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ON SOME FORMULAS OF SPHERICAL GEOMETRY

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Abstract. The article is intended the present the connection of spherical geometry with Euclidean geometry and extend basic theorems of Euclidean geometry to the spherical case.

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Spherical geometry allows us to work on the sphere surface. The article is intended to present the capability and its connection with Euclidean geometry. The first step is to start working on the sphere where the rule for measuring the angle and naming line segment to curves, which is attained by conducting the plane on the center of the sphere and its intersection with the sphere surface. This agreement has enabled us to measure the area of a spherical polygon and partly to explore the spherical triangle. In examining the accuracy of formulas plays an important role in Euclidean geometry, since each very small area on the surface of the sphere is close to the plane and we get classical formulas. In the article a sphere with a radius of one is considered, for all spheres with different radiuses are similar.

In the Euclidean and spherical geometries the triangle has certain advantages over other polygons. If we consider two geometrical properties - on the three points we can always draw the plane and each section on the sphere gives us the circle, we get that every spherical triangle can be inscribed in a circle. By connecting the three points with the Euclidean distance we also get a flat triangle inscribed in the plane and so inscribed in the circle. We name this process as the projection of a spherical polygons on the plane. We will use the polygon (in terms of spherical polygon) and the flat polygon (referring to the projection on the plane of a spherical polygon).

Theorem 1. *If we have in the given circle two chords α, β, which are connected in one point making an angle $\mathcal{A}$, and the arc length (in radians) between chords is ω, we have the formula (by the arrow is shown an analog in Euclidean geometry)*:

$$\cos\left(\frac{\omega}{2}\right) = \sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) + \cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\cos\mathcal{A} \longrightarrow \omega = 2\mathcal{A}$$

Proof.

$$\cos\left(\frac{\omega}{2}\right) = \frac{\left(2\sin\left(\frac{\alpha}{2}\right)\right)^2 + \left(2\sin\left(\frac{\beta}{2}\right)\right)^2 - \left(2\sin\left(\frac{\gamma}{2}\right)\right)^2}{2\left(2\sin\left(\frac{\alpha}{2}\right)\right)\left(2\sin\left(\frac{\beta}{2}\right)\right)}$$

$$= \frac{2\sin^2\left(\frac{\alpha}{2}\right) - \cos\beta + \cos\gamma}{4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)} = \frac{2\sin^2\left(\frac{\alpha}{2}\right) - \cos\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\mathcal{A}}{4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)}$$

$$= \sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) + \cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\cos\mathcal{A}$$

Theorem 2. *If we have triangles with sides - α, β, γ, and angles - $\mathcal{A}, \mathcal{B}, \mathcal{C}$, the law of sines is given by the formula:*

$$\frac{\sin\alpha}{\sin\mathcal{A}} = \frac{\sin\beta}{\sin\mathcal{B}} = \frac{\sin\gamma}{\sin\mathcal{C}} = 2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\tan\zeta \longrightarrow \frac{\alpha}{\sin\mathcal{A}}$$

$$= \frac{\beta}{\sin\mathcal{B}} = \frac{\gamma}{\sin\mathcal{C}} = 2\zeta,$$

where 2ζ - is the longest line segment in the circle (diameter).

For proof we use theorem 1 and the flat triangle:

$$\cos\zeta = \sqrt{1 - \sin^2\zeta} = \sqrt{1 - \left(\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right)^2}$$

$$= \sqrt{1 - \frac{1 - \cos\alpha}{1 - \cos\omega}} = \sqrt{1 - \frac{1 - \cos\alpha}{1 - \left(\cos\alpha - 2\left(\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\sin\mathcal{A}\right)^2\right)}}$$

$$= \sqrt{\frac{2\left(\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\sin\mathcal{A}\right)^2}{1 - \cos\omega}} = \frac{\sin\mathcal{A}}{\sin\left(\frac{\omega}{2}\right)}\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right) \longrightarrow \frac{\sin\alpha}{\sin\mathcal{A}}$$

$$= 2 \cdot \frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \cdot \cos\left(\frac{\alpha}{2}\right) \cdot \frac{\sin\left(\frac{\omega}{2}\right)}{\sin\mathcal{A}} = 2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\tan\zeta$$

Theorem 3. *If we have a triangles with sides - α, β, γ, and the radius of circle - ζ, we have the formula:*

$$\tan\zeta = \frac{2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)}{\sqrt{\sin p \sin(p-\alpha)\sin(p-\beta)\sin(p-\gamma)}} \longrightarrow \zeta$$

$$\frac{\alpha\beta\gamma}{4\sqrt{p(p-\alpha)(p-\beta)(p-\gamma)}} = \frac{\alpha\beta\gamma}{4\mathcal{S}},$$

where $p = \frac{\alpha+\beta+\gamma}{2}$.

Proof. The formula is based on the law of sines and law of cosines.

Theorem 4. *If we know two sides of triangle and angle between them, we have the formula (The second form is written using trigonometric conversion):*

$$\tan\left(\frac{\mathcal{S}}{2}\right) = \frac{\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\sin\mathcal{C}}{1+\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\cos\mathcal{C}} = \frac{2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\tan\left(\frac{\mathcal{C}}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)\tan^2\left(\frac{\mathcal{C}}{2}\right)}$$

Proof. We use the L'Huilier's Theorem and Law of sines.

$$\tan\left(\frac{\mathcal{S}}{4}\right) = \sqrt{\tan\left(\frac{p}{2}\right)\tan\left(\frac{p-\alpha}{2}\right)\tan\left(\frac{p-\beta}{2}\right)\tan\left(\frac{p-\gamma}{2}\right)} \longrightarrow \tan\left(\frac{S}{2}\right)$$

$$= \frac{\sqrt{\sin(p)\sin(p-\alpha)\sin(p-\beta)\sin(p-\gamma)}}{1+\cos\alpha+\cos\beta+\cos\gamma}\tan\left(\frac{\mathcal{S}}{2}\right)$$

$$= \frac{\sqrt{\sin(p)\sin(p-\alpha)\sin(p-\beta)\sin(p-\gamma)}}{1+\cos\alpha+\cos\beta+\cos\gamma}$$

$$= \frac{4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\cot\zeta}{1+\cos\alpha+\cos\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\mathcal{C}}$$

$$\frac{\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\sin\mathcal{C}}{1+\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\cos\mathcal{C}}$$

Theorem 5. *Let us take a convex polygon inscribed in a circle, then we have a point which is an equal distance from polygons points, so we have isosceles triangles. Suppose that the polygon has n sides, then its area is a sum of the triangles' areas. The equal sides of isosceless triangles are radius of the circle, which will be denoted by the letter -* ζ*, and angles between them as* $\omega_1, \omega_2, \ldots, \omega_n$ $(\omega_1+\omega_2+\ldots+\omega_n = 2\pi)$*:*

$$\frac{\mathcal{S}}{2} = \pi - \sum_{i=1}^{n}\arctan\left(\tan\left(\frac{\omega_i}{2}\right)\cos\zeta\right)$$

Proof. We need to use theorem 4 (second form) and denot the triangle's area as $\mathcal{S}_\rangle$:

$$\tan\left(\frac{\mathcal{S}_\rangle}{2}\right) = \frac{2\sin\left(\frac{\zeta}{2}\right)\sin\left(\frac{\zeta}{2}\right)\tan\left(\frac{\omega_i}{2}\right)}{\cos\left(\frac{\zeta-\zeta}{2}\right)+\cos\left(\frac{\zeta+\zeta}{2}\right)\tan^2\left(\frac{\omega_i}{2}\right)}$$

$$= \frac{\tan\left(\frac{\omega_i}{2}\right)-\tan\left(\frac{\omega_i}{2}\right)\cos\zeta}{1+\tan\left(\frac{\omega_i}{2}\right)\left(\tan\left(\frac{\omega_i}{2}\right)\cos\zeta\right)}$$

$$= \tan\left(\frac{\omega_i}{2} - \arctan\left(\tan\left(\frac{\omega_i}{2}\right)\cos\zeta\right)\right)\frac{\mathcal{S}}{2}$$

$$\frac{\omega_i}{2} - \arctan\left(\tan\left(\frac{\omega_i}{2}\right)\cos\zeta\right)\frac{\mathcal{S}}{2} = \sum_{i=1}^{n}\frac{\mathcal{S}_\rangle}{2}$$

$$= \sum_{i=1}^{n}\left(\frac{\omega_i}{2} - \arctan\left(\tan\left(\frac{\omega_i}{2}\right)\cos\zeta\right)\right) = \pi - \sum_{i=1}^{n}\arctan\left(\tan\left(\frac{\omega_i}{2}\right)\cos\zeta\right)$$

Note. If we use the identity $\sin\left(\frac{\omega_i}{2}\right) = \frac{\sin\left(\frac{\alpha_i}{2}\right)}{\sin\zeta}$ (α_i is the side of the polygon) we can rewrite the formula:

$$\arctan\left(\tan\left(\frac{\omega_i}{2}\right)\cos\zeta\right) = \arcsin\left(\frac{\tan\left(\frac{\alpha_i}{2}\right)}{\tan\zeta}\right) \longrightarrow \frac{\mathcal{S}}{2}$$

$$= \pi - \sum_{i=1}^{n} \arcsin\left(\frac{\tan\left(\frac{\alpha_i}{2}\right)}{\tan\zeta}\right).$$

Theorem 6. *Let us take a convex rectangle inscribed in a circle. Then we can calculate by formula, which analog in Euclidean geometry is formula of Brahmagupta.*

$$\tan\left(\frac{\mathcal{S}}{4}\right) = \sqrt{\frac{\sin\left(\frac{p-\alpha}{2}\right)\sin\left(\frac{p-\beta}{2}\right)\sin\left(\frac{p-\gamma}{2}\right)\sin\left(\frac{p-\delta}{2}\right)}{\cos\left(\frac{p}{2}\right)\cos\left(\frac{p-\alpha-\beta}{2}\right)\cos\left(\frac{p-\alpha-\gamma}{2}\right)\cos\left(\frac{p-\alpha-\delta}{2}\right)}}$$

Proof. The formula is obtaned by using Theorem 1 and $\mathcal{S} = \mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}-2\pi$

Theorem 7. *If we take two points on the sphere and connect them with both methods, then draw a line from the centre of the sphere towards the flat line, so it divides it with a certain ratio and cuts the spherical line into two pieces, then the sines of spherical lines will be divided with the same ratio.*

Proof. Flat line - $a = a_1 + a_2$, spherical line - $\alpha = \alpha_1 + \alpha_2$

$$\frac{a_2}{\sin\alpha_2}\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \frac{a_1}{\sin\alpha_1}\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \longrightarrow \frac{\sin\alpha_2}{\sin\alpha_1} = \frac{a_2}{a_1}$$

Theorem 8. *Let us take a triangle and draw a line from a point in the direction of a side, so that the sines of line divides with a certain ratio - λ. If we know sides of the triangle - α, β, γ, and the - λ, then we can calculate the lenght from the point towards side - ξ (ξ and angle $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$* are between β, γ).

$$\cos\xi = \frac{\cos\beta + \lambda\cos\gamma}{\sqrt{1+\lambda^2+2\lambda\cos\alpha}}$$

Proof. We use theorem 7 and law of cosines:

$$\cos^2\xi = 1 - \sin^2\xi = 1 - \left(\frac{\sin\alpha_2}{\sin\mathcal{A}_2}\sin\mathcal{C}\right)\left(\frac{\sin\alpha_1}{\sin\mathcal{A}_1}\sin\mathcal{B}\right)$$

$$= 1 - \frac{1}{\sin\mathcal{A}_1\sin\mathcal{A}_2}\sqrt{\left(\frac{1}{1+\cot^2\alpha_1}\right)\left(\frac{1}{1+\cot^2\alpha_2}\right)}\left(\frac{\sin\mathcal{A}}{\sin\alpha}\sin\beta\right)\left(\frac{\sin\mathcal{A}}{\sin\alpha}\sin\gamma\right)$$

$$1 - \frac{\lambda\sin\beta\sin\gamma}{1+\lambda^2+2\lambda\cos\alpha}\cdot\frac{(\sin\mathcal{A}_1\cos\mathcal{A}_2+\cos\mathcal{A}_1\sin\mathcal{A}_2)^2}{\sin\mathcal{A}_1\sin\mathcal{A}_2}$$

$$= 1 - \frac{\lambda\sin\beta\sin\gamma}{1+\lambda^2+2\lambda\cos\alpha}\left(\frac{\sin\mathcal{A}_1}{\sin\mathcal{A}_2} + \frac{\sin\mathcal{A}_2}{\sin\mathcal{A}_1} + 2\cos\mathcal{A}\right)$$

$$1 - \frac{\lambda\sin\beta\sin\gamma}{1+\lambda^2+2\lambda\cos\alpha}\left(\frac{\sin\beta}{\lambda\sin\gamma} + \frac{\lambda\sin\gamma}{\sin\beta} + \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \frac{(\cos\beta+\lambda\cos\gamma)^2}{1+\lambda^2+2\lambda\cos\alpha}$$

Note. Let us denote the angle between ξ, α as $\mathcal{D}$. Then

$$1)\frac{\sin\mathcal{A}_2}{\sin\mathcal{A}_1} = \frac{\sin\alpha_2}{\sin\alpha_1}\left(\frac{\sin\alpha_1}{\sin\mathcal{A}_1}\right)\left(\frac{\sin\mathcal{A}_2}{\sin\alpha_2}\right)$$

$$\lambda\left(\frac{\sin\gamma}{\sin\mathcal{D}}\right)\left(\frac{\sin\mathcal{D}}{\sin\beta}\right)=\lambda\frac{\sin\gamma}{\sin\beta}$$

$$2)\frac{\sin\alpha_2}{\sin\alpha_1}=\lambda\longrightarrow\sin\alpha_2=\lambda\sin\alpha_1=\lambda\sin(\alpha-\alpha_2)$$

$$\lambda(\sin\alpha\cos\alpha_2-\cos\alpha\sin\alpha_2)\longrightarrow\tan\alpha_2=\frac{\lambda\sin\alpha}{1+\lambda\cos\alpha}$$

$$3)\frac{\sin\alpha_2}{\sin\alpha_1}=\lambda\longrightarrow\lambda\sin\alpha_1=\sin\alpha_2=\sin(\alpha-\alpha_1)$$

$$\sin\alpha\cos\alpha_1-\cos\alpha\sin\alpha_1\longrightarrow\tan\alpha_1$$

$$=\frac{\sin\alpha}{\lambda+\cos\alpha}$$

Using the formulas given in this paper, some calculation on the sphere (see [1],[2]) can be simplified.

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