**Proceedings of I. Vekua Institute of Applied Mathematics
Vol. 64, 2014**

NEWTONIAN VOLUME POTENTIAL FOR THE HELMHOLTZ EQUATION IN UNBOUNDED DOMAINS

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Abstract. We study properties of Newtonian volume potential for the Helmholtz operator in unbounded three-dimensional domains and establish that if the density function decays at infinity as $\mathcal{O}(|x|^{-m})$ with $m > 4$ for sufficiently large $|x|$, then the volume potential satisfies the Sommerfeld radiation conditions at infinity.

Keywords and phrases: Helmholtz equation, sommerfeld radiation conditions, volume potential.

AMS subject classification (2010): 35J05, 35J15, 78A40, 78A45.

Introduction

In the wave scattering theory there often arise problems when the unknown function u satisfies the nonhomogeneous Helmholtz equation in an unbounded domain:

$$\Delta u(x) + \omega^2 u(x) = \Phi(x), \quad x \in \Omega^-, \tag{1}$$

where $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$ is the Laplace operator, Ω^- is a three-dimensional unbounded complement of a bounded domain $\Omega^+ \subset \mathbb{R}^3$ with the simply connected smooth boundary S: $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\overline{\Omega^-} = \Omega^- \cup S$, $\omega \in \mathbb{R}$ is the so called frequency parameter, and Φ is a known function with noncompact support. Without loss of generality, throughout the paper we assume that Φ is defined in the whole space $\mathbb{R}^3$. The smoothness and asymptotic properties of Φ will be specified below.

In the exterior boundary value problems along with the differential equation (1) the unknown function satisfies one of the following boundary conditions:

the Dirichlet condition:

$$\{u(x)\}^- = f_D(x) \quad \text{on} \quad S, \tag{2}$$

the Neumann condition:

$$\{\partial_n u(x)\}^- = f_N(x) \quad \text{on} \quad S, \tag{3}$$

the Robin condition:

$$\{\partial_n u(x) + \alpha(x) u(x)\}^- = f_R(x) \quad \text{on} \quad S, \tag{4}$$

or the mixed boundary condition when the surface S is divided into three disjoint sub-manifolds, $S = \overline{S}_D \cup \overline{S}_N \cup \overline{S}_R$ and on each part there are prescribed the corresponding Dirichlet, Neumann, or Robin type boundary conditions [9], [3], [5].

Moreover, the solution has to satisfy the Sommerfeld radiation conditions at infinity ([12], [11], [1], [2]).

$$u(x) = \mathcal{O}(r^{-1}) \tag{5}$$

$$\frac{\partial u(x)}{\partial r} - i\,\omega\,u(x) = o(r^{-1}), \quad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{6}$$

To investigate the above formulated boundary value problems for the non-homogeneous Helmholtz equation (1) by reduction to the homogeneous Helmholtz equation one needs to find a particular solution satisfying the Sommerfeld radiation conditions at infinity. For arbitrary smooth function Φ, satisfying some decay conditions at infinity, a particular solution to equation (1) can be represented by the Newtonian volume potential [7], [8], [10],

$$N_{\Omega^-}(\Phi)(y) := -\frac{1}{4\pi}\int_{\Omega^-} \frac{e^{i\,\omega\,|x-y|}}{|x-y|}\,\Phi(x)\,dx, \quad y \in \Omega^-, \tag{7}$$

where

$$\Gamma(x-y,\omega) := -\frac{1}{4\pi}\frac{e^{i\,\omega\,|x-y|}}{|x-y|} \tag{8}$$

is the fundamental solution of the Helmholtz operator, i.e., $(\Delta + \omega^2)\,\Gamma(x - y,\omega) = \delta(x-y)$ with $\delta(\cdot)$ being the Dirac distribution.

Our main goal in this paper is to show that the Newtonian potential (7) under some conditions for the density function Φ satisfies the Sommerfeld radiation conditions. In particular, the following assertion holds.

Theorem 1. *Let Φ be a continuous function in $\mathbb{R}^3$, $\Phi \in C(\mathbb{R}^3)$, and satisfy the following estimate*

$$|\Phi(x)| \leqslant \frac{A}{(1+|x|)^m}, \quad x \in \mathbb{R}^3, \tag{9}$$

with $m > 4$ and a positive constant A.

Then the Newtonian volume potential associated with the Helmholtz operator $\Delta + \omega^2$,

$$N(\Phi)(y) \equiv N_{\mathbb{R}^3}(\Phi)(y) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i\,\omega\,|x-y|}}{|x-y|}\,\Phi(x)\,dx, \quad y \in \mathbb{R}^3, \tag{10}$$

satisfies the Sommerfeld radiation conditions, i.e., for sufficiently large $|y|$

$$N(\Phi)(y) = \mathcal{O}(|y|^{-1}), \tag{11}$$

$$\frac{\partial N(\Phi)(y)}{\partial |y|} - i\,\omega\,N(\Phi)(y) = \mathcal{O}(|y|^{-2}). \tag{12}$$

$$\left(\frac{\partial}{\partial\,|y|} \equiv \frac{y_k}{|y|} \frac{\partial}{\partial y_k} \right)$$

Proof. First we prove the relation (12). To this end, recall that

$$\frac{\partial}{\partial|y|} \equiv \frac{y_k}{|y|} \frac{\partial}{\partial y_k}, \tag{13}$$

where summation over repeated entices is meant from 1 to 3, and denote

$$K(x,y) := \left(\frac{\partial}{\partial|y|} - i\,\omega \right) \frac{e^{i\,\omega\,|x-y|}}{|x-y|} = \left(\frac{y_k}{|y|} \frac{\partial}{\partial y_k} - i\,\omega \right) \frac{e^{i\,\omega|x-y|}}{|x-y|} \tag{14}$$

$$= \left[-\frac{y_k}{|y|} \frac{y_k - x_k}{|x-y|^3} + i\,\omega \frac{y_k}{|y|} \frac{y_k - x_k}{|x-y|^2} - \frac{i\,\omega}{|x-y|} \right] e^{i\,\omega\,|x-y|} \tag{15}$$

$$= \left[\frac{(x\cdot y) - |y|^2}{|y||x-y|^3} - \frac{i\,\omega\,(x\cdot y)}{|y||x-y|^2} + i\,\omega \left(\frac{|y|}{|x-y|^2} - \frac{1}{|x-y|} \right) \right] e^{i\,\omega\,|x-y|}$$

$$= \left[\frac{(x\cdot y) - |y|^2}{|y||x-y|^3} - \frac{i\,\omega\,(x\cdot y)}{|y||x-y|^2} + \frac{i\,\omega\,\big(2(x\cdot y) - |x|^2\big)}{|x-y|^2(|y| + |x-y|)} \right] e^{i\,\omega\,|x-y|}\,. \tag{16}$$

By simple calculations we derive

$$\frac{\partial N(\Phi)(y)}{\partial|y|} - i\,\omega\,N(\Phi)(y) = -\frac{1}{4\pi} \int_{\mathbb{R}^{\not\models}} K(x,y)\Phi(x)\,dx = -\frac{1}{4\pi} \sum_{j=1}^{4} I_j(y), \tag{17}$$

where

$$I_j(y) = -\frac{1}{4\pi} \int_{\Omega_j} K(x,y)\Phi(x)dx, \quad y \in \mathbb{R}^3, \tag{18}$$

$$\Omega_1 = B\left(y, \frac{R}{2}\right), \quad \Omega_2 = B(0,R) \setminus \left\{ B\left(0, \frac{R}{2}\right) \cup B\left(y, \frac{R}{2}\right) \right\} \tag{19}$$

$$\Omega_3 = \mathbb{R}^3 \setminus \{B(0,R) \cup \Omega_1\}, \quad \Omega_4 = B\left(0, \frac{R}{2}\right), \quad R = |y|,. \tag{20}$$

We assume that $|y| = R$ is sufficiently large. Let us estimate the integrals with respect to $|y| = R$.

Introducing the spherical co-ordinates at the point y,

$$x_1 = y_1 + \rho\cos\varphi\,\sin\theta, \quad x_2 = y_2 + \rho\sin\varphi\,\sin\theta, \quad x_3 = y_3 + \rho\cos\theta,$$

and taking into account the relations (9), (15) and (18), we get

$$
\begin{aligned}
|I_1(y)| &\leqslant C_1 \int_{\Omega_1} \left[\frac{1}{|x-y|^2} + \frac{1}{|x-y|}\right] |\Phi(x)|\, dx \\
&\leqslant C_1 \sup_{\Omega_1} |\Phi(x)| \int_{B\left(y,\frac{R}{2}\right)} \left[\frac{1}{|x-y|^2} + \frac{1}{|x-y|}\right] dx \\
&\leqslant C_1 \sup_{\Omega_1} |\Phi(x)| \int_0^{\frac{R}{2}} \int_0^{2\pi} \int_0^{\pi} \left[\frac{1}{\rho^2} + \frac{1}{\rho}\right] \rho^2 \sin\theta \, d\theta \, d\varphi \, d\rho \leqslant \frac{C_2 R^2}{(1+R)^m}.
\end{aligned}
$$

Here we employed the fact that if $x \in \Omega_1$, then $|x| \geq \frac{R}{2}$.
Thus, there is a positive constant C such that

$$
|I_1(y)| \leqslant C\, R^{2-m} = C\, |y|^{2-m} \tag{21}
$$

for sufficiently large $|y|$, $|y| = R \gg 1$.
Quite similarly, using the fact that the inclusion $x \in \Omega_2$ implies the inequalities $|x-y| \geqslant \frac{R}{2}$ and $\frac{R}{2} \leqslant |x| \leqslant R$, we get

$$
\begin{aligned}
|I_2(y)| &\leqslant C_1 \int_{\Omega_2} \left[\frac{1}{|x-y|^2} + \frac{1}{|x-y|}\right] |\Phi(x)|\, dx \\
&\leqslant \frac{C_2}{R} \sup_{\frac{R}{2} \leqslant |x| \leqslant R} |\Phi(x)| \int_{\Omega_2} dx \leqslant \frac{C_3 R^2}{(1+R)^m}\,,
\end{aligned}
$$

i.e.

$$
|I_2(y)| \leqslant C\, R^{2-m} = C\, |y|^{2-m} \tag{22}
$$

Analogously, since the inclusion $x \in \Omega_3$ implies $|x-y| \geqslant \frac{R}{2}$ and $|x| \geqslant R$, we have the following estimate

$$
\begin{aligned}
|I_3(y)| &\leqslant C_1 \int_{\Omega_3} \left[\frac{1}{|x-y|^2} + \frac{1}{|x-y|}\right] |\Phi(x)|\, dx \\
&\leqslant \frac{C_2}{R} \int_{|x| \geqslant R} |\Phi(x)|\, dx \leqslant \frac{C_3}{R} \int_R^{+\infty} \frac{\rho^2 d\rho}{\rho^m} \leqslant \frac{C_3}{m-3}\, R^{2-m},
\end{aligned}
$$

i.e.

$$
|I_3(y)| \leqslant C\, R^{2-m} = C\, |y|^{2-m} \tag{23}
$$

with some positive constant C.

Finally, we use that $|x-y| \geqslant \frac{R}{2}$ for $x \in \Omega_4$ and estimate the integral

$I_4\,(y)$ with the help of the representation (16)

$$|I_4(y)| \leqslant \int_{\Omega_4} |K(x,y)|\,|\Phi(x)|\,dx$$

$$\leqslant C_1 \int_{|x|\leq \frac{R}{2}} \left(\frac{|x|}{R^3} + \frac{1}{R^2} + \frac{|x|}{R^2} + \frac{|x|}{R^2} + \frac{|x|^2}{R^3} \right) |\Phi(x)|\,dx$$

$$\leqslant \frac{C_2}{R^2} \int_{|x|\leqslant \frac{R}{2}} (1+|x|)\,|\Phi(x)|\,dx \leqslant \frac{C_2}{R^2} \int_{\mathbb{R}^{\nvDash}} (1+|x|)\,|\Phi(x)|\,dx$$

$$\leqslant \frac{C_3}{R^2} \int_{\mathbb{R}^{\nvDash}} \frac{dx}{(1+|x|)^{m-1}}$$

Now, since $m > 4$ we deduce

$$|I_4(y)| \leqslant C\,R^{-2} = C\,|y|^{-2} \tag{24}$$

with a positive constant C.
From inequalities (22)-(24) the relation (12) follows.
The proof of inequality (11) is word for word. ■

Corollary 2. Let the conditions of Theorem 1 be satisfied and, in addition, let Φ be a locally Hölder continuous function in Ω^-, $\Phi \in C^{\,0,\alpha}(\Omega^-)$, i.e., there are positive constants B and $0 < \alpha \leqslant 1$, such that

$$|\Phi(x) - \Phi(y)| \leqslant B|x-y|^{\alpha}, \quad x, y \in \Omega^-, \quad |x-y| \leqslant 1.$$

Then the Newtonian volume potential defined by (10) has a Holder continuous second order derivatives in Ω^-, $N(\Phi) \in C^{\,0,\alpha}(\Omega^-)$, and represents a classical radiating solution of the non-homogeneous Helmholtz equation (1).

Proof. It should be mentioned that the second order partial derivatives of the Newtonian potential (1) are represented by singular integrals and the assertion immediately follows form Theorem 1 and the properties of volume potentials with Hölder continuous densities and with bounded integration domains (see e.g. [4], [9], [6]).

Remark 3. If one looks for a radiating solution u of the above formulated boundary value problems for the nonhomogeneous equation (1) in the form

$$u(x) = v(x) + N(\Phi)(x), \tag{25}$$

where v is a new unknown radiating function, then it is evident that, due to Theorem 1 and Corollary 2 the problems are reduced to the similar boundary value problems for the homogeneous Helmholtz equation:

$$\Delta v(x) + \omega^2 v(x) = 0. \tag{26}$$

Now the radiating function v satisfies the same type Dirichlet, Neumann, Robin, or mixed boundary conditions but with different boundary functions involving the boundary values of the Newtonian potential and its derivatives, e.g., in the case of Dirichlet and Neumann problems the boundary conditions (2) and (3) will be rewritten as follows

$$\{v(x)\}^- = F_D(x) := f_D(x) + N(\Phi)(x), \quad x \in S, \tag{27}$$

and

$$\{\partial_n v(x)\}^- = F_N(x) := f_n(x) + \partial_n N(\Phi)(x), \quad x \in S, \tag{28}$$

These problems for homogeneous Helmholtz equation (26) are well studied in the scientific literature.

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Received 22.04.2014; revised 29.11.2014; accepted 1.12.2014.

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