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ON ONE BOUNDARY VALUE PROBLEMS FOR A CIRCULAR RING WITH DOUBLE POROSITY

Gulua B.

Abstract. In this paper plane problems of elasticity for a circular ring with double porosity is considered. The solutions are represented by means of three analytic functions of a complex variable and one solution of the Helmholtz equation. The problems are solved when the components of the displacement vector is known on the boundary of the circular ring.

Keywords and phrases: Double porosity, a circular ring.

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1. Introduction

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary.

The physical and mathematical foundations of the theory of double porosity were considered in [1]-[3]. In part I of a series of papers on the subject, R. K. Wilson and E. C. Aifantis [2] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of this series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [3] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [4] provided a related finite element to consider the numerical solution of Aifantis equations of double porosity (see [2],[3],[4] and the references cited therein). The basic results and the historical information on the theory of porous media were summarized by Boer [5].

In [6] the full coupled linear theory of elasticity for solids with double porosity is considered. Four spatial cases of the dynamical equations are considered. The fundamental solutions are constructed by means of elementary functions and the basic properties of the fundamental solutions are established. The fundamental solution of quasi-static equations of the linear theory elasticity for double porosity solids is constructed and basic properties are established in [7]. In [8-11] the explicit solutions of the problems of porous elastostatics for an elastic circle and for the plane with a circular hole are constructed, the uniqueness theorems for regular solutions are proved and the numerical results are given for boundary value problems. Explicit solutions of the BVPs of the theory of consolidation with double porosity for the half-plane and half-space are considered in [12-15].

2. The plane deformation. Basic equations

Let V be a bounded domain in the Euclidean two-dimensional space E^2 bounded by the contour S. Suppose that $S \in C^{1,\beta}$, $0 < \beta \leq 1$. Let $x = (x_1, x_2)$ be the points of space E^2, $\partial_i = \frac{\partial}{\partial x_i}$. Let us assume that the domain V is filled with an isotropic double porosity material.

The system of homogeneous equations in the full coupled linear equilibrium theory of elasticity for materials with double porosity can be written as follows [6,15]

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \tag{1}$$

$$\begin{cases} (k_1\Delta - \gamma)p_1 + (k_{12}\Delta + \gamma)p_2 = 0, \\ (k_{21}\Delta - \gamma)p_1 + (k_2\Delta + \gamma)p_2 = 0, \end{cases} \tag{2}$$

where $\mathbf{u} = \mathbf{u}(u_1, u_2)^T$ is the displacement vector in a solid, p_α $(\alpha = 1, 2)$ are the pore and fissure fluid pressures respectively, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, λ and μ are the Lamé parameters, $k_\alpha = \dfrac{\kappa_\alpha}{\mu'}$, $k_{12} = \dfrac{\kappa_{12}}{\mu'}$, $k_{21} = \dfrac{\kappa_{21}}{\mu'}$. μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, Δ is the 2D Laplace operator.

On the plane x_1x_2, we introduce the complex variable $z = x_1 + ix_2 = re^{i\vartheta}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2)$, $\partial_{\bar{z}} = 0.5(\partial_1 + i\partial_2)$, $\bar{z} = x_1 - ix_2$, and $\Delta = 4\partial_z\partial_{\bar{z}}$.

System (1) is written in the complex form

$$2\mu\partial_{\bar{z}}\partial_z u_+ + (\lambda+\mu)\partial_{\bar{z}}\theta - \partial_{\bar{z}}(\beta_1 p_1 + \beta_2 p_2) = 0, \quad (u_+ = u_1 + iu_2). \tag{3}$$

3. The general solution of system (2)-(3)

In this section, we construct the analogues of the Kolosov-Muskhelishvili formulas [16] for system (4).

From system (3) we easily obtain the expressions for the pressures p_1 and p_2

$$p_1 = f'(z) + \overline{f'(z)} + (k_2 + k_{12})\eta(z,\bar{z}),$$

$$p_2 = f'(z) + \overline{f'(z)} - (k_1 + k_{21})\eta(z,\bar{z}),$$

where $f(z)$ is an arbitrary analytic function of a complex variable z in the domain V and $\eta(z,\bar{z})$ is an arbitrary solution of the Helmholtz equation

$$4\partial_z\partial_{\bar{z}}\eta-\zeta^2\eta=0,\quad \zeta^2=\frac{\gamma(k_1+k_2+k_{12}+k_{21})}{k_1k_2-k_{12}k_{21}}.$$

Theorem. *The general solution of the system of equations (4) is represented as follows:*

$$2\mu u_+=\varkappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}+\frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}(f'(z)+\overline{f'(z)})+\delta\partial_{\bar{z}}\eta(z,\bar{z}),$$

where

$$\varkappa=\frac{\lambda+3\mu}{\lambda+\mu},\quad \delta=\frac{4\mu((k_2+k_{12})\beta_1-(k_1+k_{21})\beta_2)}{(\lambda+2\mu)\zeta^2},$$

$\varphi(z)$ and $\psi(z)$ are arbitrary analytic functions of a complex variable z in the domain V.

4. A problem for a circular ring

In this section, we solve a concrete boundary value problem for a concentric circular ring with radius R_1 and R_2 (see fig. 1). On the boundary of the considered domain the values of pressures p_1 and p_2 and the displacement vector are given.

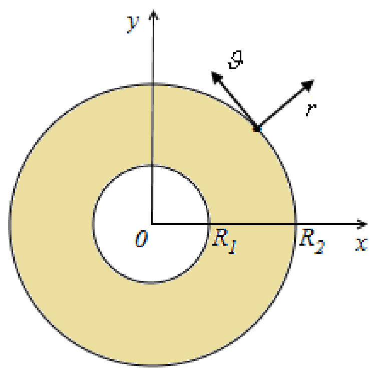

Fig. 1.

We consider the following problem

$$p_1=\begin{cases}\sum\limits_{-\infty}^{+\infty}A'_ne^{in\vartheta}, & |z|=R_1,\\ \sum\limits_{-\infty}^{+\infty}A''_ne^{in\vartheta}, & |z|=R_2,\end{cases}\qquad p_2=\begin{cases}\sum\limits_{-\infty}^{+\infty}B'_ne^{in\vartheta}, & |z|=R_1,\\ \sum\limits_{-\infty}^{+\infty}B''_ne^{in\vartheta}, & |z|=R_2,\end{cases}\tag{4}$$

$$u_+=\begin{cases}\sum\limits_{-\infty}^{+\infty}C'_ne^{in\vartheta}, & |z|=R_1,\\ \sum\limits_{-\infty}^{+\infty}C''_ne^{in\vartheta}, & |z|=R_2.\end{cases}\tag{5}$$

The analytic function $f(z)$ and the metaharmonic function $\eta(z,\bar{z})$ is represented as a series

$$f(z)=\gamma\ln z+\sum_{-\infty}^{+\infty}a_nz^n,\quad \eta(z,\bar{z})=\sum_{-\infty}^{+\infty}(\alpha_nI_n(r\zeta)+\beta_nK_n(r\zeta))e^{in\vartheta},\qquad(6)$$

where $I_n(r\zeta)$ and $K_n(r\zeta)$ are modified Bessel function of n-th order, $z=re^{i\vartheta}$, and are substituted in the boundary conditions (5) we have

$$\begin{aligned}
&(\gamma+\bar{\gamma})\ln R_1+(\gamma-\bar{\gamma})i\vartheta+\sum_{-\infty}^{+\infty}R_1^n\left(a_ne^{in\vartheta}+\bar{a}_ne^{-in\vartheta}\right)\\
&+(k_2+k_{12})\sum_{-\infty}^{+\infty}(\alpha_nI_n(R_1\zeta)+\beta_nK_n(R_1\zeta))e^{in\vartheta}=\sum_{-\infty}^{+\infty}A'_ne^{in\vartheta},\\
&(\gamma+\bar{\gamma})\ln R_2+(\gamma-\bar{\gamma})i\vartheta+\sum_{-\infty}^{+\infty}R_2^n\left(a_ne^{in\vartheta}+\bar{a}_ne^{-in\vartheta}\right)\\
&+(k_2+k_{12})\sum_{-\infty}^{+\infty}(\alpha_nI_n(R_2\zeta)+\beta_nK_n(R_2\zeta))e^{in\vartheta}=\sum_{-\infty}^{+\infty}A''_ne^{in\vartheta},\\
&(\gamma+\bar{\gamma})\ln R_1+(\gamma-\bar{\gamma})i\vartheta+\sum_{-\infty}^{+\infty}R_1^n\left(a_ne^{in\vartheta}+\bar{a}_ne^{-in\vartheta}\right)\\
&-(k_1+k_{21})\sum_{-\infty}^{+\infty}(\alpha_nI_n(R_1\zeta)+\beta_nK_n(R_1\zeta))e^{in\vartheta}=\sum_{-\infty}^{+\infty}B'_ne^{in\vartheta},\\
&(\gamma+\bar{\gamma})\ln R_2+(\gamma-\bar{\gamma})i\vartheta+\sum_{-\infty}^{+\infty}R_2^n\left(a_ne^{in\vartheta}+\bar{a}_ne^{-in\vartheta}\right)\\
&-(k_1+k_{21})\sum_{-\infty}^{+\infty}(\alpha_nI_n(R_2\zeta)+\beta_nK_n(R_2\zeta))e^{in\vartheta}=\sum_{-\infty}^{+\infty}B''_ne^{in\vartheta}.
\end{aligned}\qquad(7)$$

From the condition of displacement uniqueness it follows that $\gamma-\bar{\gamma}=0$. It is also assumed that a_0 is a real value; that is, $a_0=\bar{a}_0$.

Compare the coefficients at identical degrees. We obtain the following system of equations

$$\begin{aligned}
&2\gamma\ln R_1+2a_0+(k_2+k_{12})(\alpha_0I_0(R_1\zeta)+\beta_0K_0(R_1\zeta))=A'_0,\\
&2\gamma\ln R_2+2a_0+(k_2+k_{12})(\alpha_0I_0(R_2\zeta)+\beta_0K_0(R_2\zeta))=A''_0,\\
&2\gamma\ln R_1+2a_0-(k_1+k_{21})(\alpha_0I_0(R_1\zeta)+\beta_0K_0(R_1\zeta))=B'_0,\\
&2\gamma\ln R_2+2a_0-(k_1+k_{21})(\alpha_0I_0(R_2\zeta)+\beta_0K_0(R_2\zeta))=B''_0,
\end{aligned}\qquad(8)$$

$$\begin{aligned}
&R_1^na_n+R_1^{-n}\bar{a}_{-n}+(k_2+k_{12})(\alpha_nI_n(R_1\zeta)+\beta_nK_n(R_1\zeta))=A'_n,\\
&R_2^na_n+R_2^{-n}\bar{a}_{-n}+(k_2+k_{12})(\alpha_nI_n(R_2\zeta)+\beta_nK_n(R_2\zeta))=A''_n,\\
&R_1^na_n+R_1^{-n}\bar{a}_{-n}-(k_1+k_{21})(\alpha_nI_n(R_1\zeta)+\beta_nK_n(R_1\zeta))=B'_n,\\
&R_2^na_n+R_2^{-n}\bar{a}_{-n}-(k_1+k_{21})(\alpha_nI_n(R_2\zeta)+\beta_nK_n(R_2\zeta))=B''_n.
\end{aligned}\qquad(9)$$

From (8) and (9)

$$\alpha_n = \frac{(A'_n - B'_n)K_n(R_2\zeta) - (A''_n - B''_n)K_n(R_1\zeta)}{(k_1+k_2+k_{12}+k_{21})(I_n(R_1\zeta)K_n(R_2\zeta) - I_n(R_2\zeta)K_n(R_1\zeta))},$$

$$\beta_n = \frac{(A''_n - B''_n)I_n(R_2\zeta) - (A'_n - B'_n)I_n(R_1\zeta)}{(k_1+k_2+k_{12}+k_{21})(I_n(R_1\zeta)K_n(R_2\zeta) - I_n(R_2\zeta)K_n(R_1\zeta))},$$

$$\gamma = \frac{\tilde{A}'_0 - \tilde{A}''_0}{\ln R_1/R_2}, \quad a_0 = \frac{\tilde{A}'_0 \ln R_2 - \tilde{A}''_0 \ln R_1}{2(\ln R_2 - \ln R_1)}, \quad a_n = \frac{R_2^n \tilde{A}''_n - R_1^n \tilde{A}'_n}{R_2^{2n} - R_1^{2n}},$$

where $\tilde{A}'_n = A'_n - (k_2 + k_{12})(\alpha_n I_n(R_1\zeta) + \beta_n K_n(R_1\zeta))$, $\tilde{A}''_n = A''_n - (k_2 + k_{12})(\alpha_n I_n(R_2\zeta) + \beta_n K_n(R_2\zeta))$.

The analytic functions $\varphi(z)$ and $\psi(z)$ are represented as the series

$$\varphi(z) = \gamma_1 \ln z + \sum_{-\infty}^{\infty} b_n z^n, \quad \psi(z) = \gamma_2 \ln z + \sum_{-\infty}^{\infty} c_n z^n,$$

and are substituted in the boundary conditions (7), we have

$$(\varkappa\alpha - \bar{\beta})\ln r + (\varkappa\alpha + \bar{\beta})i\vartheta + \sum_{-\infty}^{\infty}(\varkappa b_n r^n e^{in\vartheta} - n\bar{b}_n r^n e^{-i(n-2)\vartheta} - \bar{c}_n r^n e^{-in\vartheta})$$

$$+\frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}\left(2\gamma \ln r + \sum_{-\infty}^{\infty}(a_n r^n e^{in\vartheta} + \bar{a}_n r^n e^{-in\vartheta})\right)$$

$$+\frac{\delta\zeta}{2}\sum_{-\infty}^{+\infty}(\alpha_n I_{n+1}(r\zeta) + \beta_n K_{n+1}(r\zeta))e^{i(n+1)\vartheta} = \begin{cases} \sum\limits_{-\infty}^{+\infty} C'_n e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} C''_n e^{in\vartheta}, & |z| = R_2. \end{cases}$$

From the condition of displacement uniqueness it follows that

$$\varkappa\alpha + \bar{\beta}. \tag{10}$$

Compare the coefficients at identical degrees. We obtain the following system of equations

$$\begin{cases} 2\varkappa \ln R_1 \gamma_1 - 2R_1^2 \bar{b}_2 + \varkappa b_0 - \bar{c}_0 = D'_0, \\ 2\varkappa \ln R_2 \gamma_1 - 2R_2^2 \bar{b}_2 + \varkappa b_0 - \bar{c}_0 = D''_0, \end{cases} \tag{11}$$

$$\begin{cases} -\bar{\gamma}_1 + \varkappa \ln R_1^2 b_2 - R_1^{-2}\bar{c}_{-2} = D'_2, \\ -\bar{\gamma}_1 + \varkappa \ln R_2^2 b_2 - R_2^{-2}\bar{c}_{-2} = D''_2, \end{cases} \tag{12}$$

$$\begin{cases} \varkappa R_1^n b_n + (n-2)R_1^{-n+2}\bar{b}_{-n+2} - R_1^{-n}\bar{c}_{-n} = D'_n, \\ \varkappa R_2^n b_n + (n-2)R_2^{-n+2}\bar{b}_{-n+2} - R_2^{-n}\bar{c}_{-n} = D''_n, \\ (n = \pm 1,\ -2,\ \pm 3, \ldots), \end{cases} \tag{13}$$

where

$$D_0' = C_0' - \frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}(2\gamma\ln R_1 + 2a_0) - \frac{\delta\zeta}{2}(\alpha_{-1}I_0(R_1\zeta) + \beta_{-1}K_0(R_1\zeta)),$$

$$D_0'' = C_0'' - \frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}(2\gamma\ln R_2 + 2a_0) - \frac{\delta\zeta}{2}(\alpha_{-1}I_0(R_2\zeta) + \beta_{-1}K_0(R_2\zeta)),$$

$$D_n' = C_n' - \frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}(R_1^n a_n + R_1^{-n}\bar{a}_{-n}) - \frac{\delta\zeta}{2}(\alpha_{n-1}I_n(R_1\zeta) + \beta_{n-1}K_n(R_1\zeta)),$$

$$D_n'' = C_n'' - \frac{\mu(\beta_1+\beta_2)}{\lambda+2\mu}(R_2^n a_n + R_2^{-n}\bar{a}_{-n}) - \frac{\delta\zeta}{2}(\alpha_{n-1}I_n(R_2\zeta) + \beta_{n-1}K_n(R_2\zeta)).$$

From (11) and (12) we have

$$b_2 = \frac{R_1^2 D_2' - R_2^2 D_2'' + \frac{R_1^2 - R_2^2}{2\varkappa \ln R_1/R_2 (\bar{D}_0' - \bar{D}_0'')}}{\varkappa(R_1^4 - R_2^4) - \frac{(R_1^2-R_2^2)^2}{\varkappa \ln R_1/R_2}}.$$

γ_1, γ_2, $\varkappa b_0 - c_0$ and c_{-2} are considered by (11) and (12).

From (13)

$$b_n = \frac{\varkappa(R_2^{2(2-n)} - R_1^{2(2-n)})E_n - (n-2)(R_2^2 - R_1^2)E_{2-n}}{\varkappa^2(R_2^4 + R_1^4 - R_2^{2n}R_1^{2(2-n)} - R_1^{2n}R_2^{2(2-n)}) + n(n-2)(R_2^2 - R_1^2)^2},$$

$$c_n = \varkappa R_1^{-2n}\bar{b}_{-n} + (n+2)R_1^{2n+2}b_{n+2} - R_1^n D_n',$$

$$(n = \pm 1,\ \pm 2,\ \pm 3, ...)$$

where $E_n = R_2^n D_n'' - R_1^n D_n'$.

It is easy to prove the absolute and uniform convergence of the series obtained in the circular ring (including the contours) when the functions set on the boundaries have sufficient smoothness.

Similarly the problem can be solved when on the boundary of the considered domain the values of stresses are given.

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REFERENCES

1. Cowin S.C. Bone poroelasticity. *Journal of Biomechanics,* **32** (1999), 217-238.

2. Wilson R.K., Aifantis E.C. On the theory of consolidation with double porosity-I. *International Journal of Engineering Science,* **20**, 9 (1982),1009-1035.

3. Beskos D.E., Aifantis E.C. On the theory of consolidation with double porosity-II. *International Journal of Engineering Science,* **24** (1986), 1697-1716.

4. Khaled M.Y., Beskos D.E., Aifantis E.C. On the theory of consolidation with double porosity-III. *International Journal for Numerical and Analytical Methods in Geomechanics,* **8**, 2 (1984), 101-123.

5. De Boer R. Theory of Porous Media: Highlights in the historical development and current state. *Springer, Berlin-Heidelberg- New York,* 2000.

6. Svanadze M., De Cicco S. Fundamental solutions in the full coupled theory of elasticity for solids with double porosity. *Arch. Mech.,* **65**, 5 (2013), 367-390.

7. Svanadze M. Fundamental solution in the theory of consolidation with double porosity. *J. Mech. Behavior of Materials,* **16** (2005), 123-130.

8. Tsagareli I., Svanadze M.M., Explicit solution of the boundary value problems of the theory of elasticity for solids with double porosity. *PAMM -Proc. Appl. Math. Mech.,* **10** (2010), 337-338.

9. Tsagareli I., Svanadze M. M, Explicit solution of the problems of elastostatics for an elastic circle with double porosity. *Mechanics Research Communications,* **46** (2012), 76-80.

10. Tsagareli I. The solution of the first boundary value problem of elastostatics for double porous plane with a circular hole. *Bulletin of TICMI*, **15** (2011), 1-4.

11. Tsagareli I., Svanadze M.M. The solution of the stress boundary value problem of elastostatics for double porous plane with a circular hole. *Reports of Enlarged sess. of Sem. of Appl. Math.*, **24**, (2010), 130–133.

12. Basheleishvili M., Bitsadze L., Explicit solution of the BVP of the theory of consolidation with double porosity for half-plane. *Georgian Mathematical Journal,* **19**, 1 (2012), 41-49.

13. Basheleishvili M., Bitsadze L., Explicit solutions of the BVPs of the theory of consolidation with double porosity for the half-space. *Bulletin of TICMI,* **14** (2010), 9-15.

14. Tsagareli I., Bitsadze L. The boundary value problems in the full coupled theory of elasticity for plane with double porosity with a circular hole. *Semin. I. Vekua Inst. Appl. Math. Rep.* **40** (2014), 68-79.

15. Janjgava R. Elastic equilibrium of porous Cosserat media with double porosity. *Adv. Math. Phys.*, (2016), Article ID 4792148, 9 pages http://dx.doi.org/10.1155/2016/4792148.

16. Muskhelishvili N.I. Some Basic Problems of the Mathematical Theory of Elasticity. Fundamental Equations, Plane Theory of Elasticity, Torsion and Bending. *Noordhoff International Publishing, Leiden*, 1977.

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Author's addresses:

B. Gulua
I. Javakhishvili Tbilisi State University
I. Vekua Institute of Applied Mathematics
2, University str., Tbilisi 0186
Georgia

Sokhumi State University
9, Anna Politkovskaia str., Tbilisi 0186
Georgia
E-mail: bak.gulua@gmail.com