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DIFFERENTIAL BOUNDARY VALUE PROBLEM FOR THE SECOND ORDER LINEAR ELLIPTIC SYSTEM OF DIFFERENTIAL EQUATIONS ON THE PLANE

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Abstract. In this paper differential boundary problem is considered for the system of second order differential equations of elliptic type in plane domains bounded by smooth curves. The scheme of reduction of the desired problem to the problem of Riemann-Hilbert type for generalized analytic vectors is given.

Keywords and phrases: Differential boundary value problem, generalized analytic vector, Riemann-Hilbert problem.

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We consider the following boundary value problem (BVP): Find the solution $U(x,y) = (U_1, \dots, U_n)$ of the system of differential equations

$$\Delta U + a(x,y)\frac{\partial U}{\partial x}(x,y) + b(x,y)\frac{\partial U}{\partial y}(x,y) = f(x,y), \tag{1}$$

$$z = x + iy \in D,$$

satisfying the boundary condition

$$\alpha(t)\frac{\partial U}{\partial x}(t) + \beta(t)\frac{\partial U}{\partial y}(t) = \delta(t), \quad t \in \Gamma, \tag{2}$$

where Δ is the Laplace operator, D is a finite plane domain with a smooth boundary $\Gamma, a, b,$are given real measurable bounded quadratic matrices of order n, $f = (f_1, \dots, f_n)$ is a given real vector, $f \in L_s(D)$, $s > 2$, α, β are given piecewise continuous real matrices of order n on Γ, δ is a given vector of the class $L_p(\Gamma, \rho)$, $p > 1$, U is a real desired vector, the weight function $\rho(t)$ has the form

$$\rho(t) = \prod_{k=1}^{r} |t - t_k|^{\alpha_k}, \quad -1 < \alpha_k < p - 1, \ t_k \in \Gamma,$$

$$\inf |\det(\alpha(t) + i\beta(t))| > 0.$$

Remark. The notation $A \in K$ where A is a matrix and K is some class of functions means that every element $A_{\alpha,\beta}$ of A belongs to K.

In [1] (see also [2]) Vekua investigated the problem (1)-(2) when $n = 1$ and α, β are continuous functions on Γ.

We now introduce the following notation

$$\omega(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = 2\frac{\partial U}{\partial z}.$$

Equation (1) and boundary value problem (2) will take the form:

$$\partial_{\bar{z}}\omega(z) + A(z)\omega(z) + B(z)\overline{\omega(z)} = F(z), \quad z \in D, \tag{3}$$

$$\mathrm{Re}[G(t)\omega(t)] = \delta(t) \tag{4}$$

where $A(z) = \frac{1}{4}(a(z) + ib(z)), \quad B(z) = \frac{1}{4}(a(z) - ib(z)), \quad F(z) = \frac{1}{2}f(z),$

$$G(\tau) = \alpha(\tau) + i\beta(\tau).$$

We can consider problem (3)-(4) as the Riemann-Hilbert problem for equation (3).

Mapping the domain D conformally on the unit disk we can suppose that D itself is a unit disk $D = \{z;\ |z| < 1\}$. Using a lemma of Vekua ([1], [2]) we have: If the desired vector $U(z) \in W_{1,p_0}(D),\ p_0 > 2$ and is continuous in $\bar{D}$, then it is representable by the formula

$$U(z) = c_0 + \mathrm{Re}\left[-\frac{1}{2\pi}\iint_D \left(\frac{\omega(z)}{z-\zeta} - \frac{z\omega(z)}{1-\bar{\zeta}z}\right) d\sigma_\zeta\right] \tag{5}$$

c_0 is a uniquely determined real constant vector, and $\omega(z) = 2\partial_z U(z)$.

We will seek the solution of problem (3)-(4) in the form $\omega(z) = \omega_0(z) + \omega_1(z)$, where $\omega_1(z) = R(F(z))$. R is some linear bounded operator mapping the space $L_s(D)$ in the space of Hölder continuous vectors. $\omega_0(z)$ is the solution of the homogeneous equation (3) in the class $E_p(D, A, B, \rho), p > 1$.

To introduce this class we need some terms and notations from [4]. Let the matrix $V(t, z)$ be the generalized Cauchy kernel for the homogeneous equation (3). The equation

$$\partial_{\bar{z}}\Psi - A'(z)\Psi - \overline{B'(z)\Psi} = 0 \tag{6}$$

is called conjugate to the equation (3), accent $'$ denotes transposition of a matrix. Using holomorphic vectors, generalized analytic vectors $\omega(z)$ can be represented as

$$\omega(z) = \Phi(z) + \int_D \Gamma_1(z,t)\Phi(t)d\sigma_t + \int_D \Gamma_2(z,t)\Phi(t)d\sigma_t + \sum_{k=1}^{N} c_k W_k(z), \tag{7}$$

where $\Phi(z)$ is a holomorphic vector and $\{W_k(z)\}$ $(k = 1, \ldots, N)$ is a complete system of linearly independent solutions of the Fredholm equation

$$K\omega \equiv \omega(z) - \frac{1}{\pi}\int_D V(t,z)[A(t)\omega(t) + B(t)\overline{\omega(t)}]d\sigma_t = 0.$$

The $W_k(z)$ turn out to be continuous vectors in the whole plane vanishing at infinity, and the $c_k s$ are arbitrary real constants; the kernels $\Gamma_1(z,t)$ and $\Gamma_2(z,t)$ satisfy the system of the integral equations

$$\Gamma_1(z,t) + \frac{1}{\pi}V(t,z)A(t) + \frac{1}{\pi}\int_D V(\tau,z)\{(\tau)\Gamma_1(\tau,t) + B(\tau)\overline{\Gamma_2(\tau,t)}\}d\sigma_\tau$$

$$= -\frac{1}{2}\{v_k(z), \overline{v_k(t)}\}$$

$$\Gamma_2(z,t) + \frac{1}{\pi}V(t,z)A(t) + \frac{1}{\pi}\int\limits_D V(t,z)[A(\tau)\Gamma_2(\tau,t) + B(\tau)\overline{\Gamma_1(\tau,t)}]d\sigma_\tau$$

$$= -\frac{1}{2}\sum_{k=1}^{N}\{v_k(z), \overline{v_k(t)}\},$$

where $v_\kappa(z)$ form a system of linearly independent solutions of the Fredholm integral equation

$$v(z) + \frac{\overline{A'(z)}}{\pi}\int\limits_D V'(z,t)v(t)d\sigma_t + \frac{B'(z)}{\pi}\int\limits_D V'(z,t)\bar{v}(t)d\sigma_t = 0.$$

The curly bracket $\{v, \omega\}$ means a diagonal product of the vectors u and ω.

Notice that in formula (6) $\Phi(z)$ is not an arbitrary holomorphic vector. It has to satisfy the conditions

$$Re\int\limits_D \Phi(z)v_k(z)d\sigma_z = 0, \quad k = 1, \dots, N. \tag{8}$$

It should be mentioned that, generally speaking, the Liouville theorem is not true for solution of homogeneous equation (3). This explains the appearance of the term $\sum c_k W_k(z)$ in the representation formula (6) and the fact condition (8) has be satisfied.

We can now introduce the class $E_p(D, A, B, \rho)$.

This class coincides with the class of those solutions of the homogeneous equation (3) which are representable in the form

$$\omega(z) = \frac{1}{2\pi i}\int\limits_\Gamma \{\Omega_1(z,t)\varphi(t)dt - \Omega(z,t)\overline{\varphi(t)}\ \overline{dt}\} + \sum_{k=1}^{N} c_k W_k(z) \tag{9}$$

where $c_k(z)$ are arbitrary real constants and the $W_k(z)$ are continuous vectors in the whole plane which have been introduced above, while $\varphi(t) \in L_p(\Gamma, \rho)$ satisfies the condition

$$Im\int\limits_\Gamma (\varphi(t), \Psi_j(t))dt = 0, \quad j = 1, \dots, N, \tag{10}$$

where the Ψ_j form a similar system for the conjugate equation (6).

The kernels Ω_1 and Ω_2 are representable by the resolvents Γ_1 and Γ_2 according to the formulas

$$\Omega_1(z,t) = V(t,z) + \int\limits_D \Gamma_1(z,\tau)V(t,\tau)d\sigma_\tau,$$

$$\Omega_2(z,t)=\int\limits_D \Gamma_2(z,\tau)\overline{V(t,\tau)}d\sigma_\tau.$$

Introduce also the class

$$E_q(D,-A',-\overline{B'},\rho^{1-q}),\ \ q=\frac{p}{p-1},$$

of the solutions of the conjugate equation representable in the form

$$\Psi(z)\equiv\frac{1}{2\pi i}\int\limits_\Gamma\{\Omega_1'(t,z)h(t)dt-\Omega_2'(t,z)\overline{h(t)dt}\}+\sum_{k=1}^N c_k\Psi_k(z), \tag{11}$$

where the density $h(t)\in L_q(\Gamma,\rho^{1-q})$ satisfies the conditions

$$Im\int\limits_\Gamma (h(t),W_j(t))dt=0,\quad j=1,\dots,N. \tag{12}$$

For $\omega_{(z})$ representable in the form (9) with the $\varphi(t)\in L_p(\Gamma,\rho)$ satisfying the condition (10) we have the following boundary condition

$$Re[G(t)\omega_0(t)]=H^1(t), \tag{13}$$

where $H^1(t)=\delta(t)-\mathrm{Re}[G(t)\omega_1(t)]$. From equality (13) it follows that

$$\omega_0(t)=G^{-1}(t)[H^1(t)+i\xi(t)] \tag{14}$$

where $\xi(t)$ is a desired real vector of the class $L_p(\Gamma,\rho)$. For $\xi(t)$ we obtain the real system of singular integral equations

$$\begin{gathered}\xi(t_0)\equiv\int\limits_\Gamma\Big[G(t_0)G^{-1}(t)+\frac{t_0}{t}\overline{G(t_0)G^{-1}(t)}\Big]\frac{\xi(t)dt}{t_0-t}+\int\limits_\Gamma k(t_0,t)\xi(t)ds\\ =H^2(t_0)-4\pi\sum_{k=1}^N c_k\mathrm{Re}[G(t_0)W_k(t_0\Big]\end{gathered} \tag{15}$$

and the additional conditions

$$Im\int\limits_\Gamma(G^{-1}(t)[H^1(t)+i\xi(t)],;\ \Psi_k(t))dt=0,\ \ k=1,\dots,N, \tag{16}$$

where $k(t_0,t)$ is a real kernel with weak singularity,$H^2(t)$ is a real vector, which is linearly expressible by means of $H^1(t)$.

Solving system (15) we get

$$\xi(t_0)=(KH^3)(t_0)+\sum_{k=1}\ell_k\xi_k(t_0), \tag{17}$$

where $H^3(t_0) = H^2(t_0) - 4\pi \sum\limits_{k=1}^{N} \mathrm{Re}[G(t_0)W_k(t_0)]$, $\sum\limits_{k=1}^{\ell} \ell_k\xi_k$ is a general solution of the homogeneous equation (15), K is a linear bounded operator on the space $L_p(\Gamma,\rho)$. We must also take into account the conditions of solvability of equation (15)

$$\int\limits_{\Gamma} (H^3(\tau),\ \ g^k(\tau))d\tau = 0,\ \ k = 1,\dots,\ell^*, \tag{18}$$

where g^k, $k = 1,\dots,e^*$ is a full system of linearly independent solutions of the homogeneous equation conjugate to (15) in the class $L_q(\Gamma,\rho^{1-q})$, $q = \frac{p}{p-1}$. We assume that equation (15) is Noetherian. In this case the Riemann-Hilbert boundary value problem (13) is Noetherian in the class $E_p(D,A,B,\rho)$ and necessary and sufficient conditions for solvability of (13) are

$$Im \int\limits_{\Gamma} (H^1(\tau), G'^{-1}(\tau)\eta_k(\tau))d\tau = 0,\ ;\ k = 1,\dots,\ell', \tag{19}$$

where η_k is a full system of linearly independent solutions of the Riemann Hilbert problem

$$Re[G'^{-1}(\tau)\eta(\tau)] = 0,$$

in the class $E_q(D,-A',-\bar{B},\rho^{1-q})$, $q = \frac{p}{p-1}$.

Equation (15) is Noetherian in the class $L_p(\Gamma,\rho)$ if [5]

$$\frac{1+\rho_k}{p} \neq \omega_{kj}, \quad k = 1,\dots,r, \tag{20}$$

$j = 1,\dots,n$, $\omega_{kj} = \frac{1}{2\pi}\arg\lambda_{kj}$, $0 \le \arg\lambda_{kj} < 2\pi$, λ_{kj} are the roots of the equation

$$\det[K^{-1}(t_k+0)K(t_k-0) - \lambda I)] = 0, \tag{21}$$

If G(t) is continuous on Γ, then condition (20) is dropped out.

Returning to problem (1)-(2) and using the formula of representation (5), conditions (16) and formula (17) we obtain for desired function U the equation

$$U(z) - (PU)(z) = F + \sum_{k=1}^{M} c_k M_k \tag{22}$$

where $M \le n + N + \ell$, $M_k(z)$ is linearly expressible by $\xi_j(t)$ and $W_k(z)$, $j = 1,\dots,\ell$, $k = 1,\dots,N$, and the additional conditions $\Phi_j(U) = Q_j$, where Φ_j $j = 1,\dots,N+\ell^*+\ell$ are known real linear functionals, Q_j are know real constants. P is a completely continuous operator on the space $L_r(D)$, $r > 2$. Solving equation (22) we obtain a linear algebraic system for the constants $c_k, k = 1,\dots,M$.

Remark. We can consider problem (1)-(2) similarly on a domain bounded by a simple closed piecewise smooth curve using well-known behaviour of conformal mappings of such do mains on the unit disk (see for example [6].

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