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ON THE VORTEX EQUATION ON THE COMPLEX PLANE

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Abstract. In this paper we consider the vortex equation as a particular case of Carleman-Bers-Vekua Equation and analyzed solutions space of this equation from the point of view of generalized analytic functions.

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Consider the equation (Carleman-Bers-Vekua equation below)

$$\frac{\partial w}{\partial \overline{z}} + A(z)w + B(z)\overline{w} = 0 \tag{1}$$

where $A, B \in L_{p,2}(\mathbb{C})$, $p > 2$. It is known [1] that every solution of (1) can be expressed in the form

$$w(z) = \Phi(z)e^{-T(A+B\frac{\overline{w}}{w})},$$

where $\Phi(z)$ is analytic and

$$Tf = \frac{1}{\pi}\iint_{\mathbb{C}} \frac{f(\zeta)d\xi d\eta}{\zeta - z}, \ \ \zeta = \xi + \eta,$$

if A and B are merely quasi-summable, i.e., $A_1 = \varphi^{-1}A$ and $B_1 = \psi^{-1}B$ are in $L_{p,2}(\mathbb{C}), p > 2$, for some analytic functions $\varphi(z)$ and $\psi(z)$ with arbitrary singularities (isolated in $\mathbb{C}$), then every solution of (1) can be expressed in the form

$$w(z) = \Phi(z)e^{\varphi(z)\omega(z)+\psi(z)\chi(z)} \tag{2}$$

where $\Phi(z)$ is analytic, $\omega = -T(A_1)$ and $\chi = -T(B_1\overline{w}w^{-1})$. Denote by $\mathcal{A}(A, B)$ the solutions space of (1).

The main statement of the theory of generalized analytic functions is: *for a given analytic function Φ, (2) is a solution of (1) whenever the function $\chi(z)$ satisfies the equation*

$$\chi = T_0(\chi), \tag{3}$$

where $T_0(\chi) = -T[B_*e^{-2iIm(\psi\chi)}]$, $B_* = B_1\frac{\overline{\Phi}}{\Phi}e^{-2iIm\varphi T(A_1)}$ and fixed point argument yields existence of a solution of (3). Such representation (2) is used to study the behavior of solutions of (1) near arbitrary isolated singularities of A and B.

When the Carleman-Bers-Vekua equation is *irregular* [2], it means, that if both functions A and B or at least one of them doesn't belong to $L_{p,2}(\mathbb{C})$, $p > 2$, then the analytic properties of the classes $\mathcal{A}(A, B)$ are different. In other words, *for irregular equations the dependence of the functional classes $\mathcal{A}(A, B)$ on the coefficients A and B is rigid* (see [2]).

As is well known, for every function $a \in L_{p,2}(\mathbb{C})$, $p > 2$, using the integral

$$A(z) = -\frac{1}{\pi}\iint_{\mathcal{C}} \frac{a(\zeta)d\xi\, d\eta}{\zeta - z} \quad \zeta = \xi + i\eta \tag{4}$$

we can construct a $\frac{\partial}{\partial \bar{z}}$-primitive on the whole plane with respect to a generalized derivative $\frac{\partial}{\partial \bar{z}}$ in the Sobolev sense [1]. Therefore if we consider Carleman-Bers-Vekua equations with irregular coefficients, it is necessary to investigate the problem of existence of $\frac{\partial}{\partial \bar{z}}$-primitives of functions not belonging to the class $L_{p,2}(\mathbb{C})$, $p > 2$. Note that the integral (4) is meaningless for such functions.

The following theorem is valid.

Theorem 1. *[2] Every function $a(z)$ of the class $L_p^{loc}(\mathbb{C})$, $p > 2$, has $\frac{\partial}{\partial \bar{z}}$-primitive function $Q(z)$ on the whole complex plane satisfying the Hölder condition with the exponent $\frac{p-2}{p}$ on each compact subset of the complex plane $\mathbb{C}$; moreover if $q(z)$ is one $\frac{\partial}{\partial \bar{z}}$- primitive of the function $a(z)$ then all $\frac{\partial}{\partial \bar{z}}$-primitives of this function are given by the formula*

$$Q(z) = q(z) + \Phi(z), \tag{5}$$

where $\Phi(z)$ is an arbitrary entire function.

For the detailed Proof see [2].

Introduce subclasses of the class $L_p^{loc}(\mathbb{C})$, $p > 2$, elements of which have $\frac{\partial}{\partial \bar{z}}$ primitives, satisfying certain additional asymptotic conditions. In particular, denote by $J_0(\mathbb{C})$ the set of functions $a \in L_p^{loc}(\mathbb{C})$, $p > 2$ for which there exists $\frac{\partial}{\partial \bar{z}}$-primitive $Q(z)$ satisfying the following condition

$$ReQ(z) = O(1), \quad z \in \mathbb{C}. \tag{6}$$

Denote by $J_1(\mathbb{C})$ the set of the functions $a \in L_p^{loc}(\mathbb{C})$, $p > 2$, for which there exists $\frac{\partial}{\partial \bar{z}}$ primitive $Q(z)$, satisfying the following conditions

$$z^n exp\big\{Q(z)\big\} = O(1), \quad z \in \mathbb{C}, \tag{7}$$

for every natural number n.

We used the following Theorem from [2].

Theorem 2. *The function $a(z)$ of the class $L_p^{loc}(\mathbb{C})$, $p > 2$, belongs to the class $J_1(\mathbb{C})$ if and only if its $\frac{\partial}{\partial \bar{z}}$-primitive exists and satisfies the condition*

$$\lim_{z\to\infty} z^k exp\big\{Q(z)\big\} = 0, \tag{8}$$

for every natural number k.

Let $\mathbb{R}\times\mathbb{C}$ be a trivial hermitian vector bundle with the structural group $U(1)$. Denote by $\mathcal{A}$ and Γ moduli space of gauge equivalence connections and smooth sections of this bundle, respectively.

Below we consider (two dimension) Yang-Mills-Higgs-theory on $\mathbb{R}^2$. The dynamical variables for YMH-theory are gauge potential

$$A = A_1(x_1, x_2)dx_1 + A_2(x_1, x_2)dx_2 \in \mathcal{A}$$

and a scalar - so called Higgs field

$$\Phi = \Phi(x_1, x_2) = \Phi(x_1, x_2) + i\Phi(x_1, x_2) \in \Gamma.$$

YMH-potential defines a field

$$F_A = dA + A \wedge A = \frac{\partial A_1}{\partial x_1} - \frac{\partial A_2}{\partial x_2}.$$

Denote by $\mathcal{F}$ the Yang-Mills-Higgs functional on the space $\mathcal{A} \times \Gamma$:

$$\mathcal{F} = \frac{1}{2} \int_{\mathbb{R}^2} D_A\Phi \wedge \star\overline{D_A\Phi} + F_a \wedge \star F_A + \frac{\lambda}{4} \star (\Phi\overline{\Phi} - 1)^2 dx_1 \wedge dx_2, \tag{9}$$

where $D_A = d + A$ is a covariant derivative respect to connections A, and $\star$- the Hodge star operator on the space of differential forms.

Suppose $A \to -iA$. Then $-iF_A = -idA$,

$$(\nabla_A)_1\Phi = \left(\frac{\partial}{\partial x_1} - iA_1\right)\Phi, \quad (\nabla_A)_2\Phi = \left(\frac{\partial}{\partial x_2} - iA_2\right)\Phi.$$

In these notations we have

$$D_A\Phi = -iA\Phi.$$

The **problem (A)** is to find such pair (A, Φ) for which $\mathcal{F}$ is finite.

The finiteness condition for (9) is equivalent to the conditions:

$$|\Phi| \to 1, \quad D_A = d\Phi - iA\Phi \to 0, \quad \text{as} \quad |x| \to \infty$$

and in this case the integer

$$N = \frac{1}{2\pi i} \int_{\mathcal{R}^\in} F_A$$

is a topological invariant of the line bundle.

The variational equations for the action $\mathcal{F}$ are the following equations:

$$d \star F_A = \frac{i}{2} \star (\Phi\overline{D_A\Phi} - \Phi D_A\Phi), \tag{10}$$

$$D_A \star D_A\Phi = \frac{\lambda}{2} \star (\Phi\overline{\Phi} - 1)\Phi. \tag{11}$$

By Bogomol'ny theorem [3] when $\lambda = 1$ then $\mathcal{F} \geq \pi|N|$. In case, when $N \geq 0$ the identity $\mathcal{F} = \pi|N|$ is achieved if and only if the pair (A, Φ) satisfies the following equations:

$$\left(\frac{\partial\Phi_1}{\partial x_1} + A_1\Phi_1\right) - \left(\frac{\partial\Phi_2}{\partial x_2} - A_2\Phi_1\right) = 0, \tag{12}$$

$$\left(\frac{\partial\Phi_1}{\partial x_2} + A_2\Phi_2\right) + \left(\frac{\partial\Phi_2}{\partial x_1} - A_1\Phi_1\right) = 0, \tag{13}$$

$$F_{12} + \frac{1}{2}(\Phi_1^2 + \Phi_2^2 - 1) = 0. \tag{14}$$

Proposition 1. *[3] When $\lambda = 1$, the solutions of the equation (10),(11) are the solutions of the equations (12),(13),(14) and vice versa.*

Introduce the standard notations:

$$z = x_1 + ix_2, \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Then $A = \alpha dz + \bar{\alpha} d\bar{z}$ and

$$D_A\Phi = (\partial_z - i\alpha)\Phi + (\partial\bar{z} - \bar{\alpha})\Phi d\bar{z}, \tag{15}$$

where $\alpha = \frac{1}{2}(A_1 - iA_2)$, $\bar{\alpha} = \frac{1}{2}(A_1 + iA_2)$.

Proposition 2. *The equations (12),(13) are the real and imaginary parts of the equation*

$$D_A\Phi - i \star D_A\Phi = 0. \tag{16}$$

The proof of this proposition immediately follows from (15) and the properties of Hodge $\star$- operator.

From (15) follows also that other form of the equation (16) is

$$\partial_{\bar{z}}\Phi = i\bar{\alpha}\Phi. \tag{17}$$

The last equation is particular case of the Carleman-Bers-Vekua equation ($B = 0$ in (1)). Using this observation we give simply proof of the following statement from [3].

Theorem 3. *Let $N \geq 0$ be a given integer and $z_1, z_2, ..., z_N$ are given points on the complex plane, among which may be equal points (i.e. $z_j = z_k$ when $i \neq k$ allowed.) Then there exists the solution to equations (12),(13),(14) unique up to gauge equivalence, with the following properties:*

1) The solution is smooth on the complex plane;

2) The zeros of Φ are concentrated at the points $z_1, z_2, ..., z_N$ and $\Phi(z, \bar{z}) \sim c_j(z - z_j)^{n_j}$, $\quad c_j \neq 0$;

3) $|D_A\Phi| \leq \mathtt{const}(1 - |\Phi|)$;

4) $N = \frac{1}{2\pi}\int_{\mathcal{R}^\in} F_A = \sum_{z_j, z_j \neq z_i} n_j$.

To prove this theorem we used the technique of the theory of generalized analytic functions developed in [1], [2].

The number N is the Chern number and unique analytic (and topological) invariant for the complex line bundle L on $\mathbb{C} \cup \{\infty\} \cong S^2$. It is known that there exists a one-to-one correspondence between the space of gauge equivalent Carleman-Bers-Vekua equations and the space of holomorphic structures on the bundle $L \to S^2$ [2]. From this there follows the existence of $(0, 1)$-type form ω, such that $\partial_{\bar{z}} - \omega$ is the connection of this bundle. All forms of gauge equivalence ω are solutions of the problem (A). It remains to prove that the equation $\partial_{\bar{z}}\Phi = \omega\Phi$ has the solution with zeros at the points $z_1, z_2, ..., z_N$. From Theorem 1 and Theorem 2 it follows, that the solution of the equation

$$\frac{\partial w}{\partial \bar{z}} + Aw = 0$$

on the whole plane, where $A \in L_p^{loc}(\mathbb{C})$, $p > 2$, has the form has

$$w(z) = \Phi(z)\, e^{-Q(z)},$$

where $Q(z)$ is one of the $\dfrac{\partial}{\partial \bar{z}}$-primitives of the function $A(z)$ and $\Phi(z)$ is an arbitrary entire function. Take $\Phi(z) = (z - z_1)^{n_1}(z - z_2)^{n_2}...(z - z_m)^{n_m}$, $\ z_i \neq z_j, i \neq j$ and $N = \sum_{j=1}^{m} n_j$. Then the pair (A, Φ) has all required properties from Theorem and therefore is a solution of the problem (A).

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