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CROSS-RATIOS OF PORISIC QUADRILATERALS

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Abstract. We discuss the cross-ratio map of certain one-dimensional families of planar quadrilaterals. Three types of such families are considered. First, we recall our earlier results concerning the image of cross-ratio map for the family of configurations of a planar quadrilateral linkage with generic sidelengths. Next, we find the image of cross-ratio map for one-dimensional family of bicentric quadrilaterals with prescribed incircle and circumcircle. Finally, we obtain analogous results for families of Poncelet quadrilaterals associated with a pair of confocal ellipses.

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1. Introduction

We deal with quadrilaterals in Euclidean plane $\mathbb{R}^2$ with coordinates (x, y), identified with the complex plane $\mathbb{C}$ with coordinate $z = x + \imath y$. Given such a quadrilateral Q we define *cross-ratio of* Q as the cross-ratio of the four complex numbers representing its vertices in the prescribed order. Using complex numbers in the study of polygons has a long tradition (see, e.g., [1], [2]). We present several new developments concerned with the above notion of cross-ratio of quadrilateral.

The main aim of this paper is to investigate the values of cross-ratio in certain one-parameter families of planar quadrilaterals. Three types of such families are discussed: (1) planar moduli spaces of quadrilateral linkages [3], (2) bicentric quadrilaterals with the same inscribed and circumscribed circles, and (3) poristic quadrilaterals arising from Poncelet Porism [6] for a pair of confocal ellipses.

To begin with we discuss quadrilateral linkages (or 4-*bar mechanisms* [8]). In spite of the apparent simplicity of these objects, their study is related to several deep results of algebraic geometry and function theory, in particular, to the theory of elliptic functions and Poncelet Porism [5]. Comprehensive results on the geometry of planar 4-bar mechanisms are presented in [8]. Some recent results may be found in [5], [11].

We complement results of [8] and [11] by discussing several new aspects which emerged in course of our study of extremal problems on moduli spaces of polygonal linkages (cf. [10], [11], [12]). In this context it is natural to consider polygonal linkage as a purely mathematical object defined by a collection of positive numbers and investigate its moduli spaces [3]. In this paper we deal with quadrilateral linkages and their planar moduli spaces.

We start by recalling the definition and basic geometric properties of planar moduli spaces of quadrilateral linkages (Proposition 2.1). With a generic planar quadrilateral linkage Q one can associate the *cross-ratio map* Cr_Q from its planar moduli space $M(Q)$ into the extended complex plane $\overline{\mathbb{C}}$ (Riemann sphere). As was shown in [11] the image of cross-ratio map for a generic quadrilateral linkage can be explicitly described in terms of its sidelengths (Theorem 3.1 below). This result serves as a paradigm for the second part of the paper which deals with the investigation of cross-ratios the so-called *poristic* quadrilaterals arising from Poncelet Porism [6]. Our approach yields rather complete results for bicentric poristic quadrilaterals (Theorem 5.3) and poristic quadrilaterals associated with confocal ellipses (Theorem 6.4).

In conclusion we mention several possible generalizations and research perspectives suggested by our results.

2. Moduli spaces of planar quadrilateral linkages

We freely use some notions and constructions from the mathematical theory of linkages, in particular, the concept of *planar moduli space* of a polygonal linkage [3]. Recall that *closed n-lateral linkage* $L(l)$ is defined by n-tuple l of positive real numbers l_j (*sidelengths* of L) such that the biggest of sidelengths does not exceed the sum of remaining ones. The latter condition guarantees the existence of a n-gon in Euclidean plane $\mathbb{R}^2$ with the lengths of the sides equal to numbers l_j. Each such polygon is called a *planar realization* of linkage $L(l)$.

Linkage with a *telescopic side* is defined similarly but now the last sidelength l_n is allowed to take any positive value. For brevity we will distinguish these two cases by speaking of closed and open linkages.

For a closed or open linkage L, its planar configuration space $M(L) = M_2(L)$ is defined as the set of its planar realizations (configurations) taken modulo the group of orientation preserving isometries of $\mathbb{R}^2$ [3]. It is easy to see that moduli spaces $M(L)$ have natural structures of compact real algebraic varieties. For an open n-linkage its planar moduli space is diffeomorphic to $(n-2)$-dimensional torus T^{n-2}. For a closed n-linkage with a *generic* sidelength vector l, its planar moduli space is a smooth compact $(n-3)$-dimensional manifold. As usual, here and below the term "generic" means "for an open dense subset of parameter space" (in our setting, this is the space $\mathbb{R}^n_+$ of sidelengths).

In particular, a closed 4-bar linkage $Q = Q(l)$ is defined by a quadruple of positive numbers $l = (a, b, c, d) \in \mathbb{R}^4_+$. An open planar 4-linkage (or planar robot 3-arm) is analogously defined by a triple of positive numbers $l \in \mathbb{R}^3_+$ and its planar moduli space is diffeomorphic to two-torus T^2. The complete list of possible topological types of planar moduli spaces of closed 4-linkages is also well known (see, e.g., [9]).

Proposition 2.1. *The complete list of homeomorphy types of planar moduli spaces of quadrilateral linkages is as follows: circle, disjoint union of two circles, bouquet of two circles, two circles with two common points, three circles with pairwise intersections equal to one point.*

Closed linkages with smooth moduli spaces are called non-degenerate. It is easy to show that non-degeneracy is a generic condition. In the sequel we basically deal with non-degenerate quadrilateral linkages.

3. Cross-ratio map of quadrilateral linkage

In this section we use some basic properties of cross-ratio which can be found in [2]. Recall that the complex cross-ratio of four points $p, q, z, w \in \mathbb{C}$ is defined as

$$[p, q; z, w] = \frac{z-p}{z-q} : \frac{w-p}{w-q} = \frac{p-z}{p-w}\frac{q-z}{q-w}. \tag{1}$$

Group S_4 acts by permuting points so one can obtain up to six values of the cross-ratio for a given unordered quadruple of points which are related by well-known relations [2]. For further use notice also the value of cross-ratio is real if and only if the four points lie on the same circle of a straight line [2].

Consider now a quadrilateral linkage $Q = Q(a, b, c, d)$ with smooth $M(Q)$ (i.e., without aligned configurations) and assume moreover that side-lengths are pairwise different so that Q has no configurations with coinciding vertices. Then, for each planar configuration $V = (v_1, v_2, v_3, v_4) \in \mathbb{C}^4$ of Q, put

$$Cr(V) = \mathrm{Cr}((\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4)) = [\mathrm{v}_1, \mathrm{v}_2; \mathrm{v}_3, \mathrm{v}_4] = \frac{\mathrm{v}_3 - \mathrm{v}_1}{\mathrm{v}_3 - \mathrm{v}_2} : \frac{\mathrm{v}_4 - \mathrm{v}_1}{\mathrm{v}_4 - \mathrm{v}_2}. \tag{2}$$

This obviously defines a continuous (actually, real-analytic) mapping $Cr_Q : M(Q) \to \mathbb{C}$. Our main aim in this section is to describe its image $\Gamma_Q = \mathrm{Im}\,\mathrm{Cr_Q}$ which is obviously a continuous curve in $\mathbb{C}$.

To formulate the main result of [11] it is technically more convenient to work with another map $R : M(Q) \to \mathbb{C}$ defined by the formula

$$R(v_1, v_2, v_3, v_4) = Cr(v_1, v_3, v_2, v_4) = [v_1, v_3; v_2, v_4]. \tag{3}$$

From the transformation properties of cross-ratio follows that

$$Cr(V) = 1 + R(V).$$

So the properties of Cr can be immediately derived from the properties of R. For brevity we will call R the *uniformizer* of Q.

The main advantage of map R is that, for any configuration V of closed linkage Q, the moduli of numbers $v_{i+1} - v_i$ are constant by its very definition. Consequently, for any $V \in M(Q)$, one has $|R(V)| = \frac{ac}{bd}$. In other words, the R maps $M(Q)$ into the circle of radius $\frac{ac}{bd}$ with the center at point $0 \in \mathbb{C}$. Moreover, by elementary geometric considerations it follows that the argument of complex number $R(V)$ is equal to $\alpha + \gamma$, where α and γ are the angles at vertices v_1 and v_3 of configuration V.

These observations enabled us to get a very precise description of $\mathrm{Im}\,\mathrm{R}$ proof of which can be found in [11].

Theorem 3.1 *For a quadrilateral linkage Q as above, the following statements hold:*

(1) the image $\mathrm{Im\,R}$ *is a subset of the circle of radius ac/bd centered at the point $0 \in \mathbb{C}$;*

(2) the image $\mathrm{Im\,R}$ *is connected and symmetric about the real axis containing the point $\frac{ac}{bd}$;*

(3) R is surjective if and only if $(a+b-c+d)(a-b+c-d)(a-b-c+d) \leq 0$.

Corollary 3.2. The image of Cr is a conjugation-invariant arc of the circle of radius ac/bd centered at the point $1 \in \mathbb{C}$.

This is immediate in view of the relation between Cr and R.

Corollary 3.3. Cross-ratio map of Q is surjective if and only if Q has a self-intersecting cyclic configuration.

This follows from the above proof since the argument of cross-ratio of self-intersecting cyclic configuration is equal to π.

Corollary 3.4. Cross-ratio map of Q is surjective if and only if its planar moduli space has two components. In other words, surjectivity of cross-ratio map is a topological property.

Indeed, it was shown in [10] that a self-intersecting cyclic configuration exists if and only if the moduli space has two components. Notice that these observations yield a simple criterion of connectedness of the moduli space.

Corollary 3.5. The moduli space is connected if and only if

$$(a+b-c+d)(a-b+c-d)(a-b-c+d) \leq 0.$$

If linkage $Q(a,b,c,d)$ is not generic then the description of Γ_Q looks a bit different as can be seen in the case of rhomboid linkage.

Example. Let $Q(1)$ denote the rhomboid planar linkage with sidelength vector $(1,1,1,1)$. Here, in addition to the two cyclic configurations of $Q(1)$ at which $Cr(V)$ has the same real value 1, one should take into account also the three aligned configurations $Q(1)$ at which the value of $Cr(V)$ is also real. Thus it becomes necessary to examine the image and multiplicities of its real points more carefully. In this case this can be done in a quite elementary way.

Recall that $M(Q)$ is homeomorphic to a union of three circles S^1 each pair of which has one common point [3]. Since each of these circles admits a simple parametrization we can investigate the issue by elementary computations. Let us assume that $v_1 = 0, v_2 = 1$ and consider first the part X of $M(Q(1))$ on which the second and fourth sides of $Q(1)$ remain parallel. As a (local) coordinate ϕ on X one can take the angle between positive real semi-axis and the second side. Then we have $v_3 = 1 + cos\,\phi + \imath sin\,\phi, v_4 = cos\,\phi + \imath sin\,\phi$ and we get $Cr(V(\phi)) = 1 - (cos\,\phi - \imath sin\,\phi)^2$.

This shows that cross-ratio is a surjective mapping from X onto the circle $C(1,1)$. The above formula also shows that Cr changes orientation and its mapping degree is equal to two. Similar considerations for the two other components of $M(Q(1))$ show that $Cr_Q(1)$ is surjective and each of the two real points in its image is covered 4 times.

Analogous results can be obtained for kites and degenerate quadrilaterals of other types. Notice that the image of R for a degenerate quadrilateral with $a = b+c+d$ reduces to a point. We omit a straightforward case-by-case analysis of degenerate cases and proceed by performing similar considerations in the context of *poristic quadrilaterals* introduced below

4. Poncelet theorem and poristic quadrilaterals

We are now going to obtain analogs of Theorem 3.1 in another setting concerned with families of quadrilaterals arising in connection with Poncelet Porism [6]. To give a precise description of this setting we begin with a few definitions and remarks.

Recall that Poncelet Porism states that if a pair of ellipses (E_1, E_2) is such that there exists a k-gon inscribed in E_1 and circumscribed about E_2, then, for each point p of E_1, there exists such a k-gon having p as its vertex (see, e.g., [2], [6]).

Definition 4.1. A pair of ellipses (E_1, E_2) such that there exists a k-gon inscribed in E_1 and circumscribed about E_2 is called a *Poncelet pair of ellipses of order k*. The set of all such k-gons is called the *poristic family of k-gons* $\mathcal{P}(\mathcal{E}_\infty, \mathcal{E}_\in)$ associated with (E_1, E_2).

An interesting and well-understood special case arises when both ellipses are circles.

Definition 4.2. A polygon P in the plane is called *bicentric* if there exist two circles C_1, C_2 with C_2 strictly inside C_1, such that all vertices of P lie on C_1 and each side of P is tangent to C_2 at a certain inner point of this side. The pair of circles (C_1, C_2) is called the *frame* of bicentric polygon P. Their centers O_1 and O_2 are called the *circumcenter* and *incenter* of P respectively. The family of all polygons having the same frame is called the poristic family of bicentric polygons associated with (C_1, C_2).

Thus if P is a bicentric k-gon with the frame (C_1, C_2) then there exists a whole one-dimensional family of bicentric k-gons P_t with the same frame. In fact, each point of C_1 is a vertex of such a bicentric k-gon. For example, each triangle $\triangle$ is bicentric with C_1 being the circumscribed circle (circumcircle) of $\triangle$ and C_2 the inscribed circle (incircle) of $\triangle$. Each regular polygon is obviously bicentric. Notice that this definition does not require that P is convex. So any regular *star-shaped* polygon is also bicentric.

Many results on bicentric polygons can be found in the literature. The first detailed paper on properties of bicentric n-gons was published by N.Fuss [7]. For this reason, a pair of circles constituting the frame of a bicentric k-gon will be called a *Fuss pair (of circles) of order k*. Up to a motion of the plane, a Fuss pair of circles is completely determined by a triple of non-negative numbers (R, r, d), where $R > 0$ is the radius of circumcircle, $r > 0$ is the radius of incircle, and $d \geq 0$ is the distance between the incenter and circumcenter. This triple will be called the *gauge* of Fuss pair.

It is well-known that the gauge (R, r, d) of a Fuss pair of order k satisfies an algebraic relation. For $k = 3$, it is the classical *Euler triangle formula* [2]: $R^2 - d^2 = 2Rr$. For $k \geq 4$, this relation is called *Fuss's relation* and reads:

$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}. \tag{4}$$

We will only deal with poristic bicentric quadrilaterals and investigate the cross-ratios of their vertices. Since each poristic bicentric polygon is cyclic the cross-ratio of its vertices is a real number. We explicitly calculate the image of arising cross-ratio map in terms of the gauge (R, r, d).

Another special case of Poncelet Porism fitting the context of this paper arises from the Poncelet pair of confocal ellipses of order four. In this situation we relate the image of cross-ratio map for the family $\mathcal{P}(\mathcal{E}_\infty, \mathcal{E}_\in)$ to the semi-axes of ellipses.

5. Cross-ratios of poristic bicentric quadrilaterals

To obtain an analog of Theorem 3.1 for bicentric quadrilaterals we need two lemmas. The first lemma is valid for any pair of circles with the gauge (R, r, d) and follows by elementary geometric considerations.

Lemma 5.1 *Let t_1 be the length of tangent from point $A \in C_1$ to C_2 and t_2 be the length of the segment between the point of tangency and the second point of intersection with C_2 then*

$$t_2 = \frac{2Rrt_1 + \sqrt{d}}{r^2 + t_1^2}, \text{where } \mathrm{d} = 4\mathrm{R}^2\mathrm{r}^2\mathrm{t}_1^2 - \mathrm{r}^2(\mathrm{r}^2 + \mathrm{t}_1^2)(4\mathrm{Rr} + \mathrm{r}^2 + \mathrm{t}_1^2).$$

Let ABCD be a poristic quadrilateral associated with a Fuss pair (C_1, C_2) of order four with the gauge (R, r, d). Using Lemma 5.1 and Fuss's relation (4) we can conveniently express the lengths of the sides of ABCD as follows.

Lemma 5.2 *Let $t = t_1$ be the length of tangent from point A to C_2. Then the length of the sides of $ABCD$ are given by:*

$$|AB| = t_1 + t_2, |BC| = t_2 + \frac{r^2}{t_1}, |CD| = \frac{r^2}{t_1} + \frac{r^2}{t_2}, |DA| = t_1 + \frac{r^2}{t_2},$$

where

$$t_2 = \frac{(R^2 - r^2)t_1 + \sqrt{D}}{r^2 + t_1^2}, D = (R^2 - d^2)t_1^2 - r^2(r^2 + t_1^2)^2.$$

Lemma 5.2 directly leads to the main result of this section.

Theorem 5.3 *For a Fuss pair of circles of order four with the gauge (R, r, d), the cross-ratios of associated poristic quadrilaterals fill the segment $[-\rho, -\frac{1}{\rho}]$, where*

$$\rho = \frac{2R^2 + 2r^2 - 2d^2 - 2\sqrt{R^2 - (r-d)^2}\sqrt{R^2 - (r+d)^2}}{4\sqrt{R^2 - (r-d)^2}\sqrt{R^2 - (r+d)^2}}.$$

Proof. Since the sum of opposite angles equals π for each poristic quadrilateral, its cross-ratio is a negative real number. Thus the image of cross-ratio is a closed subinterval of $(-\infty, 0)$. To find the ends of this interval it is sufficient to find the maximum and minimum of the ratio ac/bd of products of opposite sides. It is convenient to parameterize poristic quadrilateral by the length t of the tangent to the inner circle from the first vertex. Such quadrilateral will be denoted Q_t. It is easy to see that the range of values of such t is $[\sqrt{(R-d)^2 - r^2}, \sqrt{(R+d)^2 - r^2}]$. By Lemma 5.2 we can write the ratio ab/cd as an explicit function of t and it remains to find its extremal values on this segment. The proof can now be completed by elementary but lengthy computations which are omitted. ■

Notice that planar moduli spaces of bicentric quadrilaterals are always singular because of the well-known relation between the sidelengths of tangential quadrilateral. In fact, it is easy to list all possible topological types of planar moduli spaces.

Proposition 5.4. *The planar moduli space of bicentric quadrilateral is homeomorphic either to bouquet of two circles or to a union of two circles having three different points in common.*

6. Cross-ratios of poristic quadrilaterals for confocal ellipses

Our aim now is to obtain an analogous result for poristic quadrilaterals associated with confocal ellipses. So let (E_1, E_2) be a pair of confocal ellipses. Without loss of generality we can assume that their equations are given in canonical form as

$$E_1 = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}, E_2 = \left\{\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1\right\},$$

where $a^2 - b^2 = c^2 - d^2$ (as is well known, the latter equality expresses the confocality condition).

Let us now assume that (E_1, E_2) is a Poncelet pair of order 4. We obtain first a relation between parameters (a, b, c, d) which can be considered as an analog of Fuss's relation. To this end notice that, by symmetry, the Poncelet quadrilateral with the first vertex $(a, 0)$ is a rhombus with vertices $(a, 0), (0, b), (-a, 0), (0, -b)$. Hence the line $L_1 = \{(x, y) : (a-x)b - ay = 0\}$ connecting vertices $(a, 0)$ and $(0, b)$ should be tangent to E_2 at certain point $p_1 = (s, t)$. In other words, the intersection of L_1 and E_2 should consist of one point.

The system of two equations defining the coordinates of p_1 is:

$$(a - s)b - at = 0, \frac{s^2}{c^2} + \frac{t^2}{d^2} = 1,$$

and we need to find the condition that it has exactly one real solution. Using the linearity of the first equation we reduce this system to one quadratic equation with real coefficients and then write down the relation between

(a, b, c, d) which expresses vanishing of the discriminant of the latter equation. Combining this relation with the confocality condition we obtain the sought analog of Fuss's fourth relation.

Reduction to quadratic equation is obtained by substituting $t = \frac{(a-s)b}{a}$ into equation of E_2 which gives the following quadratic equation on s:

$$(a^2d^2 + b^2c^2)s^2 - 2ab^2c^2x + a^2c^2(b^2 - d^2) = 0.$$

Vanishing of its discriminant gives the following relation between (a, b, c, d):

$$a^2b^2c^4 - a^2c^2(a^2d^2 + b^2c^2)(b^2 - d^2) = 0.$$

Combining this condition with the condition of confocality and excluding c from the arising system we get the equation for d of the form: $(a^2 + b^2)d^4 - b^4d^2 = 0$, hence $d^2 = \frac{b^4}{a^2+b^2}$. From confocality condition we also get $c^2 = \frac{a^4}{a^2+b^2}$, which gives the following result.

Proposition 6.1. *The semi-axes of a pair of confocal Poncelet ellipses of order four satisfy the following equations:*

$$c = \frac{a^2}{\sqrt{a^2 + b^2}}, \quad d = \frac{b^2}{\sqrt{a^2 + b^2}}. \tag{5}$$

In fact, these equations give also a sufficient condition.

Proposition 6.2. *If equations (5) are satisfied for a pair of confocal ellipses, then these ellipses form a Poncelet pair of order four.*

Indeed, under these conditions the rectangular with vertices

$$(u, \sqrt{b^2(1 - \frac{u^2}{a^2})}\,), (-u, \sqrt{b^2(1 - \frac{u^2}{a^2})}\,),$$

$$(-u, -\sqrt{b^2(1 - \frac{u^2}{a^2})}\,), (u, -\sqrt{b^2(1 - \frac{u^2}{a^2})}\,)$$

is tangent to the inner ellipse. Hence it is a Poncelet quadrilateral for (E_1, E_2).

These two propositions show that equations (5) are analogs of Fuss's relation. In fact, it is possible to rewrite this result in a form more similar to Fuss's relation. To this end notice that, up to a motion of the plane, a pair of confocal ellipses is defined by three positive numbers $2c$ (distance between foci), $L > 2c$ (sum of distances to foci for E_1) and $l \in (2c, L)$ (sum of distances to foci for E_2). Since the sum of distances to foci is equal to doubled big semi-axis, equations (5) give the following relation between (c, L, l):

$$L^4 - L^2l^2 + 2c^2l^2 = 0$$

which may be considered as a direct analog of Fuss's relation.

Unlike the case of bicentric quadrilaterals the sidelengths of poristic quadrilaterals do not satisfy the $a+c = b+d$ relation. Nevertheless it turns out that their moduli spaces are always singular and one has an analog of Proposition 5.4. To show this we refer to the results of [4] about Poncelet quadrilaterals of confocal ellipses. In particular, it was proved in [4] that in this case all poristic quadrilaterals are parallelograms. This implies that their sidelengths do not satisfy Grashof condition. Hence their planar moduli spaces are singular and one may again use the description of singular moduli spaces given in Proposition 2.1. It is easy to verify that in this case there appear only two possible topological types of moduli spaces.

Proposition 6.3. *For a pair of confocal Poncelet ellipses of order four, all moduli spaces of poristic quadrilaterals are singular. In each such family we have two homeomorphy types of planar moduli space: bouquet of two circles and union of two circles having three different points in common.*

It is also known that in this situation the perimeter of poristic quadrilaterals $p(t)$ is constant and equal to $L = 4\sqrt{a^2+b^2}$ [4] (which according to [2] is four times the radius of the orthoptic circle of E_1). Let P_x be a poristic quadrilateral which is a parallelogram with sidelengths x and y. Hence the modulus ρ of its cross-ratio equals $x^2/y^2 = x^2/(L-x)^2$. From (6.1) it easily follows that maximum of x equals $2\frac{a^2}{\sqrt{a^2+b^2}}$ and its minimum equals $2\frac{b^2}{\sqrt{a^2+b^2}}$. Hence to find the range of value of modulus ρ of cross-ratio we just need to find the extremal values of function $x^2/(L-x)^2$ in this interval. Since this function is monotonically increasing on this interval it follows that the maximum of ρ is achieved at the rectangular

$$(c, \frac{b}{\sqrt{a^2+b^2}}), (-c, \frac{b}{\sqrt{a^2+b^2}}), (-c, -\frac{b}{\sqrt{a^2+b^2}}), (c, -\frac{b}{\sqrt{a^2+b^2}})$$

and equals $\frac{a^4}{b^4}$. Hence the minimum of ρ equals $\frac{b^4}{a^4}$.

Notice also that the cross-ratio is real only for the above rectangular. Hence we conclude that the image of cross-ratio map intersects the real axis only at the points $-\frac{a^4}{b^4}$ and $-\frac{b^4}{a^4}$. These observations together with the symmetry and connectedness of the image lead to the following result.

Theorem 6.4. *For a pair of confocal ellipses satisfying equations (5), the cross-ratios of associated poristic quadrilaterals form a smooth complex-conjugation invariant loop X passing through the points $-\frac{a^4}{b^4}$ and $-\frac{b^4}{a^4}$.*

In fact, X is a smooth algebraic loop so it would be interesting to find its explicit equation.

7. Concluding remarks

First of all, we wish to add that using stereographic projection one may introduce cross-ratio map for spherical quadrilaterals. The analogs of Theorems 3.1, 5.3 follow in a straightforward way.

It is also interesting to describe the change of cross-ratio under the action of the so-called Darboux transformation of quadrilateral linkage [5]. Taking into account a version of Poncelet Porism for quadrilateral linkages obtained in [5] one might hope to get certain insights concerning the arising discrete dynamical system in the image of cross-ratio map.

Next, one can also consider cross-ratios of families of quadrilaterals arising as the centers of circles of Steiner 4-chains [2] and try to describe the image of the corresponding cross-ratio map.

An analogous line of development arises in connection with the notion of *conformal modulus of quadrilateral* [1]. In particular, one can try describe the image and behaviour of conformal modulus for families of poristic bicentric polygons and confocal ellipses. Developments in this direction will be published elsewhere.

Finally, by a way of analogy with our Propositions 5.4 and 6.3, one can try to describe all possible topological types of planar moduli spaces of poristic n-gons for arbitrary n. Using the detailed results on the of planar moduli spaces of pentagonal linkages given in [9] one can solve this problem for $n = 5$ but in general it might appear quite hard.

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