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SOME TWO-DIMENSIONAL THERMOELASTICITY BOUNDARY VALUE PROBLEMS FOR COSSERAT CONTINUUM WITH MICROTEMPERATURE

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Abstract. In the present paper we consider the two-dimensional system of differential equations describing plane thermoelastic equilibrium for elastic bodies of Cosserat with microtemperature. The general solution of this system of equations is constructed using analytical functions of a complex variable and solutions of the Helmholtz equation.

Keywords and phrases: Microtemperature, thermoelasticity, elastic Cosserat medium, plane boundary value problems.

AMS subject classification (2000): 74A35, 74F05, 74G05.

1. Introduction

We consider the two-dimensional system of differential equations describing plane thermoelastic equilibrium for elastic bodies of Cosserat with microtemperature. The foundations of the three-dimensional micro-thermoelasticity theory were laid down in [1-5]. Various issues of termoelastic equilibrium of isotropic homogeneous bodies taking into account the microtemperature are devoted to [6-12]. The work, which considered problems of microthermoelasticity for asymmetrical elastic Cosserat medium [13-19] are unknown to us. In our opinion, such problems represent theoretical and practical interest. Therefore, we have considered thermoelastic equilibrium of elastic bodies in the case of asymmetric Cosserat theory.

2. Basic relations of plane linear theory of thermoelasticity taking into account micro-temperature

Let the homogeneous isotropic cylindrical body be related to the Cartesian coordinate system of coordinates x_1, x_2, x_3 in such a way that the generatrix coincides with the direction of the axis x_3. If in this case the temperature variation T, as well as components of the displacement vector u_1, u_2, the component of the rotation vector ω, and the vector components of microtemperature w_1, w_2 along axes x_1, x_2, x_3 don't depend on coordinates x_3, besides, components of movement and microtemperature along x_3 (u_3 and w_3 respectively), as well as components of the rotation along the axis x_1 and x_2 (w_1 and w_2) are equal to zero, then there is the case of plane strain thermoelastic state of Cosserat medium. Then the homogeneous equations of static equilibrium in a domain ω which is cross-section of the body under consideration, will have the form [6], [19]

$$
\begin{aligned}
&(\mu+\alpha)\Delta u_j+(\lambda+\mu-\alpha)\partial_j\theta+2\alpha\partial_{3-j}\omega-\gamma\partial_jT=0,\\
&(\nu+\beta)\Delta\omega+2\alpha(\partial_1u_2-\partial_2u_1)-4\alpha\omega=0,\\
&k\Delta T+k_1\vartheta=0,\\
&k_6\Delta w_j+(k_4+k_5)\partial_j\vartheta-k_3\partial_jT-k_2w_j=0,\quad j=1,2,
\end{aligned}
\tag{1}
$$

where λ and μ are the Lamé constants;

α,β,ν are the constants characterizing the microstructure of the considered elastic medium; $\Delta=\partial_{11}+\partial_{22}$ is the two-dimensional Laplace operator; $\partial_1\equiv\frac{\partial}{\partial x_1}$, $\partial_2\equiv\frac{\partial}{\partial x_2}$; γ is the coefficient depending on the thermal properties of the material; $k_1,k_2,\dots,k_6$ are constants characterizing microthermoelastic properties of the material; k is the coefficient of thermal conductivity; $\theta=\partial_1u_1+\partial_2u_2,\vartheta=\partial_1w_1+\partial_2w_2$.

3. The general solution of system (1)

On the plane Ox_1x_2 a complex variable $z=x_1+ix_2$, where i the imaginary unit, and the following operators $\partial_z=0.5(\partial_1-i\partial_2)$, $\partial_{\bar z}=0.5(\partial_1+i\partial_2)$ are introduced. Then the system consisting of the last three equations (1) can be written in complex form as follows

$$
\begin{aligned}
&k_6\Delta w_++2(k_4+k_5)\partial_{\bar z}\vartheta-k_3\partial_{\bar z}T-k_2w_+=0,\\
&k\Delta T+k_1\vartheta=0,
\end{aligned}
\tag{2}
$$

where $\Delta=4\partial_z\partial_{\bar z}$; $w_+:=w_1+iw_2$;

For the positive definiteness of the corresponding quadratic form will satisfy the conditions [6]

$$k_4+k_5+k_6>0,\quad k_2>0,\quad k_1k_3-kk_2<0,\quad k>0.$$

In [11] it is shown that the general solution of system (2) is represented as follows:

$$w_+=-\overline{\varphi'(z)}+\partial_{\bar z}[\chi_1(z\bar z)+i\chi_2(z\bar z)],\tag{3}$$

$$T=\frac{k_1}{2k_3}[\varphi(z)+\overline{\varphi(z)}]-\frac{k_1}{2k}\chi_1(z\bar z).\tag{4}$$

where $\varphi(z)$ is an arbitrary analytic function of a complex variable; $\chi_1(z\bar z)$ is a general solution of the following Helmholtz equation $\Delta\chi_1-k^*\chi_1=0$, ; $k^*=\frac{k_2k-k_1k_3}{k(k_4+k+5+k_6)}>0$; $\chi_2(z\bar z)$ is a general solution of the following Helmholtz equation $\Delta\chi_2-\tilde k\chi_2=0$, $\tilde k=\frac{k_2}{k_6}$.

The first three equations of (1) written in complex form will have the form [20], [16]

$$
\begin{cases}
2(\mu+\alpha)\partial_{\bar z}\partial_zu_++(\lambda+\mu-\alpha)\partial_{\bar z}\theta-2\alpha i\partial_{\bar z}\omega_3-\gamma\partial_{\bar z}T=0,\\
2(\nu+\beta)\partial_{\bar z}\partial_z\omega+2\alpha i(\theta-2\partial_zu_+)-2\alpha\omega=0,
\end{cases}
\tag{5}
$$

where $u_+ = u_1 + iu_2$, $\theta = \partial_z u_+ + \partial_{\bar{z}} \bar{u}_+$.

Let's factor out the operator ∂_{barz} from the brackets in the left part of equation (5)

$$\partial_{\bar{z}}(2(\mu+\alpha)\partial_z u_+(\lambda+\mu-\alpha)\theta - 2\alpha i\omega_3 - \gamma T) = 0. \tag{6}$$

therefore (6) is a system of the equations of Cauchy-Riemann

$$2(\mu+\alpha)\partial_z u_+ + (\lambda+\mu-\alpha)\theta - 2\alpha i\omega = (k+1)\phi'(z) + \gamma T. \tag{7}$$

where $\phi(z)$ are the arbitrary analytical functions of a complex variable z; $k = \frac{\lambda+3\mu}{\lambda+\mu}$

The conjugate expression of equations (7) will have the form

$$2(\mu+\alpha)\partial_{\bar{z}}\bar{u}_+ + (\lambda+\mu-\alpha)\theta + 2\alpha i\omega_3 = (k+1)\overline{\phi'(z)} + \gamma T. \tag{8}$$

If we add the formula (7) and (8), and take into account the formula we will obtain

$$\theta = \frac{1}{\lambda+\mu}(\phi'(z) + \overline{\phi'(z)}) + \frac{\gamma T}{\lambda+2\mu}. \tag{9}$$

From (7) and (8) we have

$$i(\partial_z u_+ - \partial_{\bar{z}}\bar{u}_+) = \frac{k+1}{2(\lambda+\alpha)} i(\phi'(z) - \overline{\phi'(z)}) - \frac{2\alpha}{\mu+\alpha}\omega_3. \tag{10}$$

The second equation of (5) can be written as

$$4\partial_z\partial_{\bar{z}}\omega_3 - \frac{2\alpha}{\nu+\beta} i(\partial_z u_+ - \partial_{\bar{z}}\bar{u}_+) - \frac{4\alpha}{\mu+\beta}\omega_3 = 0. \tag{11}$$

If we substitute the formula (10) in formula (11) we obtain the following equation

$$\Delta_2\omega_3 - \xi^2\omega_3 = \frac{\alpha(k+1)}{(\nu+\beta)(\mu+\alpha)} i(\phi'(z) - \overline{\phi'(z)}), \tag{12}$$

where $\xi^2 = \frac{4\mu\alpha}{(\nu+\beta)(\mu+\alpha)} > 0$.

The general solution of equation (12) is written as

$$2\mu\omega_3 = \frac{2\mu}{\nu+\beta}\eta(z\bar{z}) - \frac{k+1}{2} i(\phi'(z) - \overline{\phi'(z)}), \tag{13}$$

where $\eta(z\bar{z})$ is a general solution of the Helmoltz equation $\Delta\eta - \xi^2\eta = 0$.

If we subtituite formulas (9) and (13) in equation (8) and take into account, that $\eta(z\bar{z})$ is a solution of the equation $\Delta\eta - \xi^2\eta = 0$, we obtain

$$2\mu\partial_z u_+ = k\phi'(z) - \overline{\phi'(z)} + 2i\partial_z\partial_{\bar{z}}\eta(z\bar{z}) + \frac{\mu}{\lambda+2\mu}\gamma T.$$

When integrating the last formula over z and taking into account this equation (4), we finally obtain

$$2\mu_+ = k\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} + 2i\partial_{\bar{z}}\eta(z\bar{z})$$

$$+\frac{\mu\gamma}{\lambda+2\mu}\left\{\frac{k_2}{2k_3}\left[\int\varphi(z)dz+\overline{z\varphi(z)}\right]-\frac{2k_1}{kk^*}\partial_{\bar{z}}\chi_1(z,\bar{z})\right\}. \qquad (14)$$

where $\psi(z)$ is any analytical function of a complex z.

4. Conclusion

Thus, the general solution of (1) is represented by three arbitrary analytic functions of a complex variable $\varphi(z), \phi(z), \psi(z)$, and three solutions of the Helmholtz equations $\chi_1(z,\bar{z}), \chi_2(z,\bar{z}), \eta(z,\bar{z})$ according to (3), (4), (13), (14). Using these functions the components of stress tensor, stress moment and thermal stream are expressed. The appropriate selection of these functions can satisfy six independent classical boundary conditions.

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