Proceedings of I. Vekua Institute of Applied Mathematics Vol. 64, 2014

INITIAL DATA OPTIMIZATION PROBLEM FOR THE QUASI-LINEAR NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATION WITH VARIABLE DELAYS AND THE DISCONTINUOUS INITIAL CONDITION

Tadumadze T., Gorgodze N.

Abstract. Necessary optimality conditions are obtained for initial data of linear with respect to prehistory of the phase velocity (quasi-linear) neutral functional-differential equation. Here initial data implies the collection of initial moment and vector, delay function entering in the phase coordinates and initial function. In this paper, the essential novelty are optimality conditions of the initial moment and delay function. Discontinuity of the initial condition means that the values of the initial function and the trajectory, in general, do not coincide at the initial moment.

Keywords and phrases: Initial data optimization problem, necessary optimality conditions, neutral functional-differential equation, discontinuous initial condition.

AMS subject classification (2010): 49K21, 34K35, 34K40.

Let $I = [a, b]$ be a finite interval and let $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T is the sign of transposition. Suppose that $O \subset \mathbb{R}^n$ is an open set and $X_0 \subset O$ is a convex set. Let the function $f(t, x, y) = (f^1(t, x, y), ..., f^n(t, x, y))^T$ be defined on $I \times O^2$ and satisfy the following conditions: for almost all fixed $t \in I$ the function $f(t, x, y)$ is continuously differentiable with respect to $(x, y) \in O^2$; for any fixed $(x, y) \in O^2$ the functions $f(t, x, y), f_x(t, x, y), f_y(t, x, y)$ are measurable on I; for any compact set $K \subset O$ there exists a function $m_K(t) \in L(I, [0, \infty))$ such that

$$|f(t, x, y)| + |f_x(t, x, y)| + |f_y(t, x, y)| \le m_K(t)$$

for all $(x, y) \in K^2$ and for almost all $t \in I$.

Further, let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t), t \in [a, \infty)$, satisfying the conditions:

$$\tau(t) \le t,\ M > \dot{\tau}(t) > 0,\ \inf\{\tau(a) : \tau(t) \in D\} := \hat{\tau} > -\infty,$$
$$\sup\{\tau^{-1}(b) : \tau(t) \in D\} < +\infty,$$

where $M > 0$ is a given number and $\tau^{-1}(t)$ is the inverse function of $\tau(t)$.

Let Φ be the set of continuously differentiable initial functions $\varphi(t) \in X_1, t \in [\hat{\tau}, b]$, where $X_1 \subset O$ is a convex set.

Let $t_{01}, t_{02} \in (a, b)$ be given numbers with $t_{01} < t_{02}$ and let the scalar functions $q^i(t_0, x_0, x), i = 0, ..., l$ be continuously differentiable with respect to all arguments $t_0 \in I, x_0 \in X_0$ and $x \in O$.

The collection of initial moment $t_0 \in [t_{01}, t_{02}]$ and vector $x_0 \in X_0$, delay function $\tau(t) \in D$ and initial function $\varphi(t) \in \Phi$ is said to be initial data and will be denoted by $w = (t_0, x_0, \tau(t), \varphi(t))$.

To each initial data

$$w = (t_0, x_0, \tau(t), \varphi(t)) \in W = [t_{01}, t_{02}] \times X_0 \times D \times \Phi$$

we assign the quasi-linear neutral functional-differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f(t, x(t), x(\tau(t))), t \in [t_0, t_1], \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0), \ x(t_0) = x_0, \tag{2}$$

where $A(t)$ is a given continuous matrix function with dimension $n \times n; t_1 \in (t_{01}, b)$ is a fixed finally moment, $\sigma(t) \in D$ is a fixed delay function.The condition (2) is said to be the discontinuous initial condition since, in general, $\varphi(t_0) \neq x(t_0)$.

Definition 1. Let $w = (t_0, x_0, \tau(t), \varphi(t)) \in W$. A function $x(t) = x(t; w) \in O, t \in [\hat{\tau}, t_1]$, is called the solution of equation (1) with the discontinuous initial condition (2) or the solution corresponding to the element w, if $x(t)$ satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Definition 2. An initial data $w = (t_0, x_0, \tau(t), \varphi(t)) \in W$ is said to be admissible if the corresponding solution $x(t) = x(t; w)$ is defined on the interval $[\hat{\tau}, t_1]$ and the following conditions hold

$$q^i(t_0, x_0, x(t_1)) = 0, i = 1, ..., l.$$

The set of admissible initial data will be denoted by W_0.

Definition 3. An initial data $w_0 = (t_{00}, x_{00}, \tau_0(t), \varphi_0(t)) \in W_0$ is said to be optimal if for any $w = (t_0, x_0, \tau(t), \varphi(t)) \in W_0$ we have

$$q^0(t_{00}, x_{00}, x_0(t_1)) \leq q^0(t_0, x_0, x(t_1)),$$

where $x_0(t) = x(t; w_0), x(t) = x(t; w)$.

The initial data optimization problem consists in finding an optimal initial data w_0.

Theorem 1. *Let $w_0 \in W_0$ be an optimal initial data and $t_{00} \in [t_{01}, t_{02})$. Let the following conditions hold:*
a) $\gamma_0(t_{00}) < t_1$, where $\gamma_0(t)$ is the inverse function of $\tau_0(t)$ and

$$t_{00}, \gamma_0(t_{00}) \neq \{\sigma(t_1), \sigma(\sigma(t_1)), ...\};$$

b) for each compact set $K \subset O$ there exists a number $m_K > 0$ such that

$$|f(z)| \leq m_K, \forall z = (t, x, y) \in I \times K^2;$$

c) there exist the limits

$$\lim_{z \to z_0} f(z) = f_0^+, z \in [t_{00}, \gamma_0(t_{00})) \times O^2,$$

$$\lim_{(z_1,z_2)\to(z_{10},z_{20})} [f(z_1) - f(z_2)] = f_{01}^+, z_i \in [\gamma_0(t_{00}), t_1) \times O^2, i = 1, 2,$$

where

$$z_0 = (t_{00}, x_{00}, \varphi_0(\tau_0(t_{00})), z_{10} = (\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), x_{00}),$$

$$z_{20} = (\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \varphi_0(t_{00})).$$

Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0, \pi_0 \leq 0$ and a solution $(\chi(t), \psi(t))$ of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t) f_x[t] - \psi(\gamma_0(t)) f_y[\gamma_0(t)]\dot{\gamma}_0(t), \\ \psi(t) = \chi(t) + \psi(\rho(t))A(\rho(t))\dot{\rho}(t), t \in [t_{00}, t_1], \\ \chi(t) = \psi(t) = 0, t > t_1 \end{cases}$$

such that the conditions listed below hold:
1.1. *the condition for $\chi(t)$ and $\psi(t)$*

$$\chi(t_1) = \psi(t_1) = \pi Q_{0x},$$

where

$$Q = (q^0, ..., q^l)^T, Q_{0x} = Q_x(t_{00}, x_{00}, x_0(t_1));$$

1.2. *the condition for the optimal initial moment t_{00}*

$$\pi Q_{0t_0} + (\psi(t_{00}) - \chi(t_{00}))\dot{\varphi}_0(t_{00}) - \psi(t_{00})\Big(A(t_{00})\dot{\varphi}_0(t_{00}) + f_0^+\Big)$$

$$-\psi(\gamma_0(t_{00}))f_{01}^+\dot{\gamma}(t_{00}) \le 0;$$

1.3. the condition for the optimal initial vector x_{00}

$$\pi Q_{0x_0} + \psi(t_{00}) = 0;$$

1.4. the condition for the optimal delay function $\tau_0(t)$

$$\psi(\gamma_0(t_{00}))f_{01}^+\tau_0(t_{00}) + \int_{t_{00}}^{t_1} \psi(t)f_y[t]\dot{x}_0(\tau_0(t))\tau_0(t)dt$$

$$= \max_{\tau(t)\in D}\Big[\psi(\gamma_0(t_{00}))f_{01}^+\tau(t_{00}) + \int_{t_{00}}^{t_1} \psi(t)f_y[t]\dot{x}_0(\tau_0(t))\tau(t)dt\Big];$$

1.5. the condition for the optimal initial function $\varphi_0(t)$

$$\int_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t))f_y[\gamma_0(t)]\dot{\gamma}_0(t)\varphi_0(t)dt + \int_{\sigma(t_{00})}^{t_{00}} \psi(\rho(t))A(\rho(t))\dot{\rho}(t)\dot{\varphi}_0(t)dt$$

$$= \max_{\varphi(t)\in\Phi}\Big[\int_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t))f_y[\gamma_0(t)]\dot{\gamma}_0(t)\varphi(t)dt + \int_{\sigma(t_{00})}^{t_{00}} \psi(\rho(t))A(\rho(t))\dot{\rho}(t)\dot{\varphi}(t)dt\Big];$$

Here

$$f_x[t] = f_x(t, x_0(t), x_0(\tau_0(t))),$$

and $\rho(t)$ *is the inverse function of* $\sigma(t)$.

Some comments

The essential innovation in this work is necessary optimality condition for delay function.

Let $f(t,x,y)$ be a continuous function. Then instead of inequality 1.2 we have the equality

$$\pi Q_{0t_0} + (\psi(t_{00}) - \chi(t_{00}))\dot{\varphi}_0(t_{00}) - \psi(t_{00})\Big(A(t_{00})\dot{\varphi}_0(t_{00})$$

$$+f(t_{00}, x_{00}, \varphi_0(t_{00}))\Big) - \psi(\gamma_0(t_{00}))\Big[f(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), x_{00})$$

$$-f(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \varphi_0(t_{00}))\Big]\dot{\gamma}(t_{00}) = 0.$$

Theorem 1 is proved by a scheme described in [1].

Theorem 2. *Let $w_0 \in W_0$ be an optimal initial data and $t_{00} \in [t_{01}, t_{02})$. Let the following conditions hold:*

$$\gamma_0(t_{00}) < t_1; t_{00}, \gamma_0(t_{00}) \neq \{\sigma(t_1), \sigma(\sigma(t_1)), ...\}$$

and

$$f(t, x, y) = B(t)x + C(t)y + g(t),$$

where $B(t), C(t)$ are continuous matrix functions, $g(t)$ is a continuous vector function .Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0, \pi_0 \leq 0$ and a solution $(\chi(t), \psi(t))$ of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t)B(t) - \psi(\gamma_0(t))C(\gamma_0(t))\dot{\gamma}_0(t), \\ \psi(t) = \chi(t) + \psi(\rho(t))A(\rho(t))\dot{\rho}(t), t \in [t_{00}, t_1], \\ \chi(t) = \psi(t) = 0, t > t_1 \end{cases}$$

such that the conditions listed below hold:
2.1. the condition 1.1 for $\chi(t)$ and $\psi(t)$;
2.2. the condition for the optimal initial moment t_{00}

$$\pi Q_{0t_0} + (\psi(t_{00}) - \chi(t_{00}))\dot{\varphi}_0(t_{00}) - \psi(t_{00})\Big(A(t_{00})\dot{\varphi}_0(t_{00}) + B(t_{00})x_{00}$$

$$+C(t_{00})\varphi_0(t_{00}) + g(t_{00})\Big) - \psi(\gamma_0(t_{00}))C(\gamma_0(t_{00}))[x_{00} - \varphi_0(t_{00})]\dot{\gamma}(t_{00}) = 0;$$

2.3. the condition 1.3 for the optimal initial vector x_{00};
2.4. the condition for the optimal delay function $\tau_0(t)$

$$\psi(\gamma_0(t_{00}))C(\gamma_0(t_{00}))[x_{00} - \varphi_0(t_{00})]\tau_0(t_{00}) + \int_{t_{00}}^{t_1} \psi(t)C(t)\dot{x}_0(\tau_0(t))\tau_0(t)dt$$

$$= \max_{\tau(t)\in D} \Big[\psi(\gamma_0(t_{00}))C(\gamma_0(t_{00}))[x_{00} - \varphi_0(t_{00})]\tau(t_{00}) + \int_{t_{00}}^{t_1} \psi(t)C(t)\dot{x}_0(\tau_0(t))\tau(t)dt\Big];$$

2.5. the condition for the optimal initial function $\varphi_0(t)$

$$\int_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t))C(\gamma_0(t))\dot{\gamma}_0(t)\varphi_0(t)dt + \int_{\sigma(t_{00})}^{t_{00}} \psi(\rho(t))A(\rho(t))\dot{\rho}(t)\dot{\varphi}_0(t)dt$$

$$= \max_{\varphi(t)\in\Phi} \Big[\int_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t))C(\gamma_0(t))\dot{\gamma}_0(t)\varphi(t)dt + \int_{\sigma(t_{00})}^{t_{00}} \psi(\rho(t))A(\rho(t))\dot{\rho}(t)\dot{\varphi}(t)dt \Big].$$

The initial data optimization problem for the linear neutral functional-differential equation with constant delays and the discontinuous initial condition is considered in [2].

Acknowledgement. The work was supported by the Sh. Rustaveli National Science Foundation, Grant No. 31/23.

REFERENCES

1. Kharatishvili G. L., Tadumadze T. A. Variation formulas of solutions and optimal control problems for differential equations with retarded argument. *J. Math. Sci. (N.Y.)*, **104**, 1 (2007), 1-175.

2. Tadumadze T. Optimization of initial data for linear neutral functional-differential equations with the discontinuous initial condition. *Azerbaijan Journal of Mathematics*, **2**, 2 (2012), 84-93.

Received 30.10.2014; revised 12.11.2014; accepted 19.11.2014.

Authors' addresses:

T. Tadumadze
I. Javakhishvili Tbilisi State University
Department of Mathematics & I. Vekua Institute of Applied Mathematics
2, University St., Tbilisi 0186
Georgia
E-mail: tamaz.tadumadze@tsu.ge

N. Gorgodze
A. Tsereteli Kutaisi University
Department of Mathematics
59, King Tamari St., Kutaisi 4600
Georgia
E-mail: nika_gorgodze@yahoo.com