Proceedings of I. Vekua Institute of Applied Mathematics
Vol. 67, 2017

ON SOME SOLUTIONS IN THE THEORY OF THERMOELASTICITY FOR SOLIDS WITH DOUBLE POROSITY

Bitsadze L.

Abstract. In this paper the 2D linear theory of thermoelasticity for materials with double porosity is considered. There the fundamental and singular matrices of solutions are constructed in terms of elementary functions. The single and double layer potentials are obtained. Finally the basic properties of these potentials are established.

Keywords and phrases: Double porosity, fundamental solution, thermoelasticity.

AMS subject classification (2010): 74F10, 35E05.

1. Introduction

In recent years the concept of porous media has been used in many areas of applied science (e.g., biology, biophysics, biomechanics, geomechanics, the petroleum industry, chemical engineering, soil mechanics and engineering). The theory of thermoelasticity for double porosity materials combines the theory of heat conduction with poroelastic constitutive equations, coupling the temperature field with the stresses and the pore and fissure fluid pressure.

The theory of consolidation with double porosity was first proposed by Aifantis and co-authors in [1-3]. This theory unifies a model proposed by Biot [4] for the consolidation of deformable single porosity media with a model proposed by Barenblatt [5] for seepage in undeformable media with two degrees of porosity (see [1],[2],[3] and the references cited therein.) The basic results on the theory of porous media and the historical development of porous media theory may be found in [6],[7].

Great attention has been paid to the theories of poroelasticity taking into account the thermal effect. The basic equations of the thermo-hydro-mechanical coupling theories for elastic materials with double porosity were presented in [8,9,10,11.]

In [12] the fundamental solution for the system of steady vibrations and equilibrium equations are constructed by means of elementary functions.

The phenomenological equations of the quasi-static theory for double porous media are established in [13,14] where a method to calculate the relevant coefficients is also presented.

The problem of elastic bodies with double porosity was the subject of study for some papers more than fifty years ago. Many authors have investigated the BVPs of the theory of elasticity for materials with double porosity, that are published in a large number of papers (some of these results can be seen in [15-28] and references therein).

In this paper the 2D linear theory of thermoelasticity for materials with double porosity is considered. There the fundamental and singular matrices of solutions are constructed in terms of elementary functions. The single and double layer potentials are obtained. Finally the basic properties of these potentials are established.

2. The basic fundamental matrix

Let $\mathbf{x} = (x_1, x_2)$ be a point of the Euclidean 2D space R^2. Let D^+ be a bounded 2D domain (surrounded by the curve S) and let D^- be the complement of $D^+ \cup S$. $\partial_{\mathbf{x}} = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right)$. Let us assume that the domain D^+ is filled with an isotropic material with double porosity.

The system of homogeneous equations in the 2D linear equilibrium theory of thermoelasticity for solids with double porosity can be written as follows [8,9]

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\text{grad div}\mathbf{u} - \text{grad}(\beta_1 p_1 + \beta_2 p_2 + \gamma_0\theta) = 0,$$

$$(k_1\Delta - \gamma)p_1 + \gamma p_2 = 0, \quad \gamma p_1 + (k_2\Delta - \gamma)p_2 = 0, \quad k\Delta\theta = 0, \qquad (1)$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement vector in a solid, p_1 and p_2 are the pore and fissure fluid pressures respectively. θ is a temperature, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, λ, μ, k, k_1, k_2, γ_0 are all constitutive coefficients, Δ is the Laplacian operator. The superscript "T" denotes transposition.

We introduce the following matrix differential operator

$$A(\partial_{\mathbf{x}}) = \| A_{lj}(\partial\mathbf{x}) \|_{5x5}, \quad l, j = 1,2,3,4,5,$$

where

$$A_{lj} := \delta_{lj}\mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j}, \quad l, j = 1, 2,$$

$$A_{j3} := -\beta_1\frac{\partial}{\partial x_j}, \quad A_{j4} := -\beta_2\frac{\partial}{\partial x_j}, \quad A_{j5} := -\gamma_0\frac{\partial}{\partial x_j}, \ \ j = 1, 2,$$

$$A_{3j} := 0, \quad j = 1,2 \quad A_{33} := k_1\Delta - \gamma, \quad A_{34} := \gamma, \quad A_{35} := 0,$$

$$A_{4j} := 0, \quad A_{43} := \gamma, \quad A_{44} := k_2\Delta - \gamma, \quad A_{45} := 0,$$

$$A_{5j} := 0, \quad j = 1,2,3,4, \quad A_{55} := k\Delta.$$

$\delta_{\alpha\gamma}$ is the Kronecker delta. Then system (1) can be rewritten as

$$\mathbf{A}(\partial_{\mathbf{x}})\mathbf{U} = 0, \tag{2}$$

where $\mathbf{U} = (u_1, u_2, p_1, p_2, \theta)^T$.

We assume that $\mu\mu_0 k k_1 k_2 \neq 0$, where $\mu_0 := \lambda + 2\mu$. Obviously, if the last condition is satisfied, then $\mathbf{A}(\partial_{\mathbf{x}})$ is the elliptic differential operator. We will throughout suppose that this assumption holds true.

In the sequel the fundamental matrix of operator $\mathbf{A}(\partial_{\mathbf{x}})$ will be constructed in terms of elementary functions. To this end we consider the system of the equation

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\text{grad div}\mathbf{u} = 0,$$

$$-\beta_1 \text{div}\mathbf{u} + (k_1\Delta - \gamma p_1 + \gamma p_2 = 0,$$

$$-\beta_2 \text{div}\mathbf{u} + \gamma p_1 + (k_2\Delta - \gamma p_2 = 0,$$

$$-\gamma_0 \text{div}\mathbf{u} + k\Delta\theta = 0.$$

The latter system may be written in the form

$$A^T(\partial_{\mathbf{x}})\mathbf{U} = 0, \tag{3}$$

where $A^T(\partial_{\mathbf{x}})$ is the transpose of matrix $A(\partial_{\mathbf{x}})$.

We introduce the matrix differential operator $\mathbf{B}(\partial\mathbf{x})$ consisting of cofactors of elements of the matrix $\mathbf{A}^T$ divided on $\mu\mu_0 k k_1 k_2$:

$$\mathbf{B}(\partial_{\mathbf{x}}) = \frac{1}{\mu\mu_0 k k_1 k_2} \parallel B_{lj}(\partial_{\mathbf{x}}) \parallel_{4x4}, \quad l, j = 1, 2, 3, 4,$$

where

$$B_{ij} = k k_1 k_2 [\mu_0 \delta_{ij}\Delta - (\lambda + \mu)\xi_i\xi_j]\Delta\Delta(\Delta - \lambda_1^2), \quad i, j = 1, 2,$$

$$B_{j3} = \mu k \Delta^2 [\beta_1 k_2 \Delta - \gamma(\beta_1 + \beta_2)]\xi_j, \quad j = 1, 2, \quad \xi_j = \frac{\partial}{\partial x_j},$$

$$B_{j4} = \mu k \Delta^2 [\beta_2 k_1 \Delta - \gamma(\beta_1 + \beta_2)]\xi_j, \quad j = 1, 2,$$

$$B_{j5} = \mu\gamma_0 k_1 k_2 \Delta^2 (\Delta - \lambda_1^2)\xi_j, \quad B_{3j} = B_{4j} = B_{5j} = 0, \quad j = 1, 2,$$

$$B_{35} = B_{45} = B_{54} = B_{53} = 0, \quad B_{55} = \mu\mu_0 k_1 k_2 \Delta^3 (\Delta - \lambda_1^2),$$

$$B_{33} = \mu\mu_0 k \Delta^3 (k_2\Delta - \gamma), \quad B_{34} = B_{43} = -\gamma\mu\mu_0 k \Delta^3,$$

$$B_{44} = \mu\mu_0 k \Delta^3 (k_1\Delta - \gamma).$$

Substituting the vector $\mathbf{U}(\mathbf{x})=\mathbf{B}(\partial_{\mathbf{x}})\mathbf{\Psi}$ into (2), where $\mathbf{\Psi}$ is a five-component vector function, we get

$$\Delta\Delta\Delta\Delta(\Delta-\lambda_1^2)\mathbf{\Psi}=0, \quad \lambda_1^2=\frac{\gamma(k_1+k_2)}{k_1k_2}.$$

From here, after some calculations, we obtain

$$\Delta\Delta\Psi=\frac{\varphi_1-\varphi}{\lambda_1^4}-\frac{\varphi_0}{\lambda_1^2}, \tag{4}$$

where

$$\varphi_0=\frac{r^2(\ln r-1)}{4}, \quad \varphi=\ln r, \quad \varphi_1=\frac{\pi}{2i}H_0^{(1)}(\lambda_1 r),$$

$H_0^{(1)}(\lambda_1 r)$ is Hankel's function of the first kind with the index 0

$$\begin{aligned} H_0^{(1)}(\lambda_1 r)&=\frac{2i}{\pi}J_0(\lambda_1 r)\ln r+\frac{2i}{\pi}\left(\ln\frac{\lambda_1}{2}+C-\frac{i\pi}{2}\right)J_0(\lambda_1 r)\\ &-\frac{2i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_1 r}{2}\right)^{2k}\left(\frac{1}{k}+\frac{1}{k-1}+...+1\right), \end{aligned} \tag{5}$$

$$J_0(\lambda_1 r)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_1 r}{2}\right)^{2k}, \quad r^2=(x_1-y_1)^2+(x_2-y_2)^2.$$

Substituting (4) into $\mathbf{U}=\mathbf{B}\mathbf{\Psi}$, we obtain the matrix of fundamental solutions for equation (2) which we denote by $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$

$$\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})=\|\ \Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y})\ \|_{5\times5} \quad l,j=1,2,3,4,5. \tag{6}$$

The elements Γ_{kj} have the following form

$$\Gamma_{kj}=\frac{\delta_{kj}}{\mu}\varphi-\frac{\lambda+\mu}{\mu\mu_0}\frac{\partial^2\varphi_0}{\partial x_k\partial x_j}, \quad k,j=1,2,$$

$$\Gamma_{j3}=\frac{1}{\mu_0k_1k_2}\frac{\partial}{\partial x_j}\left[\left(\beta_1k_2-\frac{\gamma(\beta_1+\beta_2)}{\lambda_1^2}\right)\frac{\varphi_1-\varphi}{\lambda_1^2}+\gamma(\beta_1+\beta_2)\frac{\varphi_0}{\lambda_1^2}\right],$$

$$\Gamma_{j4}=\frac{1}{\mu_0k_1k_2}\frac{\partial}{\partial x_j}\left[\left(\beta_2k_1-\frac{\gamma(\beta_1+\beta_2)}{\lambda_1^2}\right)\frac{\varphi_1-\varphi}{\lambda_1^2}+\gamma(\beta_1+\beta_2)\frac{\varphi_0}{\lambda_1^2}\right],$$

$$\Gamma_{j5}=\frac{\gamma_0}{\mu_0k}\frac{\partial\varphi_0}{\partial x_j}, \qquad \Gamma_{33}=\frac{\varphi_1}{k_1}-\frac{\gamma}{k_1k_2}\frac{\varphi_1-\varphi}{\lambda_1^2},$$

$$\Gamma_{34}=\Gamma_{43}=-\frac{\gamma}{k_1k_2}\frac{\varphi_1-\varphi}{\lambda_1^2}, \quad \Gamma_{44}=\frac{\varphi_1}{k_2}-\frac{\gamma}{k_1k_2}\frac{\varphi_1-\varphi}{\lambda_1^2},$$

$$\Gamma_{45}=\Gamma_{53}=\Gamma_{54}=0, \quad \Gamma_{55}=\frac{\varphi}{k}.$$

It is evident that all elements of $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$ have a logarithmic singularity at most. It can be shown that columns of the matrix $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$ are solutions to the system (1) with respect to $\mathbf{x}$ for any $\mathbf{x} \neq \mathbf{y}$. By applying the methods, as in the classical theory of elasticity, we can similarly prove the following;

Theorem 1. *The elements of the matrix* $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$ *has a logarithmic singularity as* $\mathbf{x} \to \mathbf{y}$ *and each column of the matrix* $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$, *considered as a vector, is a solution of the system* (1) *at every point* $\mathbf{x}$, *if* $\mathbf{x} \neq \mathbf{y}$.

Remark. The operator $\mathbf{A}(\partial_{\mathbf{x}})\mathbf{U}$ is not self adjoint. Obviously, it is possible to construct the fundamental solution of adjoined operator in quite a similar manner. Let's consider the matrices $\widetilde{\mathbf{\Gamma}}(\mathbf{x}) := \mathbf{\Gamma}^T(-\mathbf{x})$ and $\widetilde{\boldsymbol{A}}(\partial_{\mathbf{x}}) := \boldsymbol{A}^T(-\partial_{\mathbf{x}})$. The following basic properties of $\widetilde{\mathbf{\Gamma}}(\mathbf{x})$ may be easily verified:

Theorem 2. *Each column of the matrix* $\widetilde{\mathbf{\Gamma}}(\mathbf{x}\text{-}\mathbf{y})$, *considered as a vector, satisfies the associated system* $\widetilde{\boldsymbol{A}}(\partial_{\mathbf{x}})\widetilde{\mathbf{\Gamma}}(\mathbf{x}\text{-}\mathbf{y}) = 0$, *at every point* $\mathbf{x}$, if $\mathbf{x} \neq \mathbf{y}$ *and the elements of the matrix* $\widetilde{\mathbf{\Gamma}}(\mathbf{x}\text{-}\mathbf{y})$ *have a logarithmic singularity as* $\mathbf{x} \to \mathbf{y}$.

3. Matrix of singular solutions

Using the matrix of fundamental solutions, we construct the so-called singular matrices of solutions by means of elementary functions.

We introduce the stress vector $\mathbf{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{U}$ which acts on the elements of the S with the normal $\mathbf{n}$,

$$\mathbf{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{U} = \mathbf{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2 + \gamma_0\theta), \tag{7}$$

where $\mathbf{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u}$ is the stress vector in the classical theory of elasticity

$$\boldsymbol{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} = \begin{pmatrix} \mu\dfrac{\partial}{\partial n} + (\lambda+\mu)n_1\dfrac{\partial}{\partial x_1} & (\lambda+\mu)n_1\dfrac{\partial}{\partial x_2} + \mu\dfrac{\partial}{\partial s} \\ (\lambda+\mu)n_2\dfrac{\partial}{\partial x_1} - \mu\dfrac{\partial}{\partial s} & \mu\dfrac{\partial}{\partial n} + (\lambda+\mu)n_2\dfrac{\partial}{\partial x_2} \end{pmatrix}\mathbf{u},$$

$$\frac{\partial}{\partial s} = n_2\frac{\partial}{\partial x_1} - n_1\frac{\partial}{\partial x_2}, \qquad \frac{\partial}{\partial \mathbf{n}} = n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2}.$$

We now introduce the following matrix-differential operators

$$\boldsymbol{R}(\partial\mathbf{x},\mathbf{n}) := \begin{pmatrix} T_{11} & T_{12} & -n_1\beta_1 & -n_1\beta_2 & -n_1\gamma_0 \\ T_{21} & T_{22} & -n_2\beta_1 & -n_2\beta_2 & -n_2\gamma_0 \\ 0 & 0 & k_1\frac{\partial}{\partial\mathbf{n}} & 0 & 0 \\ 0 & 0 & 0 & k_2\frac{\partial}{\partial\mathbf{n}} & 0 \\ 0 & 0 & 0 & 0 & k\frac{\partial}{\partial\mathbf{n}} \end{pmatrix}, \tag{8}$$

and

$$\widetilde{\boldsymbol{R}}(\partial\mathbf{x},\mathbf{n}) := \begin{pmatrix} T_{11}(\partial x,n) & T_{12}(\partial x,n) & 0 & 0 & 0 \\ T_{21}(\partial x,n) & T_{22}(\partial x,n) & 0 & 0 & 0 \\ 0 & 0 & k_1\dfrac{\partial}{\partial n} & 0 & 0 \\ 0 & 0 & 0 & k_2\dfrac{\partial}{\partial n} & 0 \\ 0 & 0 & 0 & 0 & k\dfrac{\partial}{\partial n} \end{pmatrix}, \tag{9}$$

Applying the operator $\boldsymbol{R}(\partial\mathbf{x{,}n})$ to the matrix $\boldsymbol{\Gamma}(\mathbf{x\text{-}y})$, we construct the so-called singular matrix of solutions

$$\boldsymbol{R}(\partial\mathbf{x{,}n})\boldsymbol{\Gamma}(\mathbf{x\text{-}y}) = \| \, m_{lj}(\mathbf{x\text{-}y}) \, \|_{5\times 5} \quad l,j = 1,2,3,4,5.$$

where

$$\begin{aligned}
&m_{11} = \frac{\partial\varphi}{\partial\mathbf{n}} - 2\frac{\lambda+\mu}{\mu_0}\frac{\partial}{\partial\mathbf{s}}\frac{\partial^2\varphi_0}{\partial x_1 x_2}, \quad m_{12} = \frac{\partial\varphi}{\partial\mathbf{s}} - 2\frac{\lambda+\mu}{\mu_0}\frac{\partial}{\partial\mathbf{s}}\frac{\partial^2\varphi_0}{\partial x_2^2}, \\
&m_{21} = -\frac{\partial\varphi}{\partial\mathbf{s}} + 2\frac{\lambda+\mu}{\mu_0}\frac{\partial}{\partial\mathbf{s}}\frac{\partial^2\varphi_0}{\partial x_1^2}, \ m_{22} = \frac{\partial\varphi}{\partial\mathbf{n}} + 2\frac{\lambda+\mu}{\mu_0}\frac{\partial}{\partial\mathbf{s}}\frac{\partial^2\varphi_0}{\partial x_1 x_2}, \\
&m_{13} = \frac{2\mu}{\mu_0 k_1}\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}\left[\beta_1\frac{\varphi_1-\varphi}{\lambda_1^2} - \frac{\gamma(\beta_1+\beta_2)}{k_2}\left(\frac{\varphi_1-\varphi}{\lambda_1^4} - \frac{\varphi_0}{\lambda_1^2}\right)\right], \\
&m_{23} = -\frac{2\mu}{\mu_0 k_1}\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}\left[\beta_1\frac{\varphi_1-\varphi}{\lambda_1^2} - \frac{\gamma(\beta_1+\beta_2)}{k_2}\left(\frac{\varphi_1-\varphi}{\lambda_1^4} - \frac{\varphi_0}{\lambda_1^2}\right)\right], \\
&m_{14} = \frac{2\mu}{\mu_0 k_2}\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}\left[\beta_2\frac{\varphi_1-\varphi}{\lambda_1^2} - \frac{\gamma(\beta_1+\beta_2)}{k_1}\left(\frac{\varphi_1-\varphi}{\lambda_1^4} - \frac{\varphi_0}{\lambda_1^2}\right)\right], \\
&m_{24} = -\frac{2\mu}{\mu_0 k_2}\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}\left[\beta_2\frac{\varphi_1-\varphi}{\lambda_1^2} - \frac{\gamma(\beta_1+\beta_2)}{k_1}\left(\frac{\varphi_1-\varphi}{\lambda_1^4} - \frac{\varphi_0}{\lambda_1^2}\right)\right], \\
&m_{15} = \frac{2\mu\gamma_0}{\mu_0 k}\frac{\partial}{\partial s}\frac{\partial\varphi_0}{\partial x_2}, \ m_{25} = -\frac{2\mu\gamma_0}{\mu_0 k}\frac{\partial}{\partial s}\frac{\partial\varphi_0}{\partial x_1}, \\
&m_{34} = -\frac{\gamma}{k_2}\frac{\partial}{\partial\mathbf{n}}\frac{\varphi_1-\varphi}{\lambda_1^2}, \ m_{43} = -\frac{\gamma}{k_1}\frac{\partial}{\partial\mathbf{n}}\frac{\varphi_1-\varphi}{\lambda_1^2}, \\
&m_{33} = \frac{\partial}{\partial\mathbf{n}}\left[\varphi_1 - \gamma\frac{\varphi_1-\varphi}{k_2\lambda_1^2}\right], \ m_{44} = \frac{\partial}{\partial\mathbf{n}}\left[\varphi_1 - \gamma\frac{\varphi_1-\varphi}{k_1\lambda_1^2}\right], \\
&m_{55} = \frac{\partial\varphi}{\partial\mathbf{n}}, \ m_{31} = m_{32} = m_{41} = m_{42} = m_{51} \\
&= m_{52} = m_{35} = m_{53} = m_{54} = m_{45} = 0.
\end{aligned} \tag{10}$$

In the some manner, we can obtain

$$\widetilde{\boldsymbol{R}}(\partial\mathbf{x},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{y}\text{-}\mathbf{x}) = \| \ \widetilde{m}_{lj}(\mathbf{x}\text{-}\mathbf{y}) \ \|_{5\times5}, \quad l,j=1,2,3,4,5.$$

where

$$\widetilde{m}_{ij}=m_{ij},\ \widetilde{m}_{3j}=-k_1\frac{\partial\Gamma_{j3}}{\partial\mathbf{n}},\ \widetilde{m}_{4j}=-k_2\frac{\partial\Gamma_{j4}}{\partial\mathbf{n}},\ \widetilde{m}_{5j}=-k\frac{\partial\Gamma_{j5}}{\partial\mathbf{n}},\ j=1,2,$$

$$\widetilde{m}_{33}=m_{33},\ \widetilde{m}_{43}=m_{43},\ \widetilde{m}_{34}=m_{34},\ \widetilde{m}_{44}=m_{44},\quad \widetilde{m}_{55}=m_{55},$$

$$\widetilde{m}_{i3}=\widetilde{m}_{i4}=\widetilde{m}_{i5}=\widetilde{m}_{53}=\widetilde{m}_{54}=0.\ i=1,2.$$

Let $[\mathbf{R}(\partial_\mathbf{y},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}\text{-}\mathbf{x})]^T$, be the matrix which we get from $[\mathbf{R}(\partial_\mathbf{x},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y})]$ by transposition of the columns and rows and the variables $\mathbf{x}$ and $\mathbf{y}$ (analogously $\left[\tilde{\boldsymbol{R}}(\partial\mathbf{y},\mathbf{n})\boldsymbol{\Gamma}^{\boldsymbol{T}}(\mathbf{y}\text{-}\mathbf{x})\right]^T$).

Let us introduce the following single-layer and double-layer potentials :

The vector-functions defined by the equalities

$$\mathbf{V}(\mathbf{x};\mathbf{g})=\frac{1}{\pi}\int\limits_S \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_\mathbf{y}S,$$

$$\widetilde{\boldsymbol{V}}(\mathbf{x};\mathbf{g})=\frac{1}{\pi}\int\limits_S \boldsymbol{\Gamma}^T(\mathbf{y}-\mathbf{x})\mathbf{g}(\mathbf{y})d_\mathbf{y}S$$

will be called single- layer potentials, while the vector-functions defined by the equalities

$$\mathbf{W}(\mathbf{x};\mathbf{h})=\frac{1}{\pi}\int\limits_S [\boldsymbol{R}(\partial_\mathbf{y},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})]^T\,\mathbf{h}(\mathbf{y})d_\mathbf{y}S,$$

$$\widetilde{\boldsymbol{W}}(\mathbf{x};\mathbf{h})=\frac{1}{\pi}\int\limits_S \left[\widetilde{\boldsymbol{R}}(\partial_\mathbf{y},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{y}-\mathbf{x})\right]^T\mathbf{h}(\mathbf{y})d_\mathbf{y}S$$

will be called double layer potentials. Here $\mathbf{g}$ and $\mathbf{h}$ are the continuous (or Hölder continuous) vectors and S is a closed Lyapunov curve $S\in C^{1,\alpha}$.

We can state the following:

Theorem 3. *The vectors* $\widetilde{\boldsymbol{V}}(\mathbf{x};\mathbf{g})$ *and* $\mathbf{W}(\mathbf{x};\mathbf{h})$ *are the solutions of the system* $\widetilde{\boldsymbol{A}}(\partial_\mathbf{x})\mathbf{U}=\mathbf{0}$ *at any point* $\mathbf{x}$ *and* $\mathbf{x}\neq\mathbf{y}$. *The vectors* $\mathbf{V}(\mathbf{x};\mathbf{g})$ *and* $\widetilde{\boldsymbol{W}}(\mathbf{x};\mathbf{h})$ *are the solutions of the system* $\mathbf{A}(\partial_\mathbf{x})\mathbf{U}=\mathbf{0}$ *at any point* $\mathbf{x}$ *and* $\mathbf{x}\neq\mathbf{y}$. *The elements of the matrices* $[\boldsymbol{R}(\partial_\mathbf{y},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})]^T$ and $\left[\widetilde{\boldsymbol{R}}(\partial_\mathbf{y},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{x}-\mathbf{y})\right]^T$ *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Theorem 4. *If* $S \in C^{1,\eta}(S)$, $\mathbf{g}, \mathbf{h} \in C^{0,\delta}(S)$, $\quad 0 < \delta < \eta \leq 1$, *then the vectors* $\mathbf{W}(\mathbf{x},\mathbf{h})$, $\mathbf{V}(\mathbf{x},\mathbf{g})$, $\widetilde{\boldsymbol{W}}(\mathbf{x},\mathbf{h})$ *and* $\widetilde{\boldsymbol{V}}(\mathbf{x},\mathbf{g})$ *are the regular vector-functions in* $D^+(D^-)$, *and when the point* $\mathbf{x}$ *tends to any point* $\mathbf{z}$ *of the boundary* S *from inside or from outside we have the following formulas:*

$$[\mathbf{W}(\mathbf{z},\mathbf{h})]^{\pm} = \mp\mathbf{h}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S [\boldsymbol{R}(\partial_{\mathbf{y}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{z})]^T\,\mathbf{h}(\mathbf{y})d_{\mathbf{y}}S,$$

$$[\widetilde{\boldsymbol{W}}(\mathbf{z},\mathbf{h})]^{\pm} = \mp\mathbf{h}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \left[\widetilde{\boldsymbol{R}}(\partial_{\mathbf{y}},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{y}-\mathbf{z}))\right]^T\mathbf{h}(\mathbf{y})d_{\mathbf{y}}S,$$

$$[\boldsymbol{R}(\partial_{\mathbf{z}},\mathbf{n})\mathbf{V}(\mathbf{z},\mathbf{g})]^{\pm} = \pm\mathbf{g}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \boldsymbol{R}(\partial_{\mathbf{z}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{z}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S,$$

$$[\widetilde{\boldsymbol{R}}(\partial_{\mathbf{z}},\mathbf{n})\widetilde{\boldsymbol{V}}(\mathbf{z},\mathbf{g})]^{\pm} = \pm\mathbf{g}(\mathbf{z}) + \frac{1}{\pi}\int\limits_S \widetilde{\boldsymbol{R}}(\partial_{\mathbf{z}},\mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{y}-\mathbf{z})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S.$$

Here the integrals are singular and understood as the principal value.

Theorems 1 and 2 can be proved similarly to the corresponding theorems in the classical theory of elasticity (for details see [29]).

4. The Green's formulas

Let the vector $\mathbf{u}(u_1,u_2)$, the functions p_1, p_2 and θ be the regular solutions of equations (1),(2) in D^+. Multiply the equation (1) by $\mathbf{u}$, the first equation of (2) by p_1, the second by p_2, and the third by θ_0. Integrating over D^+ and summing the results, we arrive at

$$\begin{aligned}
&\int\limits_{D^+}[E(\mathbf{u},\mathbf{u}) - (\beta_1 p_1 + \beta_2 p_2 + \gamma_0\theta)\mathrm{div}\mathbf{u}]dx = \int\limits_S \mathbf{uP}\left(\frac{\partial}{\partial\mathbf{y}},\mathbf{n}\right)\mathbf{U}d_yS,\\
&\int\limits_{D^+}\left[k_1(\mathrm{grad}p_1)^2 + k_2(\mathrm{grad}p_2)^2\right]d\mathbf{x}\\
&+\int\limits_{D^+}[\gamma(p_1-p_2)^2 = \int\limits_S \mathbf{pP}^{(1)}\left(\frac{\partial}{\partial\mathbf{y}},\mathbf{n}\right)\mathbf{p}d_yS,\\
&\int\limits_{D^+}\mathrm{grad}\theta\,\mathrm{grad}\theta_0 d_yS = \int\limits_S \theta\frac{\partial\theta_0}{\partial\mathbf{n}}d_yS,
\end{aligned} \tag{11}$$

where

$$E(\mathbf{u},\mathbf{u}) = (\lambda+\mu)(\mathrm{div}\mathbf{u})^2 + \mu\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2.$$

$$\mathbf{P}^{(1)}\left(\frac{\partial}{\partial\mathbf{x}},\mathbf{n}\right)\mathbf{p}(\mathbf{x}) = \begin{pmatrix} k_1 & 0\\ 0 & k_2\end{pmatrix}\frac{\partial\mathbf{p}(\mathbf{x})}{\partial\mathbf{n}}, \quad \mathbf{p} = (p_1,p_3) \tag{12}$$

Formula (11) can be generalized to an unbounded domain D^-, if the conditions

$$\lim_{R_1\to\infty}\int\limits_{S(0,R_1)}\mathbf{uP}\left(\frac{\partial}{\partial\mathbf{y}},\mathbf{n}\right)\mathbf{U}d_yS=0, \tag{13}$$

$$\lim_{R_1\to\infty}\int\limits_{S(0,R_1)}\left(k_1p_1\frac{\partial p_1}{\partial\mathbf{n}}+k_2p_2\frac{\partial p_2}{\partial\mathbf{n}}\right)d_yS=0$$

are fulfilled, where $S(0,R_1)$ is the circle centered at the origin and with the radius R_1; we assume that $(0,0)\in D^+$ and $S(0,R_1)$ envelopes the domain $\overline{D^+}$. Clearly, if the conditions (13) hold, we have the following formula for the unbounded domain D^-

$$\begin{gathered}\int\limits_{D^-}[E(\mathbf{u},\mathbf{u})-(\beta_1p_1+\beta_2p_2+\gamma_0\theta)\mathrm{div}\mathbf{u}]dx=-\int\limits_{S}\mathbf{uP}\left(\frac{\partial}{\partial\mathbf{y}},\mathbf{n}\right)\mathbf{U}d_yS,\\ \int\limits_{D^-}\left[k_1(\mathrm{grad}p_1)^2+k_2(\mathrm{grad}p_2)^2\right]d\mathbf{x}\\ +\int\limits_{D^-}[\gamma(p_1-p_2)^2=-\int\limits_{S}\mathbf{pP}^{(1)}\left(\frac{\partial}{\partial\mathbf{y}},\mathbf{n}\right)\mathbf{p}d_yS,\\ \int\limits_{D^-}\mathrm{grad}\theta\mathrm{grad}\theta_0d_yS=-\int\limits_{S}\theta\frac{\partial\theta_0}{\partial\mathbf{n}}d_yS,\end{gathered} \tag{14}$$

We assume that the constitutive coefficients satisfy the inequalities

$$k_1>0,\quad k_2>0,\quad \gamma>0,\quad \mu>0,\quad \lambda+\mu>0.$$

and the vector $\mathbf{U}(\mathbf{x})$ satisfies the following conditions at the infinity:

$$\mathbf{U}(\mathbf{x})=O(1),\quad \frac{\partial\mathbf{U}}{\partial x_\alpha}=O(|\mathbf{x}|^{-2}),\ |\mathbf{x}|^2=x_1^2+x_2^2>>1,\quad \alpha=1,2. \tag{15}$$

REFERENCES

1. Wilson R.K., Aifantis E.C. On the theory of consolidation with double porosity-I. *International Journal of Engineering Science*, **20** (1982), 1009-1035.

2. Khaled M.Y., Beskos D.E., Aifantis E.C. On the theory of consolidation with double porosity-III. *International Journal for Numerical and Analytical Methods in Geomechanics*, **8**, 2 (1984), 101-123.

3. Beskos D.E., Aifantis E.C. On the theory of consolidation with double porosity-II. *International Journal of Engineering Science*, **24** (1986), 1697-1716.

4. Biot M.A. General theory of three-dimensional consolidation. *J. of Applied Physics*, **12** (1941), 155-164.

5. Barenblatt G.L., Zheltov I.P., Cochina T.N. Basic concepts in the theory of seepage homogeneous liquids in fissured rocks. *J.of Applied Mathematics and Mechanics*, **24** (1960), 1286-1303.

6. De Boer R. Theory of Porous Media: Highlights in the historical development and current state. *Springer, Berlin-Heidelberg- New York*, 2000.

7. Straughan B. Stability and Wave Motion in Porous Media. Springer, New York, 2008.

8. Khalili N., Selvadurai A.P.S. On the constitutive modelling of thermo-hydro-mechanical coupling in elastic media with double porosity. *Elsevier Geo-Engineering Book Series*, **2** (2004), 559-564.

9. Khalili N., Selvadurai A.P.S. A fully coupled constitutive model for thermo-hydro-mechanical analysis in elastic media with double porosity. *Geophysical Research Letters*, **30**, (2003), SDE 7-1-7-3.

10. Bai M., Roegiers J.C. Fluid flow and heat flow in deformable fractured porous media. *Int. J. of Engineering Science*, **32** (1994), 1615-1633.

11. Masters I., Pao W.K.S., Lewis R.W. Coupling temperature to a double porosity model of deformable porous media. *Int. J. for Numerical Methods in Engineering,* **49** (2000), 421-438.

12. Scarpetta E., Svanadze M., Zampoli V. Fundamental solutions in the theory of thermoelasticity for solids with double porosity. *J.Thermal Stresses*, **37** (2014), 727-748.

13. Berryman J.G., Wang H.F. The elastic coefficients of double porosity models for fluid transport in jointed rock. *Journal of Geophysical Research*, **100** (1995), 34611-34627.

14. Berryman J.G., Wang H.F. Elastic wave propagation and attenuation in a double porosity dual-permiability medium. *International Journal of Rock Mechanics and Mining Sciences*, **37** (2000), 63-78.

15. Cowin S.C. Bone poroelasticity. *Journal of Biomechanics*, **32** (1999), 217-238.

16. Ciarletta M., Passarella F., Svanadze M., Plane waves and uniqueness theorems in the coupled linear theory of elasticity for solids with double porosity. *J. of elasticity*, **114** (2014), 55-68.

17. Straughan B. Stability and uniqueness in double porosity elasticity. *Int. J. of Engineering Science*, **65** (2013), 1-8.

18. Svanadze M., De Cicco S. Fundamental solutions in the full coupled theory of elasticity for solids with double porosity. *Arch. Mech.*, **65**, 5 (2013), 367-390.

19. Svanadze M., Scalia A. Mathematical problems in the theory of bone poroelasticity. *Int.J. on Math. Methods and Models in Biosciences*, **1**, 2 (2012), 1-4.

20. Svanadze M. The boundary value problems of the full coupled theory of poroelasticity for materials with double porosity. *PAMM. Proc. Appl. Math.and Mech.*, **12** (2012), 279-282.

21. Tsagareli I., Bitsadze L. Explicit Solution of one Boundary Value Problem in the full Coupled Theory of Elasticity for Solids with Double Porosity. *Acta Mechanica*, **226**, 5 (2015), 1409-1418, DOI: 10.1007/s00707-014-1260-8.

22. Bitsadze L., Tsagareli I. Solutions of BVPs in the fully Coupled Theory of Elasticity for the Space with Double Porosity and Spherical Cavity. *Mathematical Methods in the Applied Science*, **39**, 8 (2016), 2136-2145.

23. Basheleishvili M., Bitsadze L., The basic BVPs of the theory of consolidation with double porosity for the sphere. *Bulletin of TICMI*, **16**, 1 (2012), 15-26.

24. Bitsadze L., Tsagareli I., The solution of the Dirichlet BVP in the fully coupled theory for spherical layer with double porosity. *Meccanica*, **51**, 6 (2016), 1457-1463.

25. Basheleishvili M., Bitsadze L., Explicit solutions of the BVPs of the theory of consolidation with double porosity for the half-space. *Bulletin of TICMI*, **14** (2010), 9-15.

26. Bitsadze L., Zirakashvili N. Explicit solutions of the boundary value problems for an ellipse with double porosity. *Advances in Mathematical Physics.* **2016** (2016), Article

ID 1810795, 11 pages, doi:10.1155/2016/1810795. Hindawi Publishing Corporation.

27. Bitsadze L. The Dirichlet BVP of the Theory of Thermoelasticity with Microtemperatures for Microstretch Sphere. *Journal of Thermal Stresses*, **39**, 9, (2016), 1074-1083.

28. Bitsadze L., Jaiani G. Some basic boundary value problems of the plane thermoelasticity with microtemperatures, *Math. Meth. Appl. Sci.*, **36**, 8 (2013), 956-966.

29. Kupradze V.D., Gegelia T.G., Basheleishvili M.O., Burchuladze T.V. Three Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, 978. Noth Holland Publ. Company, Amsterdam-New-York-Oxford, 1979.

Received 13.04.2017; accepted 22.05.2017.

Author's address:

L. Bitsadze
I. Javakhishvili Tbilisi State University
I. Vekua Institute of Applied Mathematics
2, University str., Tbilisi 0186
Georgia
E-mail: lamarabitsadze@yahoo.com