Proceedings of I. Vekua Institute of Applied Mathematics
Vol. 64, 2014

AN APPOXIMATE METHOD FOR THE PLATE UNDER SYMMETRIC LOAD

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Abstract. A boundary value problem is considered for a nonlinear system of ordinary differential equations which, according to the Timoshenko theory, describes the static behavior of a symmetric plate. Using Green's function, the problem is reduced to one equation with respect to a transverse bending function w, while other unknown functions of the initial problem are expressed by means of the function w. The problem for w is solved by the Galerkin method. The Jacobi method is applied to the resulting system of discrete equations and the accuracy of this method is estimated.

Keywords and phrases: Timoshenko plate, Galerkin method, Jacobi iteration, Cardano formula, iteration error.

AMS subject classification (2010): 34B27, 47J25, 65H10, 65L60, 65L70.

1. Introduction

The nonlinear system of Timoshenko plate equations is important from the theoretical and applied viewpoints. It is used when the hypothesis that the normal is perpendicular to the midsurface after deformation fails to be fulfilled. The questions concerning the solvability of corresponding system and the justification of appropriate numerical methods for it were open [11]. Note that using the condensing operator theory [1], the existence of a solution of the Timoshenko static problem is shown for small loads only [7].

Here we consider a one-dimensional problem of elliptic type for the static Timoshenko plate.

2. Statement of the problem

Let us consider the nonlinear system of ordinary differential equations

$$
\begin{gathered}
u'' + \frac{1}{2}\left(w'^2\right)' + p(x) = 0, \\
k_0^2\,\frac{Eh_0}{2(1+\nu)}\,(w''+\psi') + \frac{Eh_0}{1-\nu^2}\left[\left(u' + \frac{1}{2}\,w'^2\right)w'\right]' + q(x) = 0, \\
\frac{h_0^2}{6(1-\nu)}\,\psi'' - k_0^2\,(w'+\psi) + r(x) = 0, \\
0 < x < 1,
\end{gathered} \tag{1}
$$

with the boundary conditions

$$
u(0) = u(1) = 0, \quad w(0) = w(1) = 0, \quad \psi'(0) = \psi'(1) = 0. \tag{2}
$$

Here the displacements $u = u(x)$, $w = w(x)$ of the plate midplane and the angle of rotation $\psi = \psi(x)$ of the normal to the midplane are the unknown functions to be determined, whereas the forces $p(x)$, $q(x)$ and $r(x)$ are the given ones. E is Young's modulus, h_0 is the plate thickness, k_0 is the lateral shear coefficient and ν is the Poisson ratio, $0 < \nu < 0.5$.

Equations (1) characterize the plate equilibrium under the action of an axially symmetric load. They are obtained from Timoshenko system for a shell [10, p. 42]. For this, we drop the variables y and t and assume $k_x = k_y = 0$. However we preserve the cubic nonlinear terms. Note that system (1) can also be obtained from Timoshenko system for a plate in [3, p. 24].

3. Reduction of the system

Using the first and the third equation from (1) and taking into account the respective boundary conditions from (2), the functions $u(x)$ and $\psi(x)$ can be expressed through the function $w(x)$ as follows

$$\begin{aligned} u(x) &= (x-1)\int_0^x \left(\frac{1}{2}\,w'^2(\xi) - \xi p(\xi)\right) d\xi \\ &\quad + x\int_x^1 \left(\frac{1}{2}\,w'^2(\xi) - (\xi-1)p(\xi)\right) d\xi, \\ \psi(x) &= -\frac{\sigma}{\sinh\sigma}\Bigg(\cosh\sigma(x-1)\int_0^x \cosh\sigma\xi\left(w'(\xi) - \frac{1}{k_0^2}\,r(\xi)\right) d\xi \\ &\quad + \cosh\sigma x\int_x^1 \cosh\sigma(\xi-1)\left(w'(\xi) - \frac{1}{k_0^2}\,r(\xi)\right) d\xi\Bigg), \end{aligned} \tag{3}$$

where $\sigma = \frac{k_0}{h_0}\sqrt{6(1-\nu)}$. Applying (3) in the second equation of system (1), we obtain the integro-differential equation with respect to $w(x)$

$$\begin{aligned} &\frac{Eh_0}{1-\nu^2}\left[\left(\frac{1-\nu}{2}\,k_0^2 + \frac{1}{2}\int_0^1 w'^2 dx + \int_0^1 (1-x)p(x)\,dx - \int_0^x p(\xi)\,d\xi\right)w'' - p(x)w'\right] \\ &\quad - \frac{3Ek_0^4}{h_0\sinh\sigma}\,\frac{1-\nu}{1+\nu}\Bigg(\sinh\sigma(x-1)\int_0^x \cosh\sigma\xi\left(w'(\xi) - \frac{1}{k_0^2}\,r(\xi)\right) d\xi \\ &\quad + \sinh\sigma x\int_x^1 \cosh\sigma(\xi-1)\left(w'(\xi) - \frac{1}{k_0^2}\,r(\xi)\right) d\xi\Bigg) + q(x) = 0, \end{aligned} \tag{4}$$

which we complement with the corresponding boundary condition from (2)

$$w(0) = w(1) = 0. \tag{5}$$

After solving problem (4),(5), we substitute $w(x)$ in (3) and find other unknown functions $u(x)$ and $\psi(x)$.

4. Algorithm

Let us consider the numerical algorithm of the solution of problem (4), (5), which in the particular cases was used in [8] [9]. The approximation of

$w(x)$ with respect to the spatial variable is written as the finite sum

$$w_n(x) = \sum_{i=1}^{n} \frac{1}{i\pi}\, w_{ni} \sin i\pi x, \tag{6}$$

where, in case we apply the Galerkin method [5], the coefficients w_{ni} satisfy the nonlinear system of equations

$$\left(p_{1i}+p_2+\sum_{j=1}^{n} w_{nj}^2\right) w_{ni}+\sum_{j=1}^{n} p_{3ij}w_{nj}+\frac{1}{i}(q_i+r_i)=0, \quad i=1,2,\dots,n. \tag{7}$$

Here the following notation is used

$$\begin{aligned}
p_{1i} &= \frac{1}{\frac{1}{2k_0^2(1-\nu)}+\frac{3}{(h_0\pi i)^2}}, \quad p_2 = 4\int_0^1 (1-x)p(x)\,dx,\\
p_{3ij} &= 8\Bigg(-\frac{j}{i}\int_0^1\left(\int_0^x p(\xi)\,d\xi\right)\sin i\pi x \sin j\pi x\,dx\\
&\quad +\frac{1}{i\pi}\int_0^1 p(x)\sin i\pi x\cos j\pi x\,dx\Bigg),\\
q_i &= -\frac{8(1-\nu^2)}{Eh_0\pi}\int_0^1 q(x)\sin i\pi x\,dx,\\
r_i &= -\frac{24k_0^2(1-\nu)^2}{h_0^2\pi\sinh\sigma}\int_0^1\Bigg(\sinh\sigma(x-1)\int_0^x r(\xi)\cosh\sigma\xi\,d\xi\\
&\quad +\sinh\sigma x\int_x^1 r(\xi)\cosh\sigma(\xi-1)\,d\xi\Bigg)\sin i\pi x\,dx.
\end{aligned} \tag{8}$$

Using the integration by parts formula we find

$$\begin{aligned}
p_{3ij} &= -8\int_0^1\left(\int_0^x p(\xi)\,d\xi\right)\cos i\pi x\cos j\pi x\,dx,\\
r_i &= \frac{24k_0^2(1-\nu)^2 i}{h_0^2(\sigma^2+(i\pi)^2)}\int_0^1 r(x)\cos i\pi x\,dx.
\end{aligned} \tag{9}$$

We will solve system (7) by using the iteration method

$$\begin{aligned}
&\left(p_{1i}+p_2+w_{ni,k+1}^2+\sum_{\substack{j=1\\ j\neq i}}^{n} w_{nj,k}^2\right)w_{ni,k+1}+p_{3ii}w_{ni,k+1}\\
&+\sum_{\substack{j=1\\ j\neq i}}^{n} p_{3ij}w_{nj,k}+\frac{1}{i}(q_i+r_i)=0, \quad k=0,1,\dots\,, \quad i=1,2,\dots,n,
\end{aligned} \tag{10}$$

where $w_{ni,k+l}$ is the $(k+l)$-th approximation of w_{ni}, $l=0,1$. This method is nothing else but the Jacobi nonlinear iteration process [6]. After defining

$w_{ni,k}$, $i = 1, 2, \ldots, n$, we use them in formula (6) instead of w_{ni} and, as a result, find the approximation of the function $w(x)$, which, when used in (3), gives the approximation for the functions $u(x)$ and $\psi(x)$.

Note that each i-th equation of system (10) is a cubic equation for $w_{ni,k+1}$. Therefore, using the Cardano formula [2], $w_{ni,k+1}$ can be written in the explicit form

$$w_{ni,k+1} = \sigma_{i,1} - \sigma_{i,2}, \quad k = 0, 1, \ldots, \quad i = 1, 2, \ldots, n, \tag{11}$$

where

$$\sigma_{i,l} = \left[(-1)^l \frac{s_i}{2} + \left(\frac{s_i^2}{4} + \frac{t_i^3}{27} \right)^{\frac{1}{2}} \right]^{\frac{1}{3}}, \quad l = 1, 2, \tag{12}$$

$$t_i = p_{1i} + p_2 + p_{3ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} w_{nj,k}^2, \quad s_i = \frac{1}{i}(q_i + r_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n} p_{3ij} w_{nj,k}\,. \tag{13}$$

5. Jacobi iteration error

Under the error of iteration method (11) we understand a difference between the function $w_n(x)$ (see (6)) and the function

$$w_{n,k}(x) = \sum_{i=1}^{n} \frac{1}{i\pi} w_{ni,k} \sin i\pi x$$

obtained during the realization of the process (11), i.e. the function

$$w_n(x) - w_{n,k}(x) = \sum_{i=1}^{n} \frac{1}{i\pi} (w_{ni} - w_{ni,k}) \sin i\pi x. \tag{14}$$

Having denoted

$$p_0 = \left(\int_0^1 p^2(x)\,dx \right)^{\frac{1}{2}}, \quad q_0 = \left(\int_0^1 q^2(x)\,dx \right)^{\frac{1}{2}}, \quad r_0 = \left(\int_0^1 r^2(x)\,dx \right)^{\frac{1}{2}}$$

we introduce into consideration the restrictions

$$c = \left(\frac{1}{k_0^2(1-\nu)} + \frac{6}{(h_0\pi)^2} \right)^{-1} - \frac{5}{\sqrt{3}} p_0 > 0 \tag{15}$$

and

$$\frac{\sqrt{2}}{c} \left[p_0(n^2 - 1) + \frac{4}{\sqrt{c}} \left(q_0 \frac{1-\nu^2}{Eh_0\pi} \sum_{i=1}^{n} \frac{1}{i} \right. \right.$$
$$\left. \left. + 3r_0 \frac{k_0^2(1-\nu)^2}{h_0^2} \sum_{i=1}^{n} \frac{1}{\sigma^2 + (i\pi)^2} \right) \right] < \Delta < 1. \tag{16}$$

Let us prove the convergence of process (11). We write system (11) in the form

$$w_{ni,k+1} = \varphi_i\left(w_{n1,k}, w_{n2,k}, \dots, w_{nn,k}\right) \tag{17}$$

and consider the Jacobi matrix

$$J = \left(\frac{\partial \varphi_i}{\partial w_{nj,k}}\right)_{i,j=1}^{n}. \tag{18}$$

By virtue of (11)–(13) and (17) we conclude that the diagonal elements of the matrix J are equal to zero. As to the nondiagonal elements, $i \neq j$, we have for them

$$\frac{\partial \varphi_i}{\partial w_{nj,k}} = -\frac{1}{6}\sum_{l=1}^{2}\frac{1}{\sigma_{i,l}^2}\left[p_{3ij} + (-1)^l\left(\frac{s_i^2}{4} + \frac{t_i^3}{27}\right)^{-\frac{1}{2}}\left(\frac{s_i}{2}p_{3ij} + \frac{2}{9}t_i^2 w_{nj,k}\right)\right]. \tag{19}$$

By (12)

$$\sigma_{i,1}\cdot\sigma_{i,2} = \frac{t_i}{3}, \quad \sigma_{i,2}^3 - \sigma_{i,1}^3 = s_i, \quad \left(\frac{s_i^2}{4} + \frac{t_i^3}{27}\right)^{\frac{1}{2}} = \frac{\sigma_{i,1}^3 + \sigma_{i,2}^3}{2}. \tag{20}$$

Using these relations in (19), we get

$$\frac{\partial \varphi_i}{\partial w_{nj,k}} = -\frac{1}{6}\,p_{3ij}\,\frac{1}{\sigma_{i,1}^2 - \frac{t_i}{3} + \sigma_{i,2}^2} + \frac{2}{3}\,w_{nj,k}\,s_i\,\frac{1}{\sigma_{i,1}^4 + (\frac{t_i}{3})^2 + \sigma_{i,2}^4}. \tag{21}$$

Taking into account the first equality of (20), we find that

$$\sigma_{i,1}^2 - \frac{t_i}{3} + \sigma_{i,2}^2 \geq \frac{1}{3}\,t_i, \quad \sigma_{i,1}^4 + \sigma_{i,2}^4 \geq \frac{2}{9}\,t_i^2. \tag{22}$$

By (21) and (22) we write

$$\left|\frac{\partial \varphi_i}{\partial w_{nj,k}}\right| \leq \phi_{1ij} + \phi_{2ij}, \tag{23}$$

where

$$\phi_{1ij} = \frac{1}{2}|p_{3ij}|\,\frac{1}{|t_i|}, \quad \phi_{2ij} = 2|w_{nj,k}|\,|s_i|\,\frac{1}{t_i^2}. \tag{24}$$

Let us estimate each ϕ_{lij}, $l = 1, 2$. Suppose that the initial data of the problem – the function $p(x)$ and the constants ν, E, h and k_0 – are such that

$$p_{1i} > |p_2 + p_{3ii}|. \tag{25}$$

As follows from (8), (9) and (25) implies the fulfillment of the relation

$$\left(\frac{1}{2k_0^2(1-\nu)} + \frac{3}{(h_0 i\pi)^2}\right)^{-1} > \left|4\int_0^1 (1-x)p(x)\,dx - 8\int_0^1\left(\int_0^x p(\xi)\,d\xi\right)\cos^2 i\pi x\,dx\right|, \quad i = 1, 2, \dots, n. \tag{26}$$

In that case, by virtue of (24) and (13) we have

$$\phi_{1ij} \le \frac{1}{2c_i} \left| p_{3ij} \right|, \tag{27}$$

where $c_i = p_{1i} - |p_2 + p_{3ii}|$. Further, using (24), (25), (13) and the fact that $\max\limits_x \frac{x}{c+x^2} = \frac{1}{2\sqrt{c}}$, $x > 0$, $c > 0$, we come to a conclusion that

$$\phi_{2ij} = 2\frac{|w_{nj,k}|}{|r_i|}\frac{|s_i|}{r_i|} \le 2\,\frac{|w_{nj,k}|}{c_i + w_{nj,k}^2}\left(\frac{1}{i}\,\frac{|q_i| + |r_i|}{c_i} + \frac{1}{c_i + \sum\limits_{\substack{l=1\\ l\ne i}}^{n} w_{nl,k}^2}\left(\sum_{\substack{l=1\\ l\ne i}}^{n} p_{3il}^2\right)^{\frac12}\right.$$

$$\left.\times\left(\sum_{\substack{l=1\\ l\ne i}}^{n} w_{nl,k}^2\right)^{\frac12}\right) \le \frac{1}{\sqrt{c_i}}\left(\frac{1}{i}\,\frac{|q_i| + |r_i|}{c_i} + \frac{1}{2\sqrt{c_i}}\left(\sum_{\substack{l=1\\ l\ne i}}^{n} p_{3il}^2\right)^{\frac12}\right). \tag{28}$$

Let us introduce the vector and matrix norms by means of the expressions $\sum\limits_{i=1}^{n} |v_i|$ and $\max\limits_{1\le j\le n} \sum\limits_{i=1}^{n} |m_{ij}|$ for $v = (v_i)_{i=1}^n$ and $M = (m_{ij})_{i,j=1}^n$.

Assume that the norm of the matrix J is smaller than a number Δ, $0 < \Delta < 1$. According to (18), (23), (27) and (28) this requirement takes place if

$$\frac12 \max_{1\le j\le n} \sum_{\substack{i=1\\ i\ne j}}^{n} \frac{1}{c_i}\,|p_{3ij}|$$

$$+ \frac12 \sum_{i=1}^{n} \frac{1}{c_i}\left(2\,\frac{|q_i| + |r_i|}{i\sqrt{c_i}} + \left(\sum_{\substack{j=1\\ j\ne i}}^{n} p_{3ij}^2\right)^{\frac12}\right) < \Delta < 1. \tag{29}$$

Then, by virtue of Banach's contraction principle [4] there exists a unique solution w_{ni}, $i = 1, 2, \dots, n$, of system (7) to which the sequence of approximations $w_{ni,k}$ of the iteration method converges as $k \to \infty$, whereas the error decreases with a geometrical progression rate

$$\sum_{i=1}^{n} |w_{ni} - w_{ni,k}| \le \frac{\Delta^k}{1-\Delta} \sum_{i=1}^{n} |w_{ni,1} - w_{ni,0}|, \quad k = 0, 1, \dots\ . \tag{30}$$

Now we replace conditions (26) and (29) by more rigid but easily verifiable conditions. For this, we require the fulfillment of $p_{1i} > |p_2| + |p_{3ii}|$ instead of (25), then apply (8), (9) and the following relations obtained by means of the Cauchy–Bunyakowski inequality

$$\left| \int_0^1 (1-x)p(x)\,dx \right| \le \frac{1}{\sqrt3}\,p_0,$$

$$\left|\int_0^1 \left(\int_0^x p(\xi)\,d\xi\right)\cos i\pi x\cos j\pi x\,dx\right| \le \frac{1}{2\sqrt{2}}\,p_0, \quad i\ne j,$$
$$\left|\int_0^1 \left(\int_0^x p(\xi)\,d\xi\right)\cos^2 i\pi x\,dx\right| \le \frac{\sqrt{3}}{4}\,p_0,$$
$$\left|\int_0^1 q(x)\sin i\pi x\,dx\right| \le \frac{1}{\sqrt{2}}\,q_0\,, \quad \left|\int_0^1 r(x)\cos i\pi x\,dx\right| \le \frac{1}{\sqrt{2}}\,r_0\,.$$

As a result, instead of requirement (26) we will have (15) and (29) will be replaced by the condition (16). It is not difficult to verify that conditions (26) and (29) as well as (15) and (16) are fulfilled for sufficiently small functions $p(x)$, $q(x)$ and $r(x)$.

Using (30), we come to the conclusion that when conditions (26) and (29) as well as (15) and (16) are fulfilled, for the $L_2(0,1)$-norm of error (14) we have

$$\left\|\frac{d^l}{dx^l}\,(w_n(x)-w_{n,k}(x))\right\|_{L_2(0,1)} \le \frac{\Delta^k}{\sqrt{2}\,\pi^{1-l}(1-\Delta)}\sum_{i=1}^n |w_{ni,1}-w_{ni,0}|,$$
$$l=0,1, \quad k=0,1,\ldots\,.$$

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Received 31.05.2014; revised 10.07.2014; accepted 12.09.2014.

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