,0,0) + \phi(2,0,0) = 3$$

and it is the coefficient of $a_1^2 a_2$ in $b_4$.

Solution of this recursion is

$$\phi(z_1,\ldots,z_n) = \frac{(z_1+z_2+\cdots+z_n)!}{z_1!z_2!\cdots z_n!}.$$

It is easy to check that it satisfies all properties, for example

$$\frac{((k_1-1)+k_2)!}{(k_1-1)!k_2!} + \frac{(k_1+(k_2-1))!}{k_1!(k_2-1)!} = (k_1+k_2)\frac{(k_1+k_2-1)!}{k_1!k_2!} = \frac{(k_1+k_2)!}{k_1!k_2!}.$$

$\triangleright$

**Example 4.** If we take

$$\frac{\sin(x)}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + O\left(x^8\right), \quad a_n = \frac{1}{(n+1)!}\cos\left(\frac{\pi n}{2}\right),$$

then

$$\frac{x}{\sin(x)} = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + O\left(x^8\right).$$

**Theorem 2.** *If we enumerate $z_1, z_2, \ldots, z_N$ so that*

$$|z_1| < |z_2| < \cdots < |z_N|,$$

*the constants in formula*

$$f(z) = w_c + C\int_0^z \prod_{k=1}^{N}\left(1-\frac{\zeta}{z_k}\right)^{-\beta_k} d\zeta$$

*are*

$$\frac{1}{C} = \frac{e^{i\theta_1}}{f(z_2) - f(z_1)}$$
$$\times \sum_{n\in\mathbb{Z}}^{\infty} (-1)^n \left( \sum_{k=n\vee 0}^{\infty} \lambda_k \left(-\beta_2, \ldots, -\beta_N; z_2, \ldots, z_N\right) \mu_{k-n}(-\beta_1; z_1) \right)$$
$$\times \frac{z_2^{n-\beta_1+1} - z_1^{n-\beta_1+1}}{n-\beta_1+1},$$
$$w_c = f(z_1) - Ce^{i\theta_2} \sum_{n=0}^{\infty} (-1)^n \lambda_n \left(-\beta_1, \ldots, -\beta_N; z_1, \ldots, z_N\right) \frac{z_1^{n+1}}{n+1},$$

*where* $\theta_1$ *and* $\theta_2$ *are some constants.*

◁ To find constants lets lake $z = z_1$ and $z = z_2$ points

$$f(z_1) = w_c + C \int_0^{z_1} \prod_{k=1}^{N} \left(1 - \frac{\zeta}{z_k}\right)^{-\beta_k} d\zeta$$

since $|\zeta| < |z_1|$

$$f(z_1) = w_c + C \int_0^{z_1} \sum_{n=0}^{\infty} (-1)^n \lambda_n \left(-\beta_1, \ldots, -\beta_N; z_1, \ldots, z_N\right) \zeta^n$$
$$= w_c + C \sum_{n=0}^{\infty} (-1)^n \lambda_n \left(-\beta_1, \ldots, -\beta_N; z_1, \ldots, z_N\right) \frac{z_1^{n+1}}{n+1}.$$

If we take the integral from $z_1$ till $z_2$, we get

$$f(z_2) - f(z_1) = C \int_{z_1}^{z_2} \prod_{k=1}^{N} \left(1 - \frac{\zeta}{z_k}\right)^{-\beta_k} d\zeta$$
$$= C \int_{z_1}^{z_2} \sum_{n\in\mathbb{Z}}^{\infty} (-1)^n \left( \zeta^{-\beta_1} \sum_{k=n}^{\infty} \lambda_k \left(-\beta_2, \ldots, -\beta_N; z_2, \ldots, z_N\right) \mu_{k-n}(-\beta_1; z_1) \right) \zeta^n d\zeta$$
$$= C \int_{z_1}^{z_2} \sum_{n\in\mathbb{Z}}^{\infty} (-1)^n \left( \sum_{k=n}^{\infty} \lambda_k \left(-\beta_2, \ldots, -\beta_N; z_2, \ldots, z_N\right) \mu_{k-n}(-\beta_1; z_1) \right) \zeta^{n-\beta_1} d\zeta$$
$$= C \sum_{n\in\mathbb{Z}}^{\infty} (-1)^n \left( \sum_{k=n}^{\infty} \lambda_k \left(-\beta_2, \ldots, -\beta_N; z_2, \ldots, z_N\right) \mu_{k-n}(-\beta_1; z_1) \right)$$
$$\times \frac{z_2^{n-\beta_1+1} - z_1^{n-\beta_1+1}}{n-\beta_1+1},$$

so

$$\frac{1}{C} = \frac{1}{f(z_2) - f(z_1)} \sum_{n\in\mathbb{Z}}^{\infty} (-1)^n \left( \sum_{k=n}^{\infty} \lambda_k \left(-\beta_2, \ldots, -\beta_N; z_2, \ldots, z_N\right) \mu_{k-n}(-\beta_1; z_1) \right)$$

$$\times \frac{z_2^{n-\beta_1+1} - z_1^{n-\beta_1+1}}{n-\beta_1+1}.$$

$\triangleright$

**Proposition 4.** *Let enumerate* $z_1, z_2, \dots, z_N$ *such, that*

$$|z_1| < |z_2| < \cdots < |z_N|,$$

*then*

$$\int_{z_M}^{z_{M+1}} \prod_{k=1}^{N} \left(1 - \frac{\zeta}{z_k}\right)^{-\beta_k} d\zeta = e^{i\theta} \sum_{n\in\mathbb{Z}} (-1)^n \xi_{n,M} \int_{z_M}^{z_{M+1}} \zeta^{n-\beta_1-\cdots-\beta_M} d\zeta,$$

*where* $\theta$ *depends on chosen* $M$*.*

**Comment 1.** We can use some manipulations with power series coefficents to get interesting results, which can be hard to prove on its own, like the Riemann zeta function for even positive arguments

$$\zeta(z) = \frac{(2\pi)^z}{2z!} |B_z|$$

is proved by using Fourier Series expansion of $B_z(x)$ Bernoulli polynomials. Besides, it is very well known how to find $\zeta(2)$ and similarly we can calculate $\zeta(6)$

$$2x^3 \prod_{k=1}^{\infty} \left(1 - \frac{x^6}{(\pi k)^6}\right) = 2x^3 - 2\left(\sum_{k=1}^{\infty} \frac{1}{(\pi k)^6}\right) x^9 + \cdots = 2x^3 - \frac{2\zeta(6)x^9}{\pi^6} + \cdots$$

$$\begin{aligned}
2x^3 \prod_{k=1}^{\infty} \left(1 - \frac{x^6}{(\pi k)^6}\right) &= 2x^3 \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(\pi k)^2}\right)\left(1 - \frac{\left(xe^{\frac{\pi}{3}i}\right)^2}{(\pi k)^2}\right)\left(1 - \frac{\left(xe^{-\frac{\pi}{3}i}\right)^2}{(\pi k)^2}\right) \\
&= 2x^3 \frac{\sin(x)}{x} \frac{\sin\left(xe^{\frac{\pi}{3}i}\right)}{xe^{\frac{\pi}{3}i}} \frac{\sin\left(xe^{-\frac{\pi}{3}i}\right)}{xe^{-\frac{\pi}{3}i}} = \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} \left(\cos\left(\frac{2\pi n}{3} + \frac{\pi}{3}\right) - \frac{1}{2}\right) \\
&= 2x^3 - \frac{2x^9}{945} + \frac{8x^{15}}{212837625} - \frac{4x^{21}}{64965492466875} + \cdots,
\end{aligned}$$

so

$$-\frac{2\zeta(6)x^9}{\pi^6} = -\frac{2x^9}{945} \Rightarrow \zeta(6) = \frac{\pi^6}{945}.$$

From Proposition  we know that

$$(1+z)^\alpha = e^{2\pi i c\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k, \quad \text{when} \quad |z| < 1.$$

It is a Taylor series expansion of $f(z) = (1+z)^\alpha$ and we don't have restrictions on $\alpha \in \mathbb{R}$. Binomial coefficients and factorials are strongly associated with classical probability theory, so it may be unusual to see

$$\left(\frac{1}{2}\right)! \quad \text{or} \quad \binom{3}{5/2}$$

since we cannot use analogs, like how many permutation $a_1, a_2, \ldots, a_n$ different letters, have which is $n!$, because it works only for natural numbers. The factorial is defined by recursion, which can be solved, by the Γ-function

$$n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt.$$

The same is for binomial coefficients. The coefficients we get from the Taylor series are

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \prod_{j=1}^{k} \frac{\alpha - j + 1}{j}$$

and then two experssions can be rewritten as

$$\binom{\alpha}{k} = \frac{\alpha!}{k!(\alpha-k)!}.$$

Therefore, arguments of factorials and binomial functions are not limited by $\mathbb{N}$.

**Example 5.**

$$(1+z)^\beta (1-z)^\beta = \left(\sum_{k=0}^\infty \binom{\beta}{k} z^k\right)\left(\sum_{j=0}^\infty \binom{\beta}{j} (-z)^j\right)$$

$$= \sum_{k,j=0}^\infty \binom{\beta}{k}\binom{\beta}{j}(-1)^j z^{k+j}$$

$$= \sum_{r=0}^\infty \left(\sum_{k+j=r} (-1)^j \binom{\beta}{k}\binom{\beta}{j}\right) z^r = \sum_{r=0}^\infty \left(\sum_{j=0}^\infty (-1)^j \binom{\beta}{r-j}\binom{\beta}{j}\right) z^r$$

$$= \sum_{r=0}^\infty \left(\binom{\beta}{r} \sum_{j=0}^\infty \frac{(-\beta)_j (-r)_j}{(\beta - r + 1)_j} \frac{(-1)^r}{j!}\right) z^r$$

$$= \sum_{r=0}^\infty (-1)^r {}_2F_1(-\beta, -r, \beta - r + 1; -1) \binom{\beta}{r} z^r,$$

on the other hand

$$(1+z)^\beta (1-z)^\beta = \left(1 - z^2\right)^\beta = \sum_{r=0}^\infty \binom{\beta}{r} (-1)^r z^{2r},$$

so if $r \geq 0$ is odd

$$ {}_2F_1\left(-\beta, -r, \beta - r + 1; -1\right) = 0 $$

and if $r \geq 0$ is even

$$ {}_2F_1\left(-\beta, -r, \beta - r + 1; -1\right) = (-1)^{r/2}\binom{\beta}{r/2}\binom{\beta}{r}^{-1}. $$

**Example 6.** If

$$ (1+z)^{\beta} = 1 + \binom{\beta}{1}z + \binom{\beta}{2}z^2 + \binom{\beta}{3}z^3 + \cdots, $$

then

$$ \frac{1}{(1+z)^{\beta}} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots, $$

where

$$ b_s = \sum_{k_1+2k_2+\cdots+sk_s=s} (-1)^{k_1+\cdots+k_s}\frac{(k_1+\cdots+k_s)!}{k_1!\cdots k_s!}\prod_{j=1}^{s}\binom{\beta}{j}^{k_j}. $$

Therefore, we get

$$ \binom{-\beta}{s} = \sum_{k_1+2k_2+\cdots+sk_s=s} (-1)^{k_1+\cdots+k_s}\frac{(k_1+\cdots+k_s)!}{k_1!\cdots k_s!}\prod_{j=1}^{s}\binom{\beta}{j}^{k_j}. $$

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