# ON THE ALGORITHMIC AND NON ALGORITHMIC SOLVABLE PROBLEMS FROM QUANTUM COMPUTING POINT OF VIEW

Giorgadze G., Makhaldiani N.

**Abstract**. Quantum computations can be implemented not only by the action of quantum circuits, but by the adiabatic evolution of a system's Hamiltonian. Quantum adiabatic statement allows to solve some classically non algorithmic problems. Our reasoning favor of this argument.

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In 1994, Shor [1] described a quantum algorithm to decompose a number in its prime factors more efficiently than any classical algorithm. It was exponentially faster than the best known classical counterpart. In 2001 the experimental development of this algorithm has had a very interesting advance: Vandersypen et al. [2] using a seven-qubit molecule manipulated with nuclear magnetic resonance techniques has reported the factorization of the number 15 into its prime factors 3 and 5. This algorithm illustrates a part of the theoretical challenge of quantum computation, i.e., to learn how to work with quantum properties to obtain more efficient algorithms. Tools such as quantum parallelism, unitary transformations, amplification techniques, interference phenomena, quantum measurements, resonances, etc., must be used by the new computation science [3-6].

In 2001 the Tien D. Kieu (see [7]) announced that a quantum computing procedure could solve a classically unsolvable problem, namely Hilberts Tenth Problem. This problem, that of deciding whether a polynomial integer-valued function of integers ever vanishes, is essentially equivalent to the standard halting problem and other such unsolvable problems as defined by Turing machines.

The main ingredient in the arguments Kieu is the adiabatic approach to quantum computation. It means that used adiabatic paradigm for construct quantum computational proccesing. By adiabatic theorem evolution of a system's Hamiltonian can be done by initializing the system into the ground state of a simple Hamiltonian, and then adiabatically evolving the Hamiltonian to one whose ground state encodes the solution to the problem. The time complexity of the problem, or more basically, the speed at which the Hamiltonian can be evolved adiabatically, is related to the separation between the energy eigenvalues. In some cases, such as Grover Search, the standard computation complexity can be recovered. Importantly, any quantum circuit can be simulated adiabatically.

By adiabatic paradigm the evolution of the quantum state is governed by a time-dependent Hamiltonian that interpolates between an initial Hamiltonian, whose ground state is easy to construct, and a final Hamiltonian, whose ground state encodes the satisfying assignment. To ensure that the system evolves to the desired final ground state, the evolution time must be big enough. The time required depends on the minimum energy difference between the two lowest states of the interpolating Hamiltonian.

**Interpolating Hamiltonian (IH).** Construction of the IH we base on the relation

$$H_1(n, s) = H_0(n)(1 - s) + sH_1(n).$$

Another choice of the Hamiltonian maybe

$$H_2(n, s) = \frac{n^{1-s} - 1}{1 - s}.$$

More precisely, for using Adiabatic theorem for computational problems we need:

I) to encode the solution of some problem P into the ground state of some suitable Hamiltonian, $H_P$ ;

II) to choose initial Hamiltonian, $H_I$, with readily obtainable ground state, $|g_I >$;

III) to deform $|g_I >$ through a process with the time depending Hamiltonian $H(t) = (1 - \frac{t}{T})H_I + \frac{t}{T}H_P$.

IV) If the deformation was sufficiently slow, then we get the desired ground state of $H_P$, $|g_P >$. Then we can compute the answer using obtained $|g_P >$.

**Diophantine equations.** Diophantine equations (DE)

$$D(n) \doteq D(n_1, ..., n_m) = 0,$$

are those equations that have integer solutions, $n \doteq (n_1, ..., n_m)$. In particular, they may be polynomials with integer coefficients.

**Quantization of DE.** For quantization of DE, (QDE), consider monomials

$$x^n \doteq x_1^{n_1}...x_m^{n_m}, \ n = n_1 + ... + n_m$$

than, for solutions of DE

$$0 = D(n)x^n = D(\delta)x^n = 0,$$

where

$$D(\delta) \doteq D(\delta_1, ..., \delta_n), \ \delta_k \doteq x_k \partial_{x_k}, \ k = 1, ..., m, \delta_k x^n = n_k x^n,$$

so, existence of the solution of DE is equivalent to existence of the eigenstate-monomials of the QDE.

**Kieus algorithm for Hilberts 10th problem.**

*Step 1.* For any given diophantine equation construct corresponding Hamiltonian $H_P$ , choose $H_I$ and find $|g_I>$;

*Step 2.* Run the adiabatic process for some time $T$;

*Step 3.* Measure the state $|f>$ obtained at T starting from $|g_I>$;

*Step 4.* Verify whether the state $|f>$ is the ground state of $H_P$. If not, we restart adiabatic evolution with new $T$.

**Harmonic oscillator.** Another formulation of QDE is given by quantum harmonic oscillator model. In the case of harmonic potential oscillator,

$$H = P^2/2 + X^2/2 = \hat{n} + 1/2$$

$$= aa^+ + a^+a, E_n = n + 1/2$$

$$X = x, P = -i\partial_x, \ [X, P] = i,$$

$$P = (a + a^+)/\sqrt{2}, \ X = i(a - a^+)/\sqrt{2},$$

$$a = (P - iX)/\sqrt{2}, a^+ = (P + iX)/\sqrt{2},$$

$$H(P, X)\psi_n(x) = E_n\psi_n(x), \ \int dx\psi_n(x)\psi_m(x) = \delta_{nm},$$

$$a\psi_0(x) = 0, \ \psi_0(x) = c\exp(-\frac{x^2}{2}), c = 1/\sqrt{\pi},$$

$$\psi_n(x) = c_n(a^+)^n\psi_0(x) = (\sqrt{\pi}a2^n n!)^{-1}\exp(-\frac{x^2}{2a^2})H_n(\frac{x}{a}),$$

$$H_n(z) = (-1)^n e^{z^2}(\frac{d^n}{dx^n}e^{-z^2}), \int_{-\infty}^{\infty} dz e^{-z^2}H_n(z)H_m(z) = \sqrt{\pi}2^n n!\delta_{nm}.$$

The Hermite polynomials, $H_n(z)$, can be defined also for fractal indexes

$$H_\alpha(z) = e^{i\pi\alpha}e^{z^2}(D^\alpha e^{-z^2}), \psi_\alpha(x) = (\sqrt{\pi}a2^\alpha\Gamma(\alpha+1))^{-1}\exp(-\frac{x^2}{2a^2})H_\alpha(\frac{x}{a})$$

**Quantum algorithm of solution of the QDE.** For quantum computing of the problem we introduce the following Hamiltonian operator,

$$H_P(\delta) = D(\delta)^2, \ H_P(\delta)x^n = H_P(n)x^n, H_P(n) = D(n)^2 \geq 0.$$

**Statement.** Quantum computer may algorithmically define if $H(n) = 0$ for some $n$. So, Quantum computer may algorithmically solve DE.

**Example.** As $H_P$ for the diophantine equation $D(x_1, ..., x_K) = 0$ we take $(D(a_1^\dagger a_1, ..., a_K^\dagger a_K))^2$. Then $H_P|n_1, ..., n_K> = D^2(n_1, ..., n_K)|n_1, ..., n_K>$ and we do not look for the zeroes of the polynomial $D(x_1, ..., x_K)$, which may not exist, but instead search for the absolute minimum of its square which exists, $0 \leq min(D(x_1, ..., x_K))^2 \leq (D(0, ..., 0))^2$, and is finite because

$$lim_{x\to\infty}(D(x_1, ..., x_K))^2$$

diverges. For example, if we have equation $(x+1)^3 + (y+1)^3 - (z+1)^3 = 0$ then $H_P = ((a_x^\dagger a_x + 1)^3 + (a_y^\dagger a_y + 1)^3 - (a_z^\dagger a_z + 1)^3)^2$ and $H_P |n_x, n_y, n_z> = ((n_x + 1)^3 + (n_y + 1)^3 - (n_z + 1)^3)^2 |n_x, n_y, n_z>$.

**Invertible Discrete Dynamical Systems (IDS).** For quantum computations it is crucial to have invertible dynamics [15]. Let us construct an example of IDS based on Hermite polynomials $H_n(z)$ :

$$H_n(z) = (-1)^n e^{z^2} \left( \frac{d^n}{dz^n} e^{-z^2} \right) = (2z - \partial_z)^n \cdot 1 = e^{-\frac{1}{4}\partial_z^2} (2z)^n$$

$$= n! \sum_{m=0}^{[n/2]} \frac{(-1)^m}{m!(n-2m)!(2z)^{n-2m}}$$

which fulfils the following recurrent relations

$$H_{n+1}(z) = 2z H_n(z) - 2n H_{n-1}(z),$$

from which we obtain

$$\Phi_{n+1}(z) = A(n,z)\Phi_n(z), \Phi_{n+1}(z) = \begin{pmatrix} H_{n+1} \\ H_{n+2} \end{pmatrix}, \Phi_n(z) = \begin{pmatrix} H_{n-1} \\ H_n \end{pmatrix},$$

$$A(n,z) = \begin{pmatrix} -2n & 2z \\ -4nz & 4z^2 - 2(n+1) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$det A = 2n2(n+1), A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} / det A$$

Note that, for $n = \alpha = -1/2$, $E_\alpha = 0$, $det A = -1$,

$$z^n = 2^{-n} e^{\frac{1}{4}\partial_z^2} H_n(z).$$

The Hermite polynomials form an orthogonal basis of the Hilbert space of functions satisfying

$$\int_{-\infty}^{\infty} w(z)|f(z)|^2 dz < \infty, \ w(z) = e^{-z^2},$$

in which the inner product is

$$< f|g> = \int_{-\infty}^{\infty} w(z) f(z) * g(z) dz.$$

The exponential generating function for Hermite polynomials is

$$G(h,z) = \sum_{n\geq 0} \frac{h^n}{n!} H_n(z) = e^{z^2} e^{-h\partial_z} e^{-z^2} = e^{-(z-h)^2}/e^{-z^2} = e^{h2z-h^2}.$$

**Quantum Field Theory Applications of the Formalism.** Quantum field theory (QFT) and Fractal calculus provide Universal language of fundamental physics (see e.g. [11]) In QFT existence of a given theory means,

that we can control its behavior at some scales by renormalization theory
[9]. If the theory exists, than we want to solve it, which means to deter-
mine what happens on other scales. This is the problem (and content) of
Renormdynamics. The result of the Renormdynamics, the solution of its
discrete or continual motion equations, is the effective QFT on a given scale
(different from the initial one).

Perturbation theory series have the following qualitative form

$$f(x) = \sum_{n \geq 0} P(n)n!x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \ \delta = x\frac{d}{dx}. \qquad (1)$$

So, we reduce previous series to the standard geometric progression series.
This series is convergent for $|x| < 1$ or for $|x|_p = p^{-k} < 1$, $x = p^k a/b$, $k \geq$
$1$, $p = 2, 3, 5, ..., 29, ..., 137, ...$ With an appropriate nomalization of the ex-
pansion parameter, the coefficients of the series are rational numbers and
if experimental data indicates for some prime value for $x$, e.g. in QED,
$x = \alpha = e^2/(4\pi) = 1/137.036...$, then we can take corresponding prime
number and consider p-adic convergence of the series.

In the Yukawa theory of strong interactions (see e.g. [8]) ), we take
$x = \alpha_{\pi N} = 13$.

So, the series is convergent. If the limit is rational number, we consider
it as an observable value of the corresponding physical quantity. In $MSSM$
(see [10] ) coupling constants unifies at $\alpha_u^{-1} = 26.3 \pm 1.9 \pm 1$. So, $23.4 <$
$\alpha_u^{-1} < 29.2$.

*Question:* how many primes are in this interval? $(24, 25, 26, 27, 28, 29)$.
Only one!

*Proposal:* take the value $\alpha_u^{-1} = 29.0...$ which will be two orders of mag-
nitude more precise prediction and find the consequences for the $SM$ scale
observables [12].

**Hamiltonization of Dynamical Systems.** Let us consider the fol-
lowing system of the ordinary differential equations [13]

$$\dot{x}_n = v_n(x), \ 1 \leq n \leq N, \qquad (2)$$

Lagrangian,

$$L = (\dot{x}_n - v_n(x))\psi_n \qquad (3)$$

and the corresponding motion equations

$$\dot{x}_n = v_n(x), \dot{\psi}_n = -\frac{\partial v_m}{\partial x_n}\psi_m. \qquad (4)$$

The system (4) extends the general system (2) by linear equation for the $\psi$.
The extended system can be put in the Hamiltonian form [16].

In the Faddeev-Jackiw formalism [14] for the Hamiltonian treatment of
systems defined by first-order Lagrangians,

$$L = f_n(x)\dot{x}_n - H(x),$$

motion equations

$$f_{mn}\dot{x}_n = \frac{\partial H}{\partial x_m},$$

for the regular structure function $f_{mn}$, can be put in the explicit Hamiltonian form

$$\dot{x}_n = f_{nm}^{-1}\frac{\partial H}{\partial x_m} = \{x_n, x_m\}\frac{\partial H}{\partial x_m} = \{x_n, H\},$$

where the fundamental Poisson (Dirac) bracket is

$$\{x_n, x_m\} = f_{nm}^{-1}, \ f_{mn} = \partial_m f_n - \partial_n f_m.$$

The system (4) is an important example of the first order regular Hamiltonian systems. Indeed, in the new variables, $y_n^1 = x_n, y_n^2 = \psi_n$, Lagrangian (3) takes the following first order form [16]

$$L = (\dot{x}_n - v_n(x))\psi_n \Rightarrow \frac{1}{2}(\dot{x}_n\psi_n - \dot{\psi}_n x_n) - v_n(x)\psi_n = \frac{1}{2}y_n^a\varepsilon^{ab}\dot{y}_n^b - H(y) =$$

$$= f_n^a(y)\dot{y}_n^a - H(y), f_n^a = \frac{1}{2}y_n^b\varepsilon^{ba}, H = v_n(y^1)y_n^2, f_{nm}^{ab} = \frac{\partial f_m^b}{\partial y_n^a} - \frac{\partial f_n^a}{\partial y_m^b} = \varepsilon^{ab}\delta_{nm};$$

corresponding motion equations and the fundamental Poisson bracket are

$$\dot{y}_n^a = \varepsilon_{ab}\delta_{nm}\frac{\partial H}{\partial y_m^b} = \{y_n^a, H\}, \ \{y_n^a, y_m^b\} = \varepsilon_{ab}\delta_{nm}.$$

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G. Giorgadze
I.Vekua Institute of Applied Mathematics of
Iv.Javakhishvili Tbilisi State University
2 University St., Tbilisi 0186
Georgia
E-mail: gia.giorgadze@tsu.ge

N. Makhaldiani
Joint Institute for Nuclear Research
6 Joliot-Curie St. Dubna, Moscow region 141980
Russia
E-mail: mnv@jinr.ru